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# **Computing Parameterized Invariants of Parameterized Petri Nets**

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**Abstract.** A fundamental advantage of Petri net models is the possibility to automatically compute useful system invariants from the syntax of the net. Classical techniques used for this are place invariants, P-components, siphons or traps. Recently, Bozga et al. have presented a novel technique for the *parameterized* verification of safety properties of systems with a ring or array architecture. They show that the statement "for every instance of the parameterized Petri net, all markings satisfying the linear invariants associated to all the P-components, siphons and traps of the instance are safe" can be encoded in WS1S and checked using tools like MONA. However, while the technique certifies that this infinite set of linear invariants extracted from P-components, siphons or traps are strong enough to prove safety, it does not return an explanation of this fact understandable by humans. We present a CEGAR loop that constructs a *finite* set of *parameterized* P-components, siphons or traps, whose infinitely many instances are strong enough to prove safety. For this we design parameterization procedures for different architectures.

Keywords: parameterized systems, logic, theorem proving, first-order, WS1S

## 1. Introduction

A fundamental advantage of Petri net system models is the possibility to automatically extract useful system invariants from the syntax of the net at low computational cost. Classical techniques used for this purpose are place invariants, P-components, siphons or traps [1, 2, 3]. All of them are syntactic objects that can be computed using linear algebra or boolean logic, and from which semantic linear

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invariants can be extracted. For example, from the fact that a set of places Q is an initially marked trap of the net one extracts the linear invariant  $\sum_{p \in Q} M(Q) \ge 1$ , which is satisfied for every reachable marking M. This information can be used to prove safety properties: Given a set S of safe markings, if every marking satisfying the invariants extracted from a set of objects is safe, then all reachable markings are safe.

Classical net invariants have been very successfully used in the verification of single systems [4, 5, 6], or as complement to state-space exploration [7]. Recently, an extension of this idea to the *parameterized* verification of safety properties of systems with a ring or array architecture has been presented in [8, 9]. The parameterized verification problem asks whether a system composed of n processes is safe for every  $n \ge 2$  [10, 11, 12]. Bozga *et al.* show in [8, 9] that the statement

"For every instance of the parameterized system, all markings satisfying the linear invariants associated to all the P-components, siphons and traps of the corresponding Petri net are safe"

can be encoded in Weak Second-order Logic With One Successor WS1S, or its analogous WS2S for two successors. This means that the statement holds iff its formula encoding is valid. This problem is decidable, and highly optimized tools exist for it, like MONA [13, 14]. The method of [9] is not complete (i.e., there are safe systems for which the invariants derived from P-components, siphons and traps are not strong enough to prove safety), but it succeeds for a remarkable set of examples. Further, incompleteness is inherent to every algorithmic method, since safety of parameterized nets is undecidable even if processes only manipulate data from a bounded domain [15, 10].

While the technique of [8, 9] is able to prove interesting properties of numerous systems, it does not yet provide an explanation of why the property holds. Indeed, when the technique succeeds for a given parameterized Petri net, the user only knows that the set of all invariants deduced from siphons, traps, and P-components together are strong enough to prove safety. However, the technique does not return a minimal set of these invariants. Moreover, since the parameterized Petri net has infinitely many instances, such a set contains infinitely many invariants. In this paper we show how to overcome this obstacle. We present a technique that automatically computes a *finite* set of *parameterized invari*ants, readable by humans. This is achieved by lifting a CEGAR (counterexample-guided abstraction refinement) loop, introduced in [16] and further developed in [5, 17, 18], to the parameterized case. Each iteration of the loop of [16, 5] first computes a counterexample, i.e., a marking that violates the desired safety property but satisfies all invariants computed so far, and then computes a P-component, siphon, or trap showing that the marking is not reachable. If no counterexample exists the property is established, and if no P-component, siphon or trap can be found the method fails. The technique is implemented on top of an SMT-solver, which receives as input a linear constraint describing the set of safe markings, and iteratively computes the set of linear invariants derived from P-components, siphons, and traps.

If we naively lift the CEGAR loop to the parameterized case, the loop never terminates. Indeed, since the loop computes one new invariant per iteration, and infinitely many invariants are needed to prove correctness of all instances, termination is not possible. So we need a procedure to extract from one single invariant for one instance a *parameterized* invariant, i.e., an infinite set of invariants for all instances, finitely represented as a WS1S-formula. We present a *semi-automatic* and an *automatic* 

*approach*. In the semi-automatic approach the user guesses the parameterized invariant, and automatically checks it, using the WS1S-checker. The automatic approach does not need user interaction, but only works for systems with symmetric structure. We provide automatic procedures for systems with a ring topology, and for barrier crowds, a class of systems closely related to broadcast protocols. We also show how to extend our results to inspection programs, a class of distributed programs in which an agent can loop through all other agents, inspecting their local states. In this extension infinite sets of invariants can no longer be represented by a WS1S-formula and we must move to a more general logical framework. While the satisfiability problem is undecidable for this extended framework, we can still prove correctness of some systems with the help of an automatic theorem prover for first-order logic. Finally, we present experimental results on a number of systems.

**Related work.** The parameterized verification problem has been extensively studied for systems whose associated transition systems are well-structured [19, 20, 21] (see e.g. [12] for a survey). In this case the verification problem reduces to a coverability problem, for which different algorithms exist [22, 23, 24, 25]; the marking equation (which is roughly equivalent to place invariants) have also been applied [26]. However, the transition systems of parametric rings and arrays are typically not well-structured.

Parameterized verification of ring and array systems has also been studied in a number of papers. Three popular techniques are regular model checking (see e.g. [27, 28, 29]), abstraction [30, 31], and automata learning [32]. All of them apply symbolic state-space exploration to try to compute a finite automaton recognizing the set of reachable markings of all instances, or an abstraction thereof. Our technique avoids any state-space exploration. Also, symbolic state-space exploration techniques are not geared towards providing explanations. Indeed, while the set of reachable markings of all instances is the strongest invariant of the system, it is also one single monolithic invariant, typically difficult to interpret by human users. Our CEGAR loop aims at finding a *collection* of invariants, each of them simple and interpretable.

Many works in the parameterized setting follow the cut-off approach, where one manually proves a *cut-off* bound  $c \ge 2$  such that correctness for at most c processes implies correctness for any number of processes (see e.g. [33, 34, 35, 36, 37], and [10] for a survey). It then suffices to prove the property for systems of up to c processes, which can be done using finite-state model checking techniques. Compared to this technique, ours is fully automatic.

## 2. Preliminaries

**WS1S.** Formulas of WS1S over first-order variables  $x, y, \ldots$  and second-order variables  $X, Y, \ldots$  have the following syntax:

$$t := \boldsymbol{x} \mid 0 \mid \text{succ}(t)$$
(terms)  
$$\phi := t_1 \le t_2 \mid \boldsymbol{x} \in \boldsymbol{X} \mid \phi_1 \land \phi_2 \mid \neg \phi_1 \mid \exists \boldsymbol{x} \colon \phi \mid \exists \boldsymbol{X} \colon \phi$$
(formulas)

An interpretation assigns elements of  $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$  to first order variables and *finite* subsets of  $\mathbb{N}_0$  to second-order variables. Given an interpretation, the semantics that assigns numbers to terms and truth values to formulas is defined in the usual way.

We extend the syntax with constants 0, 1, 2, 3, ..., and terms of the form x + c with  $c \in \mathbb{N}_0$ . Further, a term  $x \oplus_n 1$  in a formula  $\varphi$  stands for

$$(\boldsymbol{x}+1 < \boldsymbol{n} \land \varphi[\boldsymbol{x} \oplus_{\boldsymbol{n}} 1 \leftarrow \boldsymbol{x}+1]) \lor (\boldsymbol{n} = \boldsymbol{x}+1 \land \varphi[\boldsymbol{x} \oplus_{\boldsymbol{n}} 1 \leftarrow 0])$$

where  $\varphi[t \leftarrow t']$  denotes the result of substituting t' for t in  $\varphi$ . The terms  $x \oplus_n c$  for every  $1 \le c$  are defined similarly. We let  $\varphi(x_1, \ldots, x_\ell, X_1, \ldots, X_k)$  denote that  $\varphi$  uses at most  $x_1, \ldots, x_\ell$  and  $X_1, \ldots, X_k$  as free first-order resp. second-order variables. Finally, we also make liberal use of the following macros:

$oldsymbol{X}=\emptyset$		$orall oldsymbol{x} \colon  eg (oldsymbol{x} \in oldsymbol{X})$
$oldsymbol{X} = \{oldsymbol{x}\}$		$oldsymbol{x} \in oldsymbol{X} \wedge orall oldsymbol{y} \colon oldsymbol{y} \in oldsymbol{X}  o oldsymbol{y} = oldsymbol{x}$
$oldsymbol{X} = [oldsymbol{n}]$		$orall oldsymbol{x} \colon oldsymbol{x} \in oldsymbol{X} \leftrightarrow oldsymbol{x} < oldsymbol{n}$
$\boldsymbol{X} \cap \boldsymbol{Y} = \emptyset$	stands for	$orall oldsymbol{x} \colon  eg (oldsymbol{x} \in oldsymbol{X} \wedge oldsymbol{x} \in oldsymbol{Y})$
$ \boldsymbol{X}  = 1$	stanus ioi	$\exists oldsymbol{x} \colon oldsymbol{X} = \{oldsymbol{x}\}$
$ \boldsymbol{X}  \leq 1$		$\boldsymbol{X} = \emptyset \lor  \boldsymbol{X}  = 1$
$X=\overline{Y}$		$orall oldsymbol{x} \colon oldsymbol{x} \in oldsymbol{X} \leftrightarrow (oldsymbol{x} < oldsymbol{n} \wedge  eg(oldsymbol{x} \in oldsymbol{Y}))$
$\boldsymbol{Y} = \boldsymbol{X} \oplus_{\boldsymbol{n}} 1$		$orall oldsymbol{x} \colon oldsymbol{x} \oplus_{oldsymbol{n}} 1 \in oldsymbol{Y} \leftrightarrow oldsymbol{x} \in oldsymbol{X}$

**Petri nets.** We use a presentation of Petri nets equivalent to but slightly different from the standard one. A *net* is a pair  $\langle P, T \rangle$  where P is a nonempty, finite set of *places* and  $T \subseteq 2^P \times 2^P$  is a set of *transitions*. Given a transition  $t = \langle P_1, P_2 \rangle$ , we call  $P_1$  the *preset* and *postset* of t, respectively. We also denote  $P_1$  by  $\bullet t$  and  $P_2$  by  $t^{\bullet}$ . Given a place p, we denote by  $\bullet p$  and  $p^{\bullet}$  the sets of transitions  $\langle P_1, P_2 \rangle$  such that  $p \in P_2$  and  $p \in P_1$ , respectively. Given a set X of places or transitions, we let  $\bullet X := \bigcup_{x \in X} \bullet x$  and  $X^{\bullet} := \bigcup_{x \in X} x^{\bullet}$ .

A marking of  $N = \langle P, T \rangle$  is a function  $M: P \to \mathbb{N}$ . A Petri net is a pair  $\langle N, M \rangle$ , where N is a net and M is the *initial marking* of N. A transition  $t = \langle P_1, P_2 \rangle$  is enabled at a marking M if  $M(p) \ge 1$  for every  $p \in P_1$ . If t is enabled at M then it can *fire*, leading to the marking M' given by M'(p) = M(p) + 1 for every  $p \in P_2 \setminus P_1$ , M'(p) = M(p) - 1 for every  $p \in P_1 \setminus P_2$ , and M'(p) = M(p) otherwise. We write  $M \xrightarrow{t} M'$ , and  $M \xrightarrow{\sigma} M'$  for a finite sequence  $\sigma = t_1 t_2 \dots t_n$  if there are markings  $M_1, \dots, M_n$  such that  $M \xrightarrow{t_1} M_1 \xrightarrow{t_2} \dots M_{n-1} \xrightarrow{t_n} M'$ . M' is reachable from M if  $M \xrightarrow{\sigma} M'$  for some sequence  $\sigma$ .

A marking M is 1-bounded if  $M(p) \leq 1$  for every place p. A Petri net is 1-bounded if every marking reachable from the initial marking is 1-bounded. A 1-bounded marking M of a Petri net is also defined by the set of marked places; i.e.,  $(M) = \{p \in P : M(p) = 1\}$ .

## 3. Parameterized Petri nets

Intuitively, a parameterized net is a collection  $\{N_n\}_{n\geq 1}$  of nets. The places of  $N_n$  are the result of replicating a set  $\mathcal{P}$  of place names n times. For example, if  $\mathcal{P} = \{p, q\}$ , then the set of places of  $N_n$  is  $\{p(0), \ldots, p(n-1), q(0), \ldots, q(n-1)\}$ . Crucially, the transitions of all the nets in the collection

are described by a single logical formula of WS1S. Intuitively, the models of the formula are triples  $\langle n, P_1, P_2 \rangle$ , where  $P_1$  and  $P_2$  are sets of places of  $N_n$ , indicating that  $N_n$  has a transition with  $P_1$  and  $P_2$  as input and output places, respectively.

### **Definition 3.1. (Parameterized nets)**

A parameterized net is a pair  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$ , where  $\mathcal{P}$  is a finite set of place names and  $Tr(n, \mathcal{X}, \mathcal{Y})$  is a WS1S-formula over one first-order variable n which represents the size of the instance, and two tuples  $\mathcal{X}$  and  $\mathcal{Y}$  of second-order variables containing one variable for each place name of  $\mathcal{P}$ ; i.e., for a fixed enumeration  $p_1, \ldots, p_k$  of the elements of  $\mathcal{P}$  we have  $\mathcal{X} = \langle \mathcal{X}_{p_i} \rangle_{i=1}^k$  and  $\mathcal{Y} = \langle \mathcal{Y}_{p_i} \rangle_{i=1}^k$ . We call such tuples of variables *placeset variables*.

Let  $[n] = \{0, ..., n-1\}$ . A parameterized net  $\mathcal{N}$  induces a net  $\mathcal{N}(n) = \langle P_n, T_n \rangle$  for every  $n \geq 1$ , where  $P_n = \mathcal{P} \times [n]$  (i.e.,  $P_n$  consists of n copies of  $\mathcal{P}$ ), and  $T_n$  contains a transition  $\langle P_1, P_2 \rangle$  for every pair  $P_1, P_2 \subseteq P_n$  of sets of places such that " $Tr(n, P_1, P_2)$  holds". More formally, this means that  $\mu \models Tr$  for the interpretation  $\mu$  given by  $\mu(n) = n$ ,  $\mu(\mathcal{X}_p) = \{i \in [n] : \langle p, i \rangle \in P_1\}$ , and  $\mu(\mathcal{Y}_p) = \{i \in [n] : \langle p, i \rangle \in P_2\}$  for all  $p \in \mathcal{P}$ . Therefore, the intended meaning of  $Tr(n, \mathcal{X}, \mathcal{Y})$  is "the pair  $\langle \mathcal{X}, \mathcal{Y} \rangle$  of placesets is (the preset and postset of) a transition of the net  $\mathcal{N}(n)$ ". We say that  $\mathcal{N}(n)$  is an *instance* of  $\mathcal{N}$ .

In the following we use  $\langle p, i \rangle$  and p(i) as equivalent notations for the elements of  $P_n = \mathcal{P} \times [n]$ .

Equation 1: Transitions of the dining philosophers.

**Example 3.2.** We consider a version of the dining philosophers. Philosophers and forks are numbered  $0, 1, \ldots, n-1$ . For every i > 0 the *i*-th philosopher first grabs the *i*-th fork, and then the  $(i \oplus_n 1)$ -th fork, where  $\oplus_n$  denotes addition modulo n. Philosopher 0 proceeds the other way round: she first grabs fork 1, and then fork 0. After eating, a philosopher returns both forks in one single atomic step. We formalize this in the following parameterized net  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$ :

- $\mathcal{P} = \{\text{think}, \text{wait}, \text{eat}, \text{free}, \text{taken}\}$ . Intuitively,  $\{\text{think}(i), \text{wait}(i), \text{eat}(i)\}$  are the states of the *i*-th philosopher, and  $\{\text{free}(i), \text{taken}(i)\}$  the states of the *i*-th fork.
- *Tr*(*n*, *X*, *Y*) = GrabFirst ∨ GrabSecond ∨ Release. The formulas for GrabFirst, GrabSecond, and Release are shown in Equation 1.

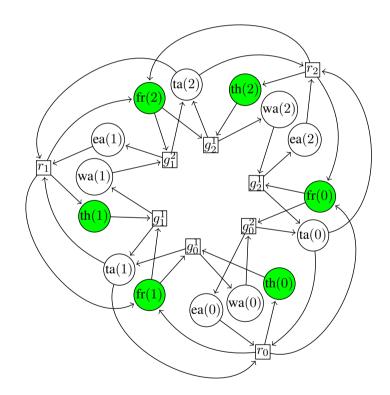


Figure 1:  $\mathcal{N}(3)$  for Example 3.2. Places which are colored green are initially marked w.r.t.  $Initial(\mathcal{X})$  from Example 3.4. Note the repeating structure for philosophers 1 and 2 while philosopher 0 grabs her forks in the opposite order. We abbreviate think(*i*) to th(*i*), and similarly with the other states.

Intuitively, the preset of GrabFirst is a philosopher in state think and her left (resp. right fork for philosopher 0) in state free; the postset puts the philosopher in state wait and the fork in state taken. The instance  $\mathcal{N}(3)$  is shown in Figure 1.

Parameterized Petri nets are parameterized nets with a WS1S-formula defining its initial markings:

#### **Definition 3.3. (Parameterized Petri nets)**

A parameterized Petri net is a pair  $\langle \mathcal{N}, Initial \rangle$ , where  $\mathcal{N}$  is a parameterized net, and  $Initial(n, \mathcal{M})$  is a WS1S-formula over a first-order variable n and a placeset variable  $\mathcal{M}$ .

A parameterized Petri net defines an infinite family of Petri nets. Loosely speaking, a Petri net  $\langle N, M \rangle$  belongs to the family if N is an instance of  $\mathcal{N}$ , i.e.,  $N = \mathcal{N}(n)$  for some  $n \geq 1$ , and M is a 1-bounded marking of N satisfying  $Initial(n, \mathcal{M})$ . For example, if  $\mathcal{P} = \{p_1, p_2\}$ , n = 3 and  $Initial(\{0, 1\}, \{0, 2\})$  holds, then the family contains a Petri net  $\langle \mathcal{N}(3), M_3 \rangle$  such that  $M_3$  is a 1-bounded marking with  $\langle M_3 \rangle = \{p_1(0), p_1(1), p_2(0), p_2(2)\}$ .

**Example 3.4.** The family of initial markings in which all philosophers think and all forks are free is modeled by:

$$Initial(\boldsymbol{n},\mathcal{M}) \coloneqq (\mathcal{M}_{think} = \mathcal{M}_{free} = [\boldsymbol{n}]) \land (\mathcal{M}_{wait} = \mathcal{M}_{eat} = \mathcal{M}_{taken} = \emptyset).$$

**Example 3.5.** Let us now model a simple version of the readers/writers system. A process can be idle, reading, or writing. An idle process can start to read if no other process is writing, and it can start to write if every other process is idle. We obtain the parameterized net  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$ , where

- $\mathcal{P} = \{ idle, rd, wr, not\_wr \}.$
- $Tr(n, \mathcal{X}, \mathcal{Y}) = \text{StartR} \lor \text{StopR} \lor \text{StartW} \lor \text{StopW}$ . We give the formulas StartR and StartW, the other two being simpler.

$$\begin{aligned} \text{Start}\mathbf{R} &\coloneqq \exists \boldsymbol{x} . \begin{pmatrix} (\mathcal{X}_{\text{idle}} = \{\boldsymbol{x}\} \land \mathcal{X}_{\text{not\_wr}} = \overline{\mathcal{X}_{\text{idle}}} \land (\mathcal{X}_{\text{rd}} = \mathcal{X}_{\text{wr}} = \emptyset) \\ \land \mathcal{Y}_{\text{rd}} = \{\boldsymbol{x}\} \land \mathcal{Y}_{\text{not\_wr}} = \overline{\mathcal{X}_{\text{idle}}} \land (\mathcal{Y}_{\text{idle}} = \mathcal{Y}_{\text{wr}} = \emptyset) \end{pmatrix} \end{aligned}$$
$$\\ \text{Start}\mathbf{W} &\coloneqq \exists \boldsymbol{x} . \begin{pmatrix} \mathcal{X}_{\text{idle}} = [\boldsymbol{n}] \land \mathcal{X}_{\text{not\_wr}} = \{\boldsymbol{x}\} \land (\mathcal{X}_{\text{rd}} = \mathcal{X}_{\text{wr}} = \emptyset) \\ \land \mathcal{Y}_{\text{idle}} = [\boldsymbol{n}] \setminus \{\boldsymbol{x}\} \land \mathcal{Y}_{\text{wr}} = \{\boldsymbol{x}\} \land (\mathcal{Y}_{\text{rd}} = \mathcal{Y}_{\text{not\_wr}} = \emptyset) \end{pmatrix} \end{aligned}$$

So the preset of a StartR transition is  $\{idle(i), not\_wr(0), ..., not\_wr(n-1)\}$  for some *i*, and the postset is  $\{rd(i), not\_wr(0), ..., not\_wr(n-1)\}$ . The initial markings in which every process is initially idle are modeled by:

$$Initial(\boldsymbol{n}, \mathcal{X}) \coloneqq \mathcal{X}_{idle} = [\boldsymbol{n}] \land \mathcal{X}_{not\_wr} = [\boldsymbol{n}] \land (\mathcal{X}_{rd} = \mathcal{X}_{wr} = \emptyset)$$

Observe that in the dining philosophers transitions have presets and postsets of size 3, independently of the number of philosophers. On the contrary, in the readers and writers problems the transitions of  $\mathcal{N}(n)$  have presets and postsets of size n.

Intuitively, our formalism allows to model transitions involving all processes or, for example, all even processes. Observe also that in both cases the formula *Initial* has exactly one model for every  $n \ge 1$ , but this is not required.

**Proving deadlock-freedom for the dining philosophers.** Let us now give a taste of what our paper achieves for Example 3.2. It is well known that this version of the dining philosophers is deadlock-free. However, finding a proof based on parameterized invariants of the systems is not so easy. Using the semi-automatic version of the approach we present, we can find the five invariants shown below, and automatically prove that they imply deadlock-freedom. The fully automatic analysis of this example gives ten properties of the system which collectively induce deadlock-freedom.

The first two invariants express that at every reachable marking M, and for every  $0 \le i \le n-1$ , the *i*-th philosopher is either thinking, waiting, or eating, and the *i*-th fork is either free or taken:

$$M(\operatorname{think}(i)) + M(\operatorname{wait}(i)) + M(\operatorname{eat}(i)) = 1$$
(1)

$$M(\text{free}(i)) + M(\text{taken}(i)) = 1.$$
(2)

The last three invariants provide the key insights; the last one holds for every  $1 \le i \le n-2$ :

$$M(wait(0)) + M(eat(0)) + M(free(1)) + M(wait(1)) + M(eat(1)) = 1$$
(3)

$$M(eat(0)) + M(free(0)) + M(eat(n-1)) = 1$$
(4)

$$M(eat(i)) + M(eat(i+1)) + M(free(i+1)) + M(wait(i+1)) = 1$$
(5)

Let us sketch why (1)-(5) imply deadlock freedom. Let  $P_i$  denote the *i*-th philosopher and  $F_i$  the *i*-th fork. If  $P_0$  is eating, then  $F_0$  and  $F_1$  are taken by (1)-(4), and there is no deadlock because  $P_0$  can return them. The same holds if  $P_1$  is eating by (1)-(3) and (5), or if any of  $P_2, \ldots, P_{n-1}$  is eating by (1)-(2) and (5). If no philosopher eats, then by (1)-(3) and (5) either  $P_{i+1}$  is thinking and  $F_{i+1}$  is free for some  $i \in \{1, \ldots, n-2\}$ , or  $P_{i+1}$  is waiting for every  $i \in \{1, \ldots, n-2\}$ . In the first case  $P_{i+1}$  can grab  $F_{i+1}$ . In the second case  $P_{n-1}$  is waiting, and since  $F_0$  is free by (1)-(2) and (4), it can grab  $F_0$ .

## 4. Checking 1-boundedness

Our techniques work for parameterized Petri nets whose instances are 1-bounded. We present a technique that automatically checks 1-boundedness of all our examples. We say that a set of places Q of a Petri net  $\langle N, M \rangle$ , where  $N = \langle P, T \rangle$ , is

- *1-balanced* if for every transition  $\langle P_1, P_2 \rangle \in T$  either  $|P_1 \cap Q| = 1 = |P_2 \cap Q|$ , or  $|P_1 \cap Q| = 0 = |P_2 \cap Q|$ , or  $|P_1 \cap Q| \ge 2$ .
- *1-bounded* at M if  $M(Q) \leq 1$ .

The following proposition is an immediate consequence of the definition:

**Proposition 4.1.** If Q is a 1-balanced and 1-bounded set of places of  $\langle N, M \rangle$ , then M'(Q) = M(Q) holds for every reachable marking M'.

We abbreviate "1-bounded and 1-balanced set" to 1BB-set, and say that N is *covered* by 1BB-sets if every place belongs to some 1BB-set at initial marking M. By the proposition above, if N is

covered by 1BB-sets at M, then  $M'(p) \leq 1$  holds for every reachable marking M' and every place p, and so N is 1-bounded.

Given a parameterized Petri net (N, *Initial*), we can check if all instances are covered by 1BB-sets with the following formula:

$$\begin{aligned} 1Bal(\boldsymbol{n},\mathcal{X}) &\coloneqq \forall \mathcal{Y}, \mathcal{Z} \colon Tr(\boldsymbol{n},\mathcal{Y},\mathcal{Z}) \to & (|\mathcal{X} \cap \mathcal{Y}| = 0 = |\mathcal{X} \cap \mathcal{Z}|) \lor \\ & (|\mathcal{X} \cap \mathcal{Y}| = 1 = |\mathcal{X} \cap \mathcal{Z}|) \lor \\ & (|\mathcal{X} \cap \mathcal{Y}| = 1 = |\mathcal{X} \cap \mathcal{Z}|) \lor \\ & (|\mathcal{X} \cap \mathcal{Y}| > 1) \end{aligned}$$
$$\begin{aligned} 1Bnd(\boldsymbol{n},\mathcal{X},\mathcal{M}) &\coloneqq & \forall \boldsymbol{n}, \forall \mathcal{M} \colon Initial(\boldsymbol{n},\mathcal{M}) \to & (\bigwedge_{p \in \mathcal{P}} \forall \boldsymbol{x} \colon \exists \mathcal{X} \colon \boldsymbol{x} \in \mathcal{X}_p \land \\ & 1Bal(\boldsymbol{n},\mathcal{X}) \land \\ & 1Bnd(\boldsymbol{n},\mathcal{X},\mathcal{M})) \end{aligned}$$

Observe that if Q is a 1BB-set then at every reachable marking exactly one of the places of Q is marked, with exactly one token. The sets of places corresponding to a philosopher, a fork, a reader, or a writer are 1BB-sets. Unsurprisingly, all our parameterized Petri net models are covered by 1BB-sets. Checking the formula *Cover* above gives us an automatic proof that all the Petri nets we consider are 1-bounded.

# 5. Checking safety properties

Let  $\langle \mathcal{N}, Initial \rangle$  be a parameterized Petri net, and let  $Safe(n, \mathcal{M})$  be a WS1S-formula describing a set of "safe" markings of the instances of  $\mathcal{N}$  (for example, "safe" could mean deadlock-free). It is easy to prove (using simulations of Turing machines by Petri nets like those of [38]) that the existence of some unsafe reachable marking in some instance of a given parameterized Petri net  $\langle \mathcal{N}, Initial \rangle$  is undecidable. In [9, 8] there is a semi-algorithm for the problem that derives from  $\langle \mathcal{N}, Initial \rangle$  a formula *PReach*( $n, \mathcal{M}$ ) describing a superset of the set of reachable markings of all instances, and checks that the formula

$$SafetyCheck := \forall n \forall \mathcal{M} \colon PReach(n, \mathcal{M}) \to Safe(n, \mathcal{M})$$

holds. We recall the main construction of [9, 8], adapted and expanded.

**1BB-sets again.** Recall that if a marking M' of some instance  $\langle N, M \rangle$  of a net  $\langle N, Initial \rangle$  is reachable from M, then  $M'(Q) \leq 1$  holds for every 1BB-set of places Q of  $\langle N, M \rangle$ . So this latter property can be interpreted as a test for potential reachability: Only markings that pass the test can be reachable. We introduce a formula  $1BBTest(n, \mathcal{M}', \mathcal{M})$  expressing that  $\mathcal{M}'$  passes the test with respect to  $\mathcal{M}$  (i.e.,  $\mathcal{M}'$  might be reachable from  $\mathcal{M}$ ).

$$1BBTest(\boldsymbol{n},\mathcal{M}',\mathcal{M}) \coloneqq \forall \mathcal{X} \colon \begin{pmatrix} 1Bal(\boldsymbol{n},\mathcal{X}) \\ \land 1Bnd(\boldsymbol{n},\mathcal{X},\mathcal{M}) \end{pmatrix} \to 1Bnd(\boldsymbol{n},\mathcal{X},\mathcal{M}')$$

**Siphons and traps.** Let  $\langle N, M \rangle$  be a Petri net with  $N = \langle P, T \rangle$  and let  $Q \subseteq P$  be a set of places. Q is a *trap* of N if  $\bullet Q \subseteq Q^{\bullet}$ , and a *siphon* of N if  $Q^{\bullet} \subseteq \bullet Q$ .

- If Q is a siphon and M(Q) = 0, then M'(Q) = 0 for all markings M' reachable from M.
- If Q is a trap and  $M(Q) \ge 1$ , then  $M'(Q) \ge 1$  for all markings M' reachable from M.

If M' is reachable from M then it satisfies the following property:  $M'(Q) \ge 1$  for every trap Q such that  $M(Q) \ge 1$ . A marking satisfying this property *passes the trap test* for  $\langle N, M \rangle$ . We construct a formula  $TrapTest(n, \mathcal{M})$  expressing that  $\mathcal{M}$  passes the trap test for some instance of a parameterized Petri net. We first introduce a formula expressing that a set  $\mathcal{X}$  of places is a trap.

$$Trap(\boldsymbol{n},\mathcal{X}) \coloneqq \forall \mathcal{Y}, \mathcal{Z} \colon (Tr(\boldsymbol{n},\mathcal{Y},\mathcal{Z}) \land \mathcal{X} \cap \mathcal{Y} \neq \emptyset) \to \mathcal{X} \cap \mathcal{Z} \neq \emptyset$$

Now we have:

$$\begin{aligned} &Marked(\boldsymbol{n},\mathcal{X},\mathcal{M}) &\coloneqq & \mathcal{X} \cap \mathcal{M} \neq \emptyset \\ &Trap\,Test(\boldsymbol{n},\mathcal{M}',\mathcal{M}) &\coloneqq & \forall \mathcal{X} \colon \begin{pmatrix} Trap(\boldsymbol{n},\mathcal{X}) \\ \wedge Marked(\boldsymbol{n},\mathcal{X},\mathcal{M}) \end{pmatrix} \to Marked(\boldsymbol{n},\mathcal{X},\mathcal{M}') \end{aligned}$$

Similarly we obtain a formula for a siphon test:

$$Empty(\boldsymbol{n}, \mathcal{X}, \mathcal{M}) \quad \coloneqq \quad \mathcal{X} \cap \mathcal{M} = \emptyset$$
  
SiphTest( $\boldsymbol{n}, \mathcal{M}', \mathcal{M}$ )  $\coloneqq \quad \forall \mathcal{X} \colon \begin{pmatrix} Siphon(\boldsymbol{n}, \mathcal{X}) \\ \wedge Empty(\boldsymbol{n}, \mathcal{X}, \mathcal{M}) \end{pmatrix} \rightarrow Empty(\boldsymbol{n}, \mathcal{X}, \mathcal{M}')$ 

We can now give the formula *PReach*:

$$PReach(\boldsymbol{n}, \mathcal{M}', \mathcal{M}) := \begin{pmatrix} 1BBTest(\boldsymbol{n}, \mathcal{M}', \mathcal{M}) \\ \wedge TrapTest(\boldsymbol{n}, \mathcal{M}', \mathcal{M}) \\ \wedge SiphTest(\boldsymbol{n}, \mathcal{M}', \mathcal{M}) \end{pmatrix}$$
$$PReach(\boldsymbol{n}, \mathcal{M}') := \exists \mathcal{M}: Initial(\boldsymbol{n}, \mathcal{M}) \wedge PReach(\boldsymbol{n}, \mathcal{M}', \mathcal{M})$$

## 5.1. Automatic computation of parameterized invariants

In [9] it was shown that many safety properties of parameterized Petri nets can be proved to hold *for all instances* by checking validity of the corresponding *PReach* formula. However, the technique does not return a set of invariants strong enough to prove the property. In this section we show how to overcome this problem. We design a CEGAR loop which, when successful, yields a finite set of *parameterized* invariants that imply the safety property being considered.

We proceed as follows. In the first part of the section, we describe a CEGAR loop for the nonparameterized case. The input to the procedure is a parameterized Petri net  $\langle N, Initial \rangle$  and a number n such that all reachable markings of all instances  $\mathcal{N}(1), \ldots, \mathcal{N}(n)$  are safe. The output is a set of invariants of  $\mathcal{N}(1), \ldots, \mathcal{N}(n)$ , derived from balanced sets, siphons, and traps, which are strong enough to prove safety. Since the set of all 1BB-sets, siphons, and traps of these instances is finite, the procedure is guaranteed to terminate even if it computes one invariant at a time. Then we modify the loop by inserting an additional *parameterization procedure* that exploits the regularity of  $\langle N, Initial \rangle$ . The procedure transforms a 1BB-set (siphon, trap) of a particular instance, say  $\mathcal{N}(4)$ , into a possibly infinite set of 1BB-sets (siphons, traps) of all instances, encoded as the set of models of a WS1S-formula. This formula is a finite representation of the infinite set.

For the sake of brevity, in the rest of the section we describe a CEGAR loop that only constructs traps. This allows us to avoid numerous repetitions of the phrase "1BB-sets, siphons, and traps". Since the structure of the loop is completely generic, this is purely a presentation issue without loss of generality<sup>1</sup>.

### 5.1.1. A CEGAR loop for the non-parameterized case.

We need some preliminaries. Let  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  be a parameterized net, and let  $\mathcal{X}$  be a placeset variable. An *interpretation* of  $\mathcal{X}$  is a pair  $\mathbf{X} = \langle n, Q \rangle$ , where  $n \geq 1$  and Q is a set of places of  $\mathcal{N}(n)$ . We identify  $\mathbf{X}$  and the tuple  $\langle \mathbf{X}_p \rangle_{p \in \mathcal{P}}$ , where  $\mathbf{X}_p \subseteq [n]$ , defined by  $j \in \mathbf{X}_p$  iff  $p(j) \in Q$ . For example, if  $\mathcal{P} = \{p, q, r\}, n = 2$ , and  $Q = \{p(0), p(1), q(1)\}$ , then  $\langle \mathbf{X}_p, \mathbf{X}_q, \mathbf{X}_r \rangle = \langle \{0, 1\}, \{1\}, \emptyset \rangle$ . Given a formula  $\phi(\ldots, \mathcal{X}, \ldots)$  and an interpretation  $\mathbf{X} = \langle n, Q \rangle$  of  $\mathcal{X}$ , we define the formula  $\phi(\ldots, \mathbf{X}, \ldots)$  as follows:

$$egin{aligned} & x \in \mathbf{X}_p & \coloneqq & \bigvee_{j \in \mathbf{X}_p} x = j \ & \mathcal{X} = \mathbf{X} & \coloneqq & n = n \land \bigwedge_{p \in \mathcal{P}} orall x \colon x < n 
ightarrow (x \in \mathcal{X}_p \leftrightarrow x \in \mathbf{X}_p) \ & \phi(\dots, \mathbf{X}, \dots) & \coloneqq & orall \mathcal{X} \colon \mathcal{X} = \mathbf{X} 
ightarrow \phi(\dots, \mathcal{X}, \dots) \end{aligned}$$

The CEGAR procedure maintains an (initially empty) set  $\mathcal{T}$  of *indexed traps* of  $\mathcal{N}(1), \mathcal{N}(2), \ldots, \mathcal{N}(n)$ , where an indexed trap is a pair  $\mathbf{T} = \langle i, Q \rangle$  such that  $1 \leq i \leq n$  and Q is a trap of  $\mathcal{N}(i)$ . After every update of  $\mathcal{T}$  the procedure constructs the formula  $SafetyCheck_{\mathcal{T}}$ , defined as follows:

$$TrapSet_{\mathcal{T}}(\boldsymbol{n},\mathcal{X}) \coloneqq \bigvee_{\mathbf{X}\in\mathcal{T}} \mathcal{X} = \mathbf{X}$$

$$PReach_{\mathcal{T}}(\boldsymbol{n},\mathcal{M}',\mathcal{M}) \coloneqq \forall \mathcal{X} \colon \begin{pmatrix} TrapSet_{\mathcal{T}}(\boldsymbol{n},\mathcal{X}) \\ \wedge Marked(\boldsymbol{n},\mathcal{X},\mathcal{M}) \end{pmatrix} \to Marked(\boldsymbol{n},\mathcal{X},\mathcal{M}') \qquad (6)$$

$$PReach_{\mathcal{T}}(\boldsymbol{n},\mathcal{M}') \coloneqq \exists \mathcal{M} \colon Initial(\boldsymbol{n},\mathcal{M}) \wedge PReach_{\mathcal{T}}(\boldsymbol{n},\mathcal{M}',\mathcal{M})$$

$$SafetyCheck_{\mathcal{T}} \coloneqq \forall \boldsymbol{n} \forall \mathcal{M} \colon (\boldsymbol{n} < \boldsymbol{n} \wedge PReach_{\mathcal{T}}(\boldsymbol{n},\mathcal{M})) \to Safe(\boldsymbol{n},\mathcal{M})$$

Intuitively,  $PReach_{\mathcal{T}}(n, \mathcal{M}', \mathcal{M})$  states that according to the set  $\mathcal{T}$  of (indexed) traps computed so far,  $\mathcal{M}'$  could still be reachable from  $\mathcal{M}$ , because every trap of  $\mathcal{T}$  marked at  $\mathcal{M}$  is also marked at  $\mathcal{M}'$ . Therefore, if  $SafetyCheck_{\mathcal{T}}$  holds then  $\mathcal{T}$  is already strong enough to show that every reachable marking is safe.

<sup>&</sup>lt;sup>1</sup>The CEGAR loop for the non-parametric case could be formulated in SAT and solved using a SAT-solver. However, we formulate it in WS1S, since this allows us to give a uniform description of the non-parametric and the parametric cases.

If  $\mathcal{T}$  is not strong enough, then the negation of  $SafetyCheck_{\mathcal{T}}$  is satisfiable. The WS1S-checker returns a counter-example, i.e., a model  $\mathbf{M} = \langle n, M \rangle$  of the formula  $PReach_{\mathcal{T}}(n, \mathcal{M}) \wedge \neg Safe(n, \mathcal{M})$ . Observe that n is a number, and M is a marking of the instance  $\mathcal{N}(n)$ , which is potentially reachable from an initial marking but not safe. In this case we search for a trap of  $\mathcal{N}(n)$  that is marked at every initial marking of  $\mathcal{N}(n)$ , but empty at  $\mathbf{M}$ . Such traps are the models of the formula

$$WTrap_{\mathbf{M}}(n, \mathcal{X}) \coloneqq \begin{pmatrix} Trap(n, \mathcal{X}) \\ \land (\forall \mathcal{M} \colon Initial(n, \mathcal{M}) \to Marked(n, \mathcal{X}, \mathcal{M})) \\ \land Empty(n, \mathcal{X}, M) \end{pmatrix}$$
(7)

and so they can also be found with the help of the WS1S-checker; notice, however, that after fixing  $n \mapsto n$  the universal quantifier of  $WTrap_{\mathbf{M}}(n, \mathcal{X})$  can be replaced by a conjunction, and so  $WTrap_{\mathbf{M}}(n, \mathcal{X})$  is equivalent to a Boolean formula.

If the formula has a model  $\mathbf{T} = \langle n, Q \rangle$ , then Q is a trap of  $\mathcal{N}(n)$ . We can now take  $\mathcal{T} := \mathcal{T} \cup \{\mathbf{T}\}$ , and iterate. Observe that after updating  $\mathcal{T}$  the interpretation  $\mathbf{M} = \langle n, M \rangle$  is no longer a model of  $PReach_{\mathcal{T}}(n, \mathcal{M}) \land \neg Safe(n, \mathcal{M})$ . Since  $\mathcal{N}(1), \ldots, \mathcal{N}(n)$  only have finitely many traps, the procedure eventually terminates.

#### 5.1.2. A CEGAR loop for the parameterized case.

In all nontrivial examples, proving safety of the infinitely many instances requires to compute infinitely many traps. Since the previous procedure only computes one trap per iteration, it does not terminate. The way to solve this problem is to insert a *parametrization step* that transforms the witness trap  $\mathbf{T} = \langle n, Q \rangle$  into a formula  $ParTrap_{\mathbf{T}}(n, \mathcal{X})$  satisfying two properties: (1) all models of the formula are traps, and (2)  $\mathbf{T}$  is a model. Since  $ParTrap_{\mathbf{T}}(n, \mathcal{X})$  can have infinitely many models, it constitutes a finite representation of an infinite set of traps. These models are also similar to each other and can be understood as capturing a single property of the system.

**Example 5.1.** Consider a parameterized net  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  exhibiting rotational symmetry: For every instance  $\mathcal{N}(n)$ , a pair  $(P_1, P_2)$  of sets is a transition of  $\mathcal{N}(n)$  iff the pair  $(P_1 \oplus_n 1, P_2 \oplus_n 1)$  is also a transition, where  $P \oplus_n 1$  denotes the result of increasing all indices by 1 modulo n. Assume that  $\mathcal{P} = \{p, q, r\}$  and  $\mathbf{T} = \langle 3, \{p(1), q(2)\} \rangle$ , i.e.,  $\{p(1), q(2)\}$  is a trap of  $\mathcal{N}(3)$ . It is intuitively plausible (and we will later prove) that, due to the rotational symmetry,  $\{p(i), q(i \oplus_m 1)\}$  is a trap of  $\mathcal{N}(j)$  for every  $m \geq 3$  and every  $0 \leq i \leq m - 1$ . We can then define the formula  $ParTrap_{\mathbf{X}}(n, \mathcal{X})$  as:

$$ParTrap_{\mathbf{T}}(\boldsymbol{n}, \mathcal{X}) \coloneqq \boldsymbol{n} \geq 3 \land \exists \boldsymbol{i} \colon \boldsymbol{i} < \boldsymbol{n}$$

$$\wedge orall oldsymbol{x} \colon oldsymbol{x} < oldsymbol{n} oldsymbol{d} \ (oldsymbol{x} \in \mathcal{X}_p \leftrightarrow oldsymbol{x} = oldsymbol{i}) \ \wedge oldsymbol{x} \in \mathcal{X}_q \leftrightarrow oldsymbol{x} = oldsymbol{i} \oplus_{oldsymbol{n}} 1) \ \wedge oldsymbol{x} 
otin \mathcal{X}_r \ \end{pmatrix}.$$

Now, in order to describe the CEGAR procedure for the parameterized case we only need to *rede-fine* the formula  $TrapSet_{\mathcal{T}}(n, \mathcal{X})$ . Instead of the formula  $TrapSet_{\mathcal{T}}(n, \mathcal{X}) := \bigvee_{\mathbf{T} \in \mathcal{T}} \mathcal{X} = \mathbf{T}$ , which

holds only when  $\mathcal{X}$  is one of the finitely many traps in  $\mathcal{T}$ , we insert the parametrization procedure and define

$$TrapSet_{\mathcal{T}}(\boldsymbol{n},\mathcal{X}) \coloneqq \bigvee_{\mathbf{T}\in\mathcal{T}} ParTrap_{\mathbf{T}}(\boldsymbol{n},\mathcal{X})$$

$$PReach_{\mathcal{T}}(\boldsymbol{n},\mathcal{M}',\mathcal{M}) \coloneqq \forall \mathcal{X} \colon \begin{pmatrix} TrapSet_{\mathcal{T}}(\boldsymbol{n},\mathcal{X}) \\ \land Marked(\boldsymbol{n},\mathcal{X},\mathcal{M}) \end{pmatrix} \to Marked(\boldsymbol{n},\mathcal{X},\mathcal{M}')$$

$$PReach_{\mathcal{T}}(\boldsymbol{n},\mathcal{M}') \coloneqq \exists \mathcal{M} \colon Initial(\boldsymbol{n},\mathcal{M}) \land PReach_{\mathcal{T}}(\boldsymbol{n},\mathcal{M}',\mathcal{M})$$

$$SafetyCheck_{\mathcal{T}} \coloneqq \forall \boldsymbol{n} \forall \mathcal{M} \colon PReach_{\mathcal{T}}(\boldsymbol{n},\mathcal{M}) \to Safe(\boldsymbol{n},\mathcal{M})$$

$$(8)$$

Notice the two differences with (6): the definition of  $TrapSet_{\mathcal{T}}(n, \mathcal{X})$ , and the absence of the condition n < n in the definition of  $SafetyCheck_{\mathcal{T}}$ . The question is how to obtain the formula  $ParTrap_{\mathbf{T}}(n, \mathcal{X})$  from  $\mathbf{T}$ . We discuss this point in the rest of the section.

A semi-automatic approach If we guess the formula  $ParTrap_{\mathbf{T}}(n, \mathcal{X})$  we can use the WS1Schecker to automatically prove that the guess is correct. Indeed, it suffices to check that all models of  $ParTrap_{\mathbf{T}}(n, \mathcal{X})$  are traps, which reduces to proving that the formula

$$\forall \boldsymbol{n} \forall \mathcal{X} \colon ParTrap_{\mathbf{T}}(\boldsymbol{n}, \mathcal{X}) \to Trap(\boldsymbol{n}, \mathcal{X})$$

holds. Let us see how this works in Example 3.2. Assume that the CEGAR procedure produces a trap  $\mathbf{T} = \langle 3, \{p(1), q(2)\} \rangle$ . The user finds it plausible that, due to the identical behavior of philosophers  $1, 2, \ldots, n-1$ , the set  $\{p(i), q(i \oplus 1)\}$  will be a trap of  $\mathcal{N}(n)$  for every  $n \ge 3$  and for every  $1 \le i \le n-2$  (i.e., the user excludes the case in which *i* or  $i \oplus_n 1$  are equal to 0). So the user guesses a new formula

$$\begin{aligned} ParTrap_{\mathbf{T}}(\boldsymbol{n},\mathcal{X}) \coloneqq & \boldsymbol{n} \geq 3 \land \exists \boldsymbol{i} \colon (1 \leq \boldsymbol{i} \leq \boldsymbol{n} - 2) \land \forall \boldsymbol{x} \colon \\ & (\boldsymbol{x} \in \mathcal{X}_p \leftrightarrow \boldsymbol{x} = \boldsymbol{i}) \land (\boldsymbol{x} \in \mathcal{X}_q \leftrightarrow \boldsymbol{x} = \boldsymbol{i} \oplus_{\boldsymbol{n}} 1) \land \boldsymbol{x} \notin \mathcal{X}_r \end{aligned}$$

The user now automatically checks that all models of  $ParTrap_{\mathbf{T}}(\mathbf{n}, \mathcal{X})$  are traps. The formula can then be safely added to  $TrapSet_{\mathcal{T}}(\mathbf{n}, \mathcal{X})$  as a new disjunct.

An automatic approach for specific architectures. Parameterized Petri nets usually have a regular structure. For example, in the readers-writers problem all processes are indistinguishable, and in the philosophers problem, all right-handed processes behave in the same way.

In the next sections we show how the structural properties of ring topologies and crowds (two common structures for parameterized systems) can be exploited to automatically compute the formula  $ParTrap_{T}(n, \mathcal{X})$  for each witness trap T.

## 6. Trap parametrization in rings

Intuitively, a parameterized net  $\mathcal{N}$  is a ring if for every transition of every instance  $\mathcal{N}(n)$  there is an index  $i \in [n]$  and sets  $\mathcal{P}_L, \mathcal{P}_R, \mathcal{Q}_L, \mathcal{Q}_R \subseteq \mathcal{P}$  such that the preset of the transition is  $(\mathcal{P}_L \times \{i\}) \cup (\mathcal{P}_R \times \{i \oplus_n 1\})$  and the postset is  $(\mathcal{Q}_L \times i) \cup (\mathcal{Q}_R \times i \oplus_n 1)$ . In other words, every transition involves only two neighbor-processes of the ring. In a fully symmetric ring all processes behave identically, while in a headed ring there is one distinguished process, as in Example 3.2. To ease presentation in this section we only consider fully symmetric rings. The extension to headed rings can be found in [39].

The informal statement "all processes behave identically" is captured by requiring the existence of a finite set of *transition patterns*  $\langle \mathcal{P}_L, \mathcal{P}_R, \mathcal{Q}_L, \mathcal{Q}_R \rangle$  such that the transitions of  $\mathcal{N}(n)$  are the result of "instantiating" each pattern with all pairs *i* and  $i \oplus_n 1$  of consecutive indices.

**Definition 6.1.** A parameterized net  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  is a *fully symmetric ring* if there is a finite set of transition patterns of the form  $\langle \mathcal{P}_L, \mathcal{P}_R, \mathcal{Q}_L, \mathcal{Q}_R \rangle$ , where  $\mathcal{P}_L, \mathcal{P}_R, \mathcal{Q}_L, \mathcal{Q}_r \subseteq \mathcal{P}$ , such that for every instance  $\mathcal{N}(n)$  the following condition holds:  $\langle P, Q \rangle$  is a transition of  $\mathcal{N}(n)$  iff there is  $i \in [n]$  and a pattern such that  $P = \mathcal{P}_L \times \{i\} \cup \mathcal{P}_R \times \{i \oplus_n 1\}$  and  $Q = \mathcal{Q}_L \times \{i\} \cup \mathcal{Q}_R \times \{i \oplus_n 1\}$ .

It is possible to decide if a given parameterized Petri net is a fully symmetric ring:

**Proposition 6.2.** There is a formula of WS1S such that a parameterized net is a fully symmetric ring iff the formula holds.

#### **Proof:**

We introduce a WS1S formula describing symmetric rings in several steps. To avoid dealing with edge cases we assume that any transition formula  $Tr(n, \mathcal{X}, \mathcal{Y})$  enforces a minimal size of its models; i.e.,  $Tr(n, \mathcal{X}, \mathcal{Y}) \models n > 3$ . This streamlines the argument and formulas. However, it is straightforward to adapt the formulas to the full generality.

The following formula expresses that for every transition of every instance there is an index i such that all places in the preset and postset of the transition have index i or  $i \oplus_n 1$ . We call i the index of the transition.

$$\varphi \coloneqq \forall \boldsymbol{n}, \mathcal{X}, \mathcal{Y} \colon Tr(\boldsymbol{n}, \mathcal{X}, \mathcal{Y}) \longrightarrow$$
$$\exists \boldsymbol{i} : \boldsymbol{i} < \boldsymbol{n} \land \forall \boldsymbol{x} : \boldsymbol{x} < \boldsymbol{n} \rightarrow \begin{pmatrix} \bigvee \boldsymbol{x} \in \mathcal{X}_p \lor \boldsymbol{x} \in \mathcal{Y}_p \\ p \in \mathcal{P} \\ \leftrightarrow [\boldsymbol{x} = \boldsymbol{i} \lor \boldsymbol{x} = \boldsymbol{i} \oplus_{\boldsymbol{n}} 1] \end{pmatrix}$$
(9)

Now we express that if some instance, say  $\mathcal{N}(n)$ , contains a transition with index *i*, then for every other instance, say  $\mathcal{N}(m)$ , and for every index  $0 \le j \le m$ , substituting *j* for *i* yields

a transition of  $\mathcal{N}(m)$ :

$$\psi \coloneqq \forall \boldsymbol{n}, \boldsymbol{i}, \mathcal{X}, \mathcal{Y}, \boldsymbol{m}, \boldsymbol{j} \colon (\boldsymbol{i} < \boldsymbol{n} \land Tr(\boldsymbol{n}, \mathcal{X}, \mathcal{Y}) \land \boldsymbol{j} < \boldsymbol{m}) \longrightarrow \\ \exists \mathcal{X}', \mathcal{Y}' \colon Tr(\boldsymbol{m}, \mathcal{X}', \mathcal{Y}') \land \\ \bigwedge_{p \in \mathcal{P}} \begin{pmatrix} \boldsymbol{i} \in \mathcal{X}_p \leftrightarrow \boldsymbol{j} \in \mathcal{X}'_p \\ \land \boldsymbol{i} \in \mathcal{Y}_p \leftrightarrow \boldsymbol{j} \in \mathcal{Y}'_p \\ \land \boldsymbol{i} \oplus_{\boldsymbol{n}} 1 \in \mathcal{X}_p \leftrightarrow \boldsymbol{j} \oplus_{\boldsymbol{m}} 1 \in \mathcal{X}'_p \\ \land \boldsymbol{i} \oplus_{\boldsymbol{n}} 1 \in \mathcal{Y}_p \leftrightarrow \boldsymbol{j} \oplus_{\boldsymbol{m}} 1 \in \mathcal{Y}'_p \end{pmatrix}.$$
(10)

We prove that  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  is a fully symmetric ring iff its associated formula  $\varphi \wedge \psi$  is valid.

First, we show that  $\varphi$  is valid if and only if for all n and every transition  $\langle P_1, P_2 \rangle$  of  $T_n$  there is an index  $0 \le i \le n-1$  such that  $P_1 \cup P_2 \subseteq \mathcal{P} \times \{i, i \oplus_n 1\}$ .

Assume for all *n* and every transition  $\langle P_1, P_2 \rangle$  of  $T_n$  there is an index  $0 \le i \le n-1$  such that  $P_1 \cup P_2 \subseteq \mathcal{P} \times \{i, i \oplus_n 1\}$ . Then for any interpretation  $\mu$  of  $n, \mathcal{X}, \mathcal{Y}$  with  $\mu \models Tr(n, \mathcal{X}, \mathcal{Y})$  we have  $\langle P, Q \rangle$  and an index  $j < \mu(n)$  such that  $\mu(\mathcal{X}_p) = P \cap \{p\} \times \{j, j \oplus_{\mu(n)} 1\}$  and  $\mu(\mathcal{Y}_p) = Q \cap \{p\} \times \{j, j \oplus_{\mu(n)} 1\}$  for all  $p \in \mathcal{P}$ .

Consequently,

$$\mu[\boldsymbol{i} \mapsto i] \models \forall \boldsymbol{x} : \boldsymbol{x} < \boldsymbol{n} \rightarrow \begin{pmatrix} \bigvee_{p \in \mathcal{P}} x \in \mathcal{X}_p \lor x \in \mathcal{Y}_p \\ p \in \mathcal{P} \\ \leftrightarrow [x = i \lor x = i \oplus_{\boldsymbol{n}} 1] \end{pmatrix}.$$

Which renders  $\varphi$  valid in general.

On the other hand, if  $\varphi$  is valid careful examining  $\varphi$  gives the desired result: let  $\mu \models Tr(n, \mathcal{X}, \mathcal{Y})$ . Then fix any  $i \in [\mu(n)]$  such that

$$\mu[\boldsymbol{i} \mapsto i] \models \forall \boldsymbol{x} : \boldsymbol{x} < \boldsymbol{n} \rightarrow \begin{pmatrix} \bigvee x \in \mathcal{X}_p \lor x \in \mathcal{Y}_p \\ p \in \mathcal{P} \\ \leftrightarrow [x = i \lor x = i \oplus_{\boldsymbol{n}} 1] \end{pmatrix}.$$

For the transition  $\langle P, Q \rangle$  of  $\mu$ ; i.e.,

$$P = \{ \langle p, i \rangle \in \mathcal{P} \times [\mu(\boldsymbol{n})] \mid i \in \mu(\mathcal{X}_p) \}, \ Q = \{ \langle p, i \rangle \in \mathcal{P} \times [\mu(\boldsymbol{n})] \mid i \in \mu(\mathcal{Y}_p) \}$$

we see that  $P \subseteq \mathcal{P} \times \{i, i \oplus_{\mu(n)} 1\}$  and  $Q \subseteq \mathcal{P} \times \{i, i \oplus_{\mu(n)} 1\}$ .

Using this observation we restrict the remaining argument to the case that every transition of  $\mathcal{N}(n)$  has an index  $i \in [n]$ . It remains to show that – under this condition –  $\psi$  is valid if and only if  $\mathcal{N}$  is a fully symmetric ring: assume  $\mathcal{N}$  to be a fully symmetric ring. Let  $\mu$  be an arbitrary interpretation of  $n, m, \mathcal{X}, \mathcal{Y}, i, j$ . If  $\mu \not\models i < n \land Tr(n, \mathcal{X}, \mathcal{Y}) \land j < m$  then there is nothing to show. Let now

 $\mu \models i < n \land Tr(n, \mathcal{X}, \mathcal{Y}) \land j < m$ . Let  $n = \mu(n), m = \mu(m), i = \mu(i), j = \mu(j)$  and  $\langle P, Q \rangle$ such that  $P = \{\langle p, i \rangle \in \mathcal{P} \times [\mu(n)] \mid i \in \mu(\mathcal{X}_p)\}$  and  $Q = \{\langle p, i \rangle \in \mathcal{P} \times [\mu(n)] \mid i \in \mu(\mathcal{Y}_p)\}$ . Since  $\mathcal{N}$  is assumed to be a fully symmetric ring we know that  $\langle P, Q \rangle$  is an instance of the pattern  $\langle \mathcal{P}_L, \mathcal{P}_R, \mathcal{Q}_L, \mathcal{Q}_R \rangle$  at an index *i*. More formally,  $P = P_L \times \{i\} \cup P_R \times \{i \oplus_n 1\}$  and  $Q = Q_L \times \{i\} \cup Q_R \times \{i \oplus_n 1\}$ . If  $\mu(i) \notin \{i, i \oplus_n 1\}$ , then expanding the interpretation  $\mu$  to an interpretation  $\mu'$  which chooses values  $\mu'(\mathcal{X}')$  and  $\mu'(\mathcal{Y}')$  which yield a transition  $\langle P', Q' \rangle$  as an instance of  $\langle \mathcal{P}_L, \mathcal{P}_R, \mathcal{Q}_L, \mathcal{Q}_R \rangle$  for an index j' such that  $\{j', j' \oplus_m 1\} \cap \{j, j \oplus_m 1\} = \emptyset$ . (Note that we use implicitly here that the formula Tr enforces models of sufficient size. Adapting  $\psi$  such that i has to be the index of  $\langle P, Q \rangle$  is straightforward.)

On the other hand, if  $\mu(i) = i$  then expanding  $\mu$  to  $\mu'$  with values for  $\mu'(\mathcal{X}')$  and  $\mu'(\mathcal{Y}')$  such that the associated  $\langle P', Q' \rangle$  is an instance of  $\langle \mathcal{P}_L, \mathcal{P}_R, \mathcal{Q}_L, \mathcal{Q}_R \rangle$  at index  $\mu(\mathbf{j})$  yields the desired result. Analogously, for  $\mu(\mathbf{i}) = i \oplus_n 1$ . It follows that  $\psi$  is valid.

Now, assume  $\psi$  to be valid. The result follows from carefully examining  $\psi$ . For any transition  $\langle P, Q \rangle$  in an instance  $\mathcal{N}(n)$  we can extract its structure; i.e., a pattern  $\langle \mathcal{P}_L, \mathcal{P}_R, \mathcal{Q}_L, \mathcal{Q}_R \rangle$  such that  $P = P_L \times \{i\} \cup P_R \times \{i \oplus_n 1\}$  and  $Q = Q_L \times \{i\} \cup Q_R \times \{i \oplus_n 1\}$  for an appropriate *i* (remember that we assume  $\varphi$  to be valid). By the validity of  $\psi$  we see that the same pattern can be instantiated (represented by the choice of  $\mathcal{X}'$  and  $\mathcal{Y}'$ ) at all other indices (corresponding to the choice for  $\mu(\boldsymbol{j})$ ) for all other instances (corresponding to the choice for  $\mu(\boldsymbol{m})$ ).

We need to distinguish between *global* and *local* traps of an instance. Loosely speaking, a global trap contains places of all processes, while a local trap does not. To understand why this is relevant, consider a fully symmetric ring  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  where  $\mathcal{P} = \{p, q\}$  and the transitions of each instance  $\mathcal{N}(n)$  are the pairs  $\langle \{p(i), q(i \oplus_n 1)\}, \{p(i \oplus_n 1), q(i)\} \rangle$  for every  $i \in [n]$ . The sets  $\{p(0), q(0)\}$  and  $\{p(0), p(1), p(2), p(3)\}$  are both traps of  $\mathcal{N}(4)$  (they are even 1-balanced sets). However, they are of different nature. Intuitively, in order to decide that  $\{p(0), q(0)\}$  is a trap it is not necessary to inspect all of  $\mathcal{N}(4)$ , but only process 0 and its neighborhood. On the contrary,  $\{p(0), p(1), p(2), p(3)\}$  involves all processes. This has consequences when parametrizing. Due to the symmetry of the ring,  $\{p(i), q(i)\}$  is a trap of every instance  $\mathcal{N}(n)$  for every  $i \in [n]$ . However,  $\{p(i), p(i \oplus_n 1), \ldots, p(i \oplus_n 3)\}$  is *not* a trap of every instance for every  $i \in [n]$ , for example  $\{p(0), p(1), p(2), p(3)\}$  is not a trap of  $\mathcal{N}(5)$ . The correct parametrization is a different one, namely  $\{p(0), p(1), \dots, p(n-1)\}$ . The difference between the two traps is captured by the following definition.

**Definition 6.3.** Let  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  be a parameterized net. An indexed trap  $\mathbf{T} = \langle n, Q \rangle$  of  $\mathcal{N}$  is global if  $Q \cap (\mathcal{P} \times \{i\}) \neq \emptyset$  for every  $i \in [n]$ , otherwise  $\mathbf{T}$  is *local*.

### 6.1. Parametrizing local traps

We first observe that local indexed traps can be "shifted" locally while maintaining their trap property.

**Lemma 6.4.** Let  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  be a fully symmetric ring and let  $\langle n, Q \rangle$  be a local indexed trap of  $\mathcal{N}$ . Then  $\langle n, Q' \rangle$  with  $Q' = \{ \langle p, i \oplus_n 1 \rangle : \langle p, i \rangle \in Q \}$  is a local indexed trap of  $\mathcal{N}$ .

### **Proof:**

Assume Q' is not an indexed local trap. Then there is  $t \in T_n$  such that  ${}^{\bullet}t \cap Q' \neq \emptyset = t^{\bullet} \cap Q'$ . Since  $\mathcal{N}$  is a fully symmetric ring, there is a pattern  $\langle \langle P_L, P_R \rangle, \langle Q_L, Q_R \rangle \rangle$  and an index  $i \in [n]$  such that t is the instance of  $\langle \langle P_L, P_R \rangle, \langle Q_L, Q_R \rangle \rangle$  with index i. Let t' be the transition obtained from the same pattern with index  $i \oplus_n (n-1)$ ; i.e., moved one index to the left. It follows  ${}^{\bullet}t' = \{\langle p, j \oplus_n (n-1) \rangle : \langle p, j \rangle \in {}^{\bullet}t\}$  and  $t'^{\bullet} = \{\langle p, j \oplus_n (n-1) \rangle : \langle p, j \rangle \in t^{\bullet}\}$ . By definition of Q' we have  $Q = \{\langle p, j \oplus_n (n-1) \rangle : \langle p, j \rangle \in Q'\}$ . That, however, gives  ${}^{\bullet}t' \cap Q \neq \emptyset = t'^{\bullet} \cap Q$  in contradiction to Q being a local indexed trap.

Our second lemma states that for any indexed local traps  $\langle n, Q \rangle$  with  $Q \cap (\mathcal{P} \times \{n-1\})$ , the set Q remains a trap in any instance  $\mathcal{N}(n')$  with  $n \leq n'$ .

**Lemma 6.5.** Let  $\mathcal{N}$  be a fully symmetric ring and  $\langle n, Q \rangle$  a local indexed trap such that  $Q \cap (\mathcal{P} \times \{n-1\}) = \emptyset$ . Then  $\langle n', Q \rangle$  is a local indexed trap for all  $n' \ge n$ .

#### **Proof:**

Assume the statement is false. That is,  $\langle n', Q \rangle$  is not a local indexed trap. If n' = n there is an immediate contradiction with the assumption that  $\langle n, Q \rangle$  is a local indexed trap. Hence, let n' > n minimal such that  $\langle n', Q \rangle$  is not a local indexed trap. So there is a transition t in  $\mathcal{N}(n')$  such that  ${}^{\bullet}t \cap Q \neq \emptyset = t^{\bullet} \cap Q$ . Since  $Q \cap (\mathcal{P} \times \{n-1\}) = \emptyset$  by assumption of the lemma and n' > n by case distinction we have  $Q \cap (\mathcal{P} \times \{n'-2, n'-1\}) = \emptyset$ . With this and the facts that fully symmetric rings only allow for transitions using places of two adjacent indices and  ${}^{\bullet}t \cap Q \neq \emptyset$  we get  ${}^{\bullet}t \cap \mathcal{P} \times \{n'-1\} = \emptyset$  and  $t^{\bullet} \cap \mathcal{P} \times \{n'-1\} = \emptyset$ . That means, however, that t is also a transition in  $\mathcal{N}(n'-1)$  because  $\mathcal{N}$  is a fully symmetric ring and, consequently,  $\langle n'-1, Q \rangle$  already is not a local indexed trap. This contradicts that n' was chosen minimal and concludes the proof.

We can now show how to obtain a sound parameterization of a given indexed trap. The formula  $ParTrap_{\mathbf{T}}(\mathcal{X})$  states that  $\mathcal{X}$  is the result of "shifting"  $\mathbf{T} = \langle n, Q \rangle$  in  $\mathcal{N}(n')$  for some  $n' \geq n$ .

**Theorem 6.6.** Let  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  be a fully symmetric ring and let  $\langle n, Q \rangle$  be a local indexed trap of  $\mathcal{N}(n)$  such that  $Q \subseteq (\mathcal{P} \times I)$  for a minimal set  $I \subset [n]$ . Assume  $I = \{i_0, \ldots, i_{k-1}\}$  with  $0 \leq i_0 < i_1 < \ldots < i_{k-1} < n-1$ . Then every model of the formula

$$ParTrap_{\mathbf{T}}(\boldsymbol{n},\mathcal{X}) \coloneqq n \leq \boldsymbol{n} \land \exists \boldsymbol{y} \colon \boldsymbol{y} < \boldsymbol{n} \land \bigwedge_{p \in \mathcal{P}} \forall \boldsymbol{x} \colon \boldsymbol{x} < \boldsymbol{n} 
ightarrow$$
 $\begin{pmatrix} & \bigvee \boldsymbol{x} = \boldsymbol{y} \\ & & \bigvee & & \\ & & \lor & \bigvee & \\ & & & \bigvee & & \\ & & & \downarrow > 0, \langle i_j, p 
angle \in Q \end{pmatrix} \boldsymbol{x} = \boldsymbol{y} \oplus_{\boldsymbol{n}} (i_j - i_{j-1}) \end{pmatrix} \end{pmatrix}$ 

is an indexed trap of  $\mathcal{N}$ .

#### **Proof:**

Assume  $\mu \models ParTrap_{\mathbf{T}}(\mathbf{n}, \mathcal{X})$ . Then there exists a tuple  $\langle k, P \rangle$  such that  $\mu(\mathbf{n}) = k$  and  $\mu(\mathcal{X}_p) = \{i \in [k] \mid \langle p, i \rangle \in P\}$  for every  $p \in \mathcal{P}$ . We have  $n \leq k$  by the first conjunct of  $ParTrap_{\mathbf{T}}$ . Let  $j \in [k]$  be any value such that assigning j to y satisfies the existentially quantified subformula of  $ParTrap_{\mathbf{T}}$ .

Since  $i_{k-1} < n-1$  we have  $Q \cap (\mathcal{P} \times \{n-1\}) = \emptyset$ . So we can apply Lemma 6.5 to  $\langle n, Q \rangle$ , and in fact we can apply it (n-k) times, yielding a local trap  $\langle k, Q \rangle$ . Now, fix indices  $i'_0, \ldots, i'_{k-1}$  such that  $i'_0 = j$  and  $i'_{\ell+1} = i'_{\ell} \oplus_k (i_{\ell+1} - i_{\ell})$  for  $0 \le \ell < k-1$ . Carefully examining  $ParTrap_{\mathbf{T}}$  one can now observe that

$$\{p \in \mathcal{P} \mid \langle p, i_{\ell}' \rangle \in P\} = \{p \in \mathcal{P} \mid \langle p, i_{\ell} \rangle \in P\} \text{ for all } 0 \le \ell < k$$

and  $\emptyset = \mathcal{P} \times ([k] \setminus \{i'_0, \dots, i'_{k-1}\})$ . We can now apply  $(k - i_0) + i'_0$  times Lemma 6.4 to the local trap  $\langle k, Q \rangle$ , which shows that  $\langle k, P \rangle$  is a local trap.

**Remark 6.7.** Since Theorem 6.6 requires  $i_{k-1} < n-1$ , it can only be applied to local traps  $\langle n, Q \rangle$  such that  $Q \cap (\mathcal{P} \times \{n-1\}) = \emptyset$ . However, for every local trap  $\langle n, Q \rangle$  Lemma 6.4 allows us to find a local trap  $\langle n, Q' \rangle$  satisfying  $Q' \cap (\mathcal{P} \times \{n-1\}) = \emptyset$ , which we can then parameterize via Theorem 6.6.

#### 6.2. Parametrizing global traps

In contrast to local traps, global traps involve all indices [n] of the instance  $\mathcal{N}(n)$ . Let  $\langle n, Q \rangle$  be an indexed global trap. We denote with Q[i] the set  $P \subseteq \mathcal{P}$  such that  $P \times \{i\} = Q \cap (\mathcal{P} \times \{i\})$ ; i.e., the set of places in Q at index i. Moreover, we say Q has period p if p is the smallest divisor of n such that for all  $0 \leq j < p$  we have  $Q[j] = Q[k \cdot p + j]$  for all  $0 \leq k < \frac{n}{p}$ . That is, Q is a repetition of the same p sets in a row. Since n is a period of Q we know that every Q has a period, which we denote  $p_Q$ . Recall the global trap  $Q = \{p(0), p(1), p(2), p(3)\}$  from before. Then,  $Q[0] = Q[1] = Q[2] = Q[3] = \{p\}$  and, consequently,  $p_Q = 1$ . Intuitively, we can repeat a period over and over again and still obtain a trap. So we can parameterize global traps by capturing the repetition of periodic behavior:

**Theorem 6.8.** Let  $\langle n, Q \rangle$  be an indexed global trap with  $n \geq 2$ . Then every model of the formula

$$ParTrap_{\mathbf{T}}(\boldsymbol{n},\mathcal{X}) \coloneqq \exists \boldsymbol{P} : 0 \in \boldsymbol{P} \land \boldsymbol{n} \in \boldsymbol{P}$$

$$egin{aligned} &\wedge \ orall oldsymbol{x}: oldsymbol{x} \leq oldsymbol{n} 
ightarrow oldsymbol{\left( egin{aligned} &egin{aligned} &oldsymbol{x} + k \notin oldsymbol{P} \ &0 \leq k < p_Q \ &\wedge oldsymbol{x} + p_Q \in oldsymbol{P} \ &\wedge oldsymbol{x} + p_Q \in oldsymbol{P} \ &\wedge oldsymbol{x} + p_Q \in oldsymbol{P} \ &\wedge oldsymbol{x}_{k-1} + 1 = oldsymbol{x}_k \ &0 < k \leq p_Q - 1 \ &\wedge oldsymbol{x}_{pQ-1} < n \wedge oldsymbol{x}_0 \in oldsymbol{P} \ &\to egin{aligned} &oldsymbol{A} & oldsymbol{x}_{k} \in \mathcal{X}_p \land &oldsymbol{A} & oldsymbol{x}_{pQ-1} < n \wedge oldsymbol{x}_0 \in oldsymbol{P} \ &\to egin{aligned} &oldsymbol{A} & oldsymbol{A} & oldsymbol{A} & oldsymbol{x}_{pQ-1} < n \wedge oldsymbol{x}_0 \in oldsymbol{P} \ &oldsymbol{A} & oldsymbol{A} & oldsymbol{x}_{pQ} & oldsymbol{P} & oldsymbol{A} \ &oldsymbol{A} & oldsymbol{A} \ &oldsymbol{A} & oldsymbol{A} & olds$$

is an indexed global trap.

#### **Proof:**

Let  $\mu$  be a model of  $ParTrap_{\mathbf{T}}(\mathbf{n}, \mathcal{X})$ . Observe that we have  $\mu(\mathbf{P}) = \{0, p_Q, 2 \cdot p_Q, \dots, \ell \cdot p_Q\}$  for some  $\ell > 0$ . Let  $k := \ell \cdot p_Q = \mu(\mathbf{n})$  and let P be the set of places of  $\mathcal{N}(k)$  such that  $\mu(\mathcal{X}_p) = \{i \in [k] : \langle p, i \rangle \in P\}$ . Examining  $ParTrap_{\mathbf{T}}(\mathbf{n}, \mathcal{X})$  further we observe that  $P[j \cdot p_Q + o] = Q[o]$ holds for all  $0 \le o < p_Q$  and  $0 \le j < \ell$ .

It remains to show that P is indeed a trap. Assume the contrary. Then there is a transition tin  $\mathcal{N}(k)$  such that  $P \cap {}^{\bullet}t \neq \emptyset = P \cap t^{\bullet}$ . Since  $\mathcal{N}$  is a fully symmetric ring there is an index  $i \in [k]$  such that  ${}^{\bullet}t \cup t^{\bullet} \subseteq \mathcal{P} \times \{i, i \oplus_k 1\}$ . Pick j such that  $i = j \cdot p_Q + o$  for  $0 \leq o < p_Q$ . Observe that P[i] = Q[o] and  $P[i \oplus_k 1] = Q[o \oplus_n 1]$ . Again, by  $\mathcal{N}$  being a symmetric ring, we can find a transition t' such that  ${}^{\bullet}t' = \{\langle p, o \rangle : \langle p, i \rangle \in {}^{\bullet}t\} \cup \{\langle p, o \oplus_n 1 \rangle : \langle p, i \oplus_k 1 \rangle \in {}^{\bullet}t\}$  and  $t'^{\bullet} = \{\langle p, o \rangle : \langle p, i \rangle \in t^{\bullet}\} \cup \{\langle p, o \oplus_n 1 \rangle : \langle p, i \oplus_k 1 \rangle \in t^{\bullet}\}$ . This, however, yields a contradiction since t' is a witness for Q not being a trap in contradiction to the assumptions.  $\Box$ 

## 7. Trap parametrization in barrier crowds

Barrier crowds are parameterized systems in which communication happens by means of global steps in which each process makes a move. An initiator process decides to start a step, and all the other processes get a chance to veto it; if the step is not blocked (if all the processes accept it), all the processes, including the initiator, update their local state. Barrier crowds are slightly more general than broadcast protocols [40], which, loosely speaking, correspond to the special case in which no process makes use of the veto capability. Like broadcast protocols, barrier crowds can be used to model cache coherence protocols [41].

As for fully symmetric rings, transitions of the instances of a barrier crowd are generated from a finite set of "transition patterns". A transition pattern of a barrier crowd  $\mathcal{N}$  is a pair  $\langle \mathcal{I}, \mathbb{A} \rangle$ , where  $\mathcal{I} \in 2^{\mathcal{P}} \times 2^{\mathcal{P}}$  and  $\mathbb{A} \subseteq 2^{\mathcal{P}} \times 2^{\mathcal{P}}$ . Assume for example that each process can be in states p, q, r, and maintains a boolean variable with values  $\{false, true\}$ . The corresponding parameterized net has  $\mathcal{P} = \{p, q, r, false, true\}$  as set of places. Consider the transition pattern with  $\mathcal{I} = \langle \{p, false\}, \{q, true\} \rangle$ , and  $\mathbb{A} = \{\langle \{p\}, \{p\} \rangle, \langle \{q, false\}, \{r, false\} \rangle, \langle \{q, true\}, \{r, false\} \rangle \}$ . This pattern models that the initiator process, say process *i*, proposes a step that takes it from *p* to *q*, setting its variable to *true*. Each other process reacts as follows, depending on its current state: if in *p*, it stays in *p*, leaving the variable unchanged; if in *q*, it moves to *r*, setting the variable to *false*; if in *r*, it vetoes the step (because  $\mathbb{A}$  does not offer a way to accept from state *r*). We depict an instance with three agents for this example in Figure 2.

**Definition 7.1.** A parameterized Petri net  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  is a *barrier crowd* if there is a finite set of transition patterns of the form  $\langle \mathcal{I}, \mathbb{A} \rangle$  such that for every instance  $\mathcal{N}(n)$  the following condition holds: a pair  $\langle P, Q \rangle$  is a transition of  $\mathcal{N}(n)$  iff there exists a pattern  $\langle \mathcal{I}, \mathbb{A} \rangle$  and  $i \in [n]$  such that:

- $P \cap (\mathcal{P} \times \{i\}) = P_I \times \{i\}$  and  $Q \cap (\mathcal{P} \times \{i\}) = Q_I \times \{i\}$ , where  $\mathcal{I} = \langle P_I, Q_I \rangle$ .
- for every  $j \neq i$  there is  $\langle P_A, Q_A \rangle \in \mathbb{A}$  such that  $P \cap (\mathcal{P} \times \{j\}) = P_A \times \{j\}$  and  $Q \cap (\mathcal{P} \times \{j\}) = Q_A \times \{j\}$ .

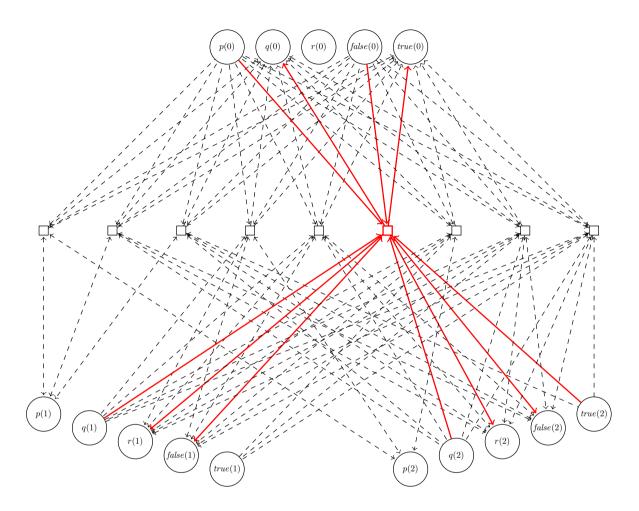


Figure 2: An example  $\mathcal{N}(3)$  of an instance of a crowd  $\mathcal{N}$ . The places of  $\mathcal{N}$  are  $\mathcal{P} = \{p, q, r, false, true\}$  and we consider only the transition pattern with  $\mathcal{I} = \langle \{p, false\}, \{q, true\} \rangle$  and  $\mathbb{A} = \{\langle \{p\}, \{p\} \rangle, \langle \{q, false\}, \{r, false\} \rangle, \langle \{q, true\}, \{r, false\} \rangle\}$ . We give the transitions only for the case that the agent with index 0 executes the pattern of  $\mathcal{I}$ . This means we give 9 transitions where agents 1 and 2 execute one of the patterns  $\langle \{p\}, \{p\} \rangle, \langle \{q, false\}, \{r, false\} \rangle, \langle \{q, true\}, \{r, false\} \rangle$ . Specifically, we only drew the sixth transition from the left continuously and red. This transition corresponds to agent 1 using the pattern  $\langle \{q, false\}, \{r, false\} \rangle$  and agent 2 using the pattern  $\langle \{q, true\}, \{r, false\} \rangle$ .

Note that the number of transitions of  $\mathcal{N}(n)$  grows quickly in *n*, even though the structure of the system remains simple, making parameterized verification particularly attractive.

In the rest of the section we present an automatic parametrization procedure for traps of barrier crowds. First we show that barrier crowds satisfy two important structural properties.

Given a set of places  $P \subseteq \mathcal{P} \times [n]$  and a permutation  $\pi : [n] \to [n]$ , let  $\pi(P)$  denote the set of places  $\{p(\pi(i)) : p(i) \in P\}$ . Given an index  $0 \leq k < n$ , let  $\operatorname{drop}_{k,n}(P)$  denote the set of places defined as follows:  $p(i) \in \operatorname{drop}_{k,n}(P)$  iff either  $0 \leq i < k$  and  $p(i) \in P$ , or  $k < i \leq n-1$  and  $p(i+1) \in P$ .

**Definition 7.2.** Let  $\mathcal{N}$  be a parameterized Petri net. A transition  $\langle P_1, P_2 \rangle$  of  $\mathcal{N}(n)$  is:

- order invariant if  $\langle \pi(P_1), \pi(P_2) \rangle$  is also a transition of  $\mathcal{N}(n)$  for every permutation  $\pi \colon [n] \to [n]$ .
- homogeneous if there is an index  $0 \le i < n$  such that for every  $k \in [n] \setminus \{i\}$  the pair  $\langle \operatorname{drop}_{k,n}(P_1), \operatorname{drop}_{k,n}(P_2) \rangle$  is a transition of  $\mathcal{N}(n-1)$ .

 $\mathcal{N}$  is homogeneous (order invariant) if all transitions of all instances  $\mathcal{N}(n)$  is homogeneous (order invariant).

Intuitively, order invariance indicates that processes are indistinguishable. Homogeneity indicates that transitions in the large instances are not substantially different from the transitions in the smaller ones.

Proposition 7.3. Barrier crowds are order invariant and homogeneous.

#### **Proof:**

Let  $\mathcal{N}$  be a barrier crowd. For order invariance, let  $\langle P, Q \rangle$  be a transition of an instance  $\mathcal{N}(n)$ , and let  $\pi \colon [n] \to [n]$  be a permutation. We show that  $\langle \pi(P), \pi(Q) \rangle$  is also a transition of  $\mathcal{N}(n)$ . By the definition of barrier crowds there is a pattern  $\langle \mathcal{I}, \mathbb{A} \rangle$ , where  $\mathcal{I} = \langle P_I, Q_I \rangle$ , and an index *i* such that

- $P \cap (\mathcal{P} \times \{i\}) = P_I \times \{i\}$  and  $Q \cap (\mathcal{P} \times \{i\}) = Q_I \times \{i\}$ ; and
- for every  $j \neq i$  there is  $\left\langle P_A^j, Q_A^j \right\rangle \in \mathbb{A}$  such that  $P \cap (\mathcal{P} \times \{j\}) = P_A^j \times \{j\}$  and  $Q \cap (\mathcal{P} \times \{j\}) = Q_A^j \times \{j\}$ .

Intuitively, by the definition of barrier crowds, the result of instantiating  $\langle \mathcal{I}, \mathbb{A} \rangle$  with the index  $\pi(i)$  instead of *i* is also a transition of  $\mathcal{N}(n)$ . Formally, the pair  $\langle P', Q' \rangle$  given by

- $P' \cap (\mathcal{P} \times \{\pi(i)\}) = P_I \times \{\pi(i)\}$  and  $Q' \cap (\mathcal{P} \times \{\pi(i)\}) = Q_I \times \{\pi(i)\}$ , and
- $P' \cap (\mathcal{P} \times \{\pi(j)\}) = P_A^j \times \{\pi(j)\} \text{ and } Q' \cap (\mathcal{P} \times \{\pi(j)\}) = Q_A^j \times \{\pi(j)\} \text{ for every } j \neq i$

is a transition of  $\mathcal{N}(n)$ . By construction we have  $\pi(P) = P'$  and  $\pi(Q) = Q'$ . So  $\langle \pi(P), \pi(Q) \rangle$  is a transition of  $\mathcal{N}(n)$ , and we are done.

For homogeneity, let  $\langle P, Q \rangle$  be a transition of  $\mathcal{N}(n)$ . Let  $\langle \mathcal{I}, \mathbb{A} \rangle$  with  $\mathcal{I} = \langle P_I, Q_I \rangle$  be the pattern of which  $\langle P, Q \rangle$  is an instance, i.e.,  $P \cap (\mathcal{P} \times \{i\}) = P_I \times \{i\}$  and  $Q \cap (\mathcal{P} \times \{i\}) = Q_I \times \{i\}$  for every  $i \in [n]$ . By the definition of barrier crowds, for every  $j \in [n] \setminus \{i\}$  there is  $\langle P_I^j, Q_I^j \rangle \in \mathbb{A}$  such that  $P \cap (\mathcal{P} \times \{j\}) = P_I^j$  and  $Q \cap (\mathcal{P} \times \{j\}) = Q_I^j$ . For every  $k \in [n] \setminus \{i\}$ , we carefully instantiate  $\langle \mathcal{I}, \mathbb{A} \rangle$ to obtain a transition  $\langle P', Q' \rangle$  satisfying  $\langle P', Q' \rangle = \langle \operatorname{drop}_{k,n}(P), \operatorname{drop}_{k,n}(Q) \rangle$ , which concludes the proof. We need to consider two cases:

- If i < k, then:
  - $P' \cap (\mathcal{P} \times \{i\}) = P_I$  and  $Q' \cap (\mathcal{P} \times \{j\}) = P_I^j$  for all  $i \neq j < k$ ; -  $P' \cap (\mathcal{P} \times \{j\}) = P_I^j$  and  $P' \cap (\mathcal{P} \times \{j\}) = P_I^{j-1}$  for all k < j; -  $Q' \cap (\mathcal{P} \times \{i\}) = Q_I$  and  $Q' \cap (\mathcal{P} \times \{j\}) = Q_I^j$  for all  $i \neq j < k$ ; and -  $Q' \cap (\mathcal{P} \times \{j\}) = Q_I^j$  and  $Q' \cap (\mathcal{P} \times \{j\}) = Q_I^{j-1}$  for all k < j.
- If k < i, then  $\langle P', Q' \rangle$  is defined as for the case i < k, with the exception that now  $P' \cap (\mathcal{P} \times \{i-1\}) = P_I$  and  $Q' \cap \mathcal{P} \times \{i-1\} = Q_I$ .

In both cases  $\langle P', Q' \rangle$  is a transition in  $\mathcal{N}(n-1)$  by definition of barrier crowds.  $P' = \operatorname{drop}_{k,n}(P)$ and  $Q' = \operatorname{drop}_{k,n}(Q)$  which concludes the argument.

### 7.1. Parametrizing traps for barrier crowds

By order invariance, if Q is a trap of an instance, say  $\mathcal{N}(n)$ , then  $\pi(Q)$  is also a trap for every permutation  $\pi$ . The set of all traps that can be obtained from Q by permutations can be described as a multiset  $Q: 2^{\mathcal{P}} \to [n]$ . For example, assume  $\mathcal{P} = \{p, q\}, n = 5$ , and  $Q = \{p(0), p(1), q(1), p(2), q(2), q(4)\}$ . Then  $\mathcal{Q}(\{p, q\}) = 2$  (because of indices 1 and 2),  $\mathcal{Q}(\{p\}) = \mathcal{Q}(\{q\}) = 1$  (index 0 and 4, respectively), and  $\mathcal{Q}(\emptyset) = 1$  (index 3). Any assignment of indices to the elements of Q results in a trap. We call Q the *trap family of* Q.

**Proposition 7.4.** Let  $\mathcal{N}$  be an order invariant and homogeneous parameterized Petri net, let Q be a trap of an instance  $\mathcal{N}(n)$ , and let  $\mathcal{Q}: 2^{\mathcal{P}} \to [n]$  be the trap family of Q. We have:

- If Q(Ø) ≥ 1 and Q' is obtained from Q by increasing the multiplicity of Ø, then Q' is also a trap family of another instance of N.
- For every S ∈ 2<sup>P</sup>, if Q(S) ≥ 2 and Q' is obtained from Q by increasing the multiplicity of S, then Q' is also a trap family of another instance of N.

#### **Proof:**

First consider increasing the multiplicity of  $\emptyset$ . It suffices to consider the case of the family Q' obtained from Q by setting  $Q'(\emptyset) = Q(\emptyset) + 1$ , since the general statement follows by induction in a straightforward manner. Assume that  $\langle n + 1, Q' \rangle$  is not a trap in  $\mathcal{N}(n+1)$ , but has the multiplicities of Q'. Let t' be a transition of  $\mathcal{N}(n+1)$  such that  $Q' \cap {}^{\bullet}t' \neq \emptyset = Q' \cap t'^{\bullet}$ . Since  $Q'(\emptyset) \ge 2$ , there are at least two distinct indices i, k such that  $Q' \cap (\mathcal{P} \times \{i, k\}) = \emptyset$ . By homogeneity of  $\mathcal{N}$  we can choose i, k so that a transition t in  $\mathcal{N}(n)$  satisfies  ${}^{\bullet}t = \operatorname{drop}_{k,n}({}^{\bullet}t')$  and  $t^{\bullet} = \operatorname{drop}_{k,n}(t'^{\bullet})$ . Further, let  $Q = \operatorname{drop}_{k,n}(Q')$ . Note that Q is an instance of the trap family Q. However,  ${}^{\bullet}t \cap Q \neq \emptyset = t^{\bullet} \cap Q$  by the definition of t and  $\operatorname{drop}_{k,n}$  in contradiction to Q being a trap family.

In the case of increasing the multiplicity of a non-empty set S we know that  $Q(S) \ge 2$  and  $Q'(S) = Q(S) + 1 \ge 3$ . The argument is analogous to the previous case. First we assume  $\langle n + 1, Q' \rangle$  is an instance of Q' that is not a trap. For Q', let  $k_1, k_2, k_3$  be three distinct indices in [n + 1] such that  $Q' \cap \mathcal{P} \times \{k_1, k_2, k_3\} = S \times \{k_1, k_2, k_3\}$ . Then, we find a transition t' in  $\mathcal{N}(n + 1)$  witnessing that Q' is not a trap. We consider two cases:

• •  $t' \cap S \times \{k_1, k_2, k_3\} = \emptyset.$ 

By homogeneity, there is  $k \in \{k_1, k_2, k_3\}$  such that the result of applying the drop<sub>k,n</sub> operation to t' is a transition t of  $\mathcal{N}(n)$ . The set  $Q = \operatorname{drop}_{k,n}(Q')$  is a set of places of  $\mathcal{N}(n)$  with trap family Q. We have  $\bullet t \cap Q \neq \emptyset$  since  $\bullet t' \cap Q' \neq \emptyset$  and  $Q' \cap (\mathcal{P} \times \{k\}) = \emptyset$ ; further,  $t^{\bullet} \cap Q = \emptyset$ since  $t'^{\bullet} \cap Q' = \emptyset$ . So Q is not a trap, contradicting the assumption.

• •  $t' \cap S \times \{k_1, k_2, k_3\} \neq \emptyset.$ 

By homogeneity there are  $k'_1, k'_2 \in \{k_1, k_2, k_3\}$  such that the result of applying  $\operatorname{drop}_{k'_1, n}$  and  $\operatorname{drop}_{k'_2, n}$  to t' are two transitions  $t_1$  and  $t_2$  of  $\mathcal{N}(n)$ . Since  $t'^{\bullet} \cap Q' = \emptyset$  we also have  $t_1^{\bullet} \cap Q = \emptyset$  and  $t_1^{\bullet} \cap Q = \emptyset$ . Let  $k_o$  denote the only element in  $\{k_1, k_2, k_3\} \setminus \{k'_1, k'_2\}$ . If  $S \times \{k_o\} \cap {}^{\bullet}t' \neq \emptyset$  then  ${}^{\bullet}t_1 \cap Q \neq \emptyset$  and  ${}^{\bullet}t_1 \cap Q \neq \emptyset$  because  $k_o$  is not dropped. If, on the other hand,  $S \times \{k_o\} \cap {}^{\bullet}t' = \emptyset$ , then either  $S \times \{k_1\} \cap {}^{\bullet}t' \neq \emptyset$  or  $S \times \{k_2\} \cap {}^{\bullet}t' \neq \emptyset$ . By symmetry we can assume w.l.o.g.  $S \times \{k_1\} \cap {}^{\bullet}t' \neq \emptyset$ . Then  ${}^{\bullet}t_2 \cap Q \neq \emptyset$ . So Q is not a trap although its trap family is Q, which contradicts the assumption.

Proposition 7.4 leads to a parameterization procedure for barrier crowds. Given a trap Q of some instance  $\mathcal{N}(n)$  and its trap family Q, consider all multisets obtained from Q by applying the operations of Proposition 7.4. We call this set of multisets the *extended trap family* of Q. Observe that Q represents a set of traps of  $\mathcal{N}(n)$ , while the extended family represents a set of traps across all instances  $\mathcal{N}(n')$  with  $n' \ge n$ .

Give an indexed trap  $\mathbf{T} = \langle n, Q \rangle$ , we choose the formula  $ParTrap_{\mathbf{T}}(\mathcal{X})$  so that its models correspond to the traps of the extended family of Q. For this, we capture the minimal required multiplicities of  $\langle n, Q \rangle$  by quantifying for every  $S \subseteq \mathcal{P}$  with  $\mathcal{Q}(S) > 0$  indices  $\mathbf{i}_{S,1}, \ldots, \mathbf{i}_{S,\mathcal{Q}(S)}$  for which precisely the places in S are marked. Making all indices introduced this way pairwise distinct ensures that any model of the formula at least covers the multiset Q. Additionally, we can capture that the subset S of  $\mathcal{P}$  which are marked in every other index are chosen such that Proposition 7.4 ensures that we still obtain a trap.

$$egin{aligned} & ParTrap_{\mathbf{T}}(m{n},\mathcal{X}) \coloneqq \exists_{S\subseteq\mathcal{P}}m{i}_{S,1},\ldots,m{i}_{S,\mathcal{Q}(S)} \colon \left\{ \left( igwedge _{(S,k)
eq (S',k')}(m{i}_{S,k}
eq m{i}_{S',k'}) 
ight) \land \\ & orall m{j} \colon m{j} < m{n} 
ightarrow \left[ \left( igwedge _{S\subseteq\mathcal{P},k=1,\ldots,\mathcal{Q}(S)}(m{j} = m{i}_{S,k} \land igwedge _{p\in S}(m{j} \in \mathcal{X}_p) \\ & \land igwedge _{p\in \mathcal{P}\setminus S}(m{j} \notin \mathcal{X}_p) \end{pmatrix} 
ight) 
ight) \lor \\ & \left( \left( igwedge _{S\subseteq\mathcal{P},k=1,\ldots,\mathcal{Q}(S)}(m{j} \neq m{i}_{S,k}) \land igwedge _{S\subseteq\mathcal{P}\colon\mathcal{Q}(S)\geq 2}(m{j}_{S\in\mathcal{Q}}(S)\geq 2}(m{k}) \in \mathcal{X}_p \\ & \land igwedge _{p\in \mathcal{P}\setminus S}(m{j} \notin \mathcal{X}_p) \end{pmatrix} 
ight) 
ight) 
ight\}. \end{aligned}$$

We immediately get:

**Theorem 7.5.** Let  $\mathcal{N} = \langle \mathcal{P}, Tr \rangle$  be a barrier crowd and let  $\langle n, Q \rangle$  be a local indexed trap of  $\mathcal{N}(n)$ . Then every model of the formula  $ParTrap_{\mathbf{T}}(n, \mathcal{X})$  defined above is an indexed trap of  $\mathcal{N}$ . **Remark 7.6.** This theorem applies to all order invariant and homogeneous systems. It is easy to see that order invariance and homogeneity of a given parameterized net can be expressed in WS1S and verified automatically.

## 8. Generalized parameterized Petri nets

Recall that the set of places of a parameterized net has the form  $\mathcal{P} \times [n]$ , i.e., *n* copies of the set  $\mathcal{P}$  of place names. Intuitively, the net  $N_n$  consists of *n* communicating processes, each of them with (a copy of)  $\mathcal{P}$  as states.

Unfortunately, this setting is not powerful enough to model many classical distributed algorithms. In particular, it cannot model any parameterized mutual exclusion algorithm, like Dekker's, Dijkstra's, Knuth's and others [42]. The reason is that in all these algorithms agents need to execute loops in which they inspect the current value of a flag in all other agents. We explain this point in detail taking Dijkstra's mutual exclusion algorithm as an example.

**Example 8.1.** In Dijkstra's algorithm, each agent maintains a boolean variable *flag* that indicates whether the agent wants to access the critical section or not. The flag is initially set to *false*. At any moment the agent *i* may set it to *true*, after which agent *i* iteratively inspects the *flag* variable of the other agents. Crucially, the inspection takes *n* atomic steps, one for each agent. If some flag has value *true*, then the agent sets its *flag* to *false* and starts over; if all flags have value *false* (at the respective times at which the agent inspects them), then the agent moves to the critical section. If we assume that agents have identities  $0, 1, \ldots, n - 1$ , then agent *i* can be modeled by the code shown in Figure 3.

```
init: flag[i] = true;
for(j = 0; j < n; j++) {
    if(i != j and flag[j] == true) {flag[i] = false; goto init;}
}
/* critical section */
restart: flag[i] = false; goto init;
```

Figure 3: Pseudocode for Dijkstra's mutual exclusion algorithm for agent *i*.

When agent *i* inspects the flag of agent *j*, we say that agent *i* points to agent *j*. Assume that when an agent is not executing the loop the variable *j* has a special value  $\perp$  (points to null). At every moment in time the local state of each agent is determined by its current position in the code, the value of its flag, and the agent it is pointing to (or null). We distinguish six positions: *initial* (corresponds to the label init above); *loop* (before the for loop); *looping* (before the if statement); *break* (before the body of the if statement); *crit* (critical section); and *done* (label restart). The flag has two values, and *j* can have n + 1 different values.

In the Petri net model for an instance of Dijkstra's algorithm with n agents, each agent is assigned six places for the positions, two places for the values of flag, and n + 1 places for the values of j. The net has a total of  $n \cdot (8 + n)$  places. The crucial difference with respect to the net for the philosophers is that the number of places *per agent* depends on n.

Since some places now involve two agents (the places indicating that agent i points to agent j), sets of places must be modeled as relations, which leads to the definition of generalized parameterized nets:

#### **Definition 8.2. (Generalized Parameterized nets)**

A generalized parameterized net is a triple  $\mathcal{N} = \langle \mathcal{P}, \mathcal{R}, Tr \rangle$ , where  $\mathcal{P}$  is a finite set of place names,  $\mathcal{R}$  is a finite set of relation names, and  $Tr(n, \mathcal{X}, \mathcal{U}, \mathcal{Y}, \mathcal{S})$  is a second-order formula over one first-order variable n which represents the considered size of the instance, a tuple  $\mathcal{X}$  and  $\mathcal{Y}$  of monadic second-order variables for each place name of  $\mathcal{P}$ , a tuple  $\mathcal{U}$  and  $\mathcal{S}$  of dyadic second-order variables for each relation name of  $\mathcal{R}$ . Moreover, we require Tr to only quantify first-order and *monadic* second-order variables.

For every  $n \ge 1$ , the *n*-instance of  $\mathcal{N}$  is the net  $\mathcal{N}(n) = \langle P_n, T_n \rangle$  given by:

- $P_n = (\mathcal{P} \times [n]) \cup (\mathcal{R} \times [n] \times ([n] \cup \{\bot\})),$
- $\langle P_1 \cup R_1, P_2 \cup R_2 \rangle \in T_n$  if and only if " $Tr(n, P_1, R_1, P_2, R_2)$ " holds; i.e., if the interpretation  $\mu$  given by

$$\begin{split} \mu(\mathcal{X}_p) &= \left\{ i \in [n] \mid \langle p, i \rangle \in P_1 \right\}, \\ \mu(\mathcal{U}_r) &= \left\{ \langle i, j \rangle \in [n] \times ([n] \cup \{\bot\}) \mid \langle r, i, j \rangle \in P_1 \right\}, \\ \mu(\mathcal{Y}_p) &= \left\{ i \in [n] \mid \langle p, i \rangle \in P_2 \right\}, \text{ and} \\ \mu(\mathcal{S}_r) &= \left\{ \langle i, j \rangle \in [n] \times ([n] \cup \{\bot\}) \mid \langle r, i, j \rangle \in P_2 \right\}. \end{split}$$

is a model of Tr.

We now define generalized parameterized Petri nets

#### **Definition 8.3. (Generalized parameterized Petri nets)**

A generalized Petri net is a pair  $\langle \mathcal{N}, Initial \rangle$  where

- $\mathcal{N} = \langle \mathcal{P}, \mathcal{R}, Tr \rangle$  is a generalized net, and
- Initial (n, X, U) is a second-order formula over the first-order variable n, monadic second-order variables X, and dyadic second-order variables U.

As for Tr before, we restrict *Initial* to only quantify first-order variables and monadic second-order variables.

As before, we consider  $\langle \mathcal{N}(n), M \rangle$  to be the marked *n*-instance if "Initial(n, M) is true"; that is M is a 1-safe marking of  $\mathcal{N}(n)$  for which  $\mu$  with  $\mu(n) = n$  and  $\mu(\mathcal{X}_p) = \{i \in [n] \mid M(p(i)) = 1\}$  and  $\mu(\mathcal{U}_r) = \{\langle i, j \rangle \in [n] \cup ([n] \cup \{\bot\}) \mid M(r(i, j)) = 1\}$  satisfies  $\mu \models Initial$ .

**Example 8.4.** We model Dijkstra's mutual exclusion algorithm as a generalized parameterized Petri net  $\langle \mathcal{N}, Initial \rangle$  where  $\mathcal{N} = \langle \mathcal{P}, \mathcal{R}, Tr \rangle$ . We define

$$\mathcal{P} = \{initial, loop, looping, break, crit, done\} \cup \{idle, trying\} \qquad \mathcal{R} = \{ptr\}$$

Equation 2: Part of the formula Tr for Dijkstra's mutual exclusion algorithm.

$$\exists \mathbf{i} . \exists \mathbf{j} . \qquad \mathbf{i} < \mathbf{n} \land \mathbf{j} \neq \bot \land 1 \leq \mathbf{j} < \mathbf{n}$$

$$\land \quad \mathcal{X}_{looping} = \{\mathbf{i}\} \land \mathcal{U}_{ptr} = \{\langle \mathbf{i}, \mathbf{j} \rangle\}$$

$$\begin{pmatrix} \mathbf{i} = \mathbf{j} < \mathbf{n} - 1 \land \begin{bmatrix} \mathcal{S}_{ptr} = \{\langle \mathbf{i}, \mathbf{j} + 1 \rangle\} \\ \land \mathcal{Y}_{looping} = \{\mathbf{i}\} \end{bmatrix} \land Allfrom_{-}\{looping\}$$

$$\lor \quad \mathbf{i} \neq \mathbf{j} < \mathbf{n} - 1 \land \begin{bmatrix} \mathcal{S}_{ptr} = \{\langle \mathbf{i}, \mathbf{j} + 1 \rangle\} \\ \land \mathcal{X}_{idle} = \{\mathbf{j}\} \\ \land \mathcal{Y}_{idle} = \{\mathbf{j}\} \\ \land \mathcal{Y}_{looping} = \{\mathbf{i}\} \end{bmatrix} \land Allfrom_{-}\{looping, idle\}$$

$$\land \quad \mathbf{i} = \mathbf{j} = \mathbf{n} - 1 \land \begin{bmatrix} \mathcal{Y}_{crit} = \{\mathbf{i}\} \\ \land \mathcal{Y}_{looping} = \{\mathbf{i}\} \\ \land \mathcal{Y}_{looping} = \{\langle \mathbf{i}, \bot \rangle\} \end{bmatrix} \land Allfrom_{-}\{crit, looping\}$$

$$\land \quad \mathbf{i} \neq \mathbf{j} = \mathbf{n} - 1 \land \begin{bmatrix} \mathcal{Y}_{crit} = \{\mathbf{i}\} \\ \land \mathcal{Y}_{looping} = \emptyset \\ \land \mathcal{S}_{ptr} = \{\langle \mathbf{i}, \bot \rangle\} \end{bmatrix} \land Allfrom_{-}\{crit, idle, \\ looping\}$$

The set  $\mathcal{P}$  contains the positions in the code plus the elements *idle* and *trying*, which are abbreviations for "the value of *flag* is *true*", and "the value of *flag* is *false*", respectively. The relation symbol *ptr* (for pointing) indicates which agent is pointing to which one. The set of places of the instance  $\mathcal{N}_n$ is  $(\mathcal{P} \times [n]) \cup (\{ptr\} \times [n] \times ([n] \cup \{\bot\}))$ .

We only describe one part of the formula Tr modeling the transitions that correspond to some agent *successfully* advancing in its loop, either because the agent inspects itself, or because another agent that has not set its *flag* variable to *true*. For every  $\mathcal{P}' \subseteq \mathcal{P}$ , we first introduce an auxiliary formula indicating that a set of places only contains places of type  $\mathcal{P}' \subseteq \mathcal{P}$ .

$$Allfrom_{\mathcal{P}} = \bigwedge_{p \notin \mathcal{P}'} \mathcal{X}_p = \mathcal{Y}_p = \emptyset$$

The transitions are given by the formula shown in Equation 2. The formula states that there is a transition  $\langle P_1 \cup R_1, P_2 \cup R_2 \rangle$  if

•  $P_1 = \{looping(i)\}, R_1 = \{ptr(i, j)\}, P_2 = \{looping(i)\}, \text{ and } R_2 = \{ptr(i, j + 1)\} \text{ for some } i = j < n - 1; \text{ or }$ 

- $P_1 = \{looping(i), idle(j)\}, R_1 = \{ptr(i, j)\}, P_2 = \{looping(i), idle(j)\}, and, for some i \neq j < n-1, R_2 = \{ptr(i, j+1)\}; or$
- $P_1 = \{looping(i)\}, R_1 = \{ptr(i, j)\}, P_2 = \{crit(i)\}, \text{ and } R_2 = \{ptr(i, \perp)\} \text{ for } i = j = n-1; \text{ or,}$
- for some  $i \neq j = n-1$ ,  $P_1 = \{looping(i), idle(j)\}, R_1 = \{ptr(i, j)\}, P_2 = \{crit(i), idle(j)\},$ and  $R_2 = \{ptr(i, \bot)\}.$

Finally, the initial markings are given by:

### 8.1. A CEGAR approach

In Section 5.1.2 we have described a CEGAR loop for the analysis of parameterized Petri nets. We extend the approach to generalized parameterized Petri nets.

Recall that the CEGAR loop takes a parameterized Petri net  $\langle \mathcal{N}, Initial \rangle$ , and a safety property described by a formula  $Safe(n, \mathcal{M})$ , as inputs. The loop maintains a set  $\mathcal{T}$  of traps of the instances of  $\mathcal{N}$ , initially empty. In every iteration in the loop, the procedure first constructs the formula  $ParTrap_{\mathbf{T}}(n, \mathcal{X})$ , then the formula  $SafetyCheck_{\mathcal{T}}$  of (8), and then sends  $SafetyCheck_{\mathcal{T}}$  to a WS1Schecker. If the checker returns that  $SafetyCheck_{\mathcal{T}}$  holds, then every instance of  $\mathcal{N}$  satisfies the safety property, and the loop terminates. Otherwise, the checker returns a model  $\mathbf{M} = \langle n, M \rangle$  of the formula  $PReach_{\mathcal{T}}(n, \mathcal{M}) \land \neg Safe(n, \mathcal{M})$ , and searches for a witness trap  $\mathcal{X}$  that is marked at every initial marking, but empty at  $\mathbf{M}$ , with the help of the SAT-formula  $WTrap_{\mathbf{M}}(n, \mathcal{X})$  defined in (7).

Let us see how to extend the CEGAR loop to generalized parameterized Petri nets. We assume again that the generalized parameterized Petri net belongs to a class with a special topology that allows one to compute the formula  $ParTrap_{\mathbf{T}}$  for a given trap T. The obstacle is that the formula  $SafetyCheck_{\mathcal{T}}$  no longer belongs to WS1S. Indeed, since the places of a generalized Petri net are of the form p(i) or r(i, j), in (8) we have to add to the placeset parameters  $\mathcal{X}$  and  $\mathcal{M}$  relationset parameters  $\mathcal{U}$  and  $\mathcal{L}$ , i.e., sequences of dyadic predicate symbols, one for each relation in  $\mathcal{R}$ . For example,  $ParTrap_{\mathbf{T}}(\mathbf{n}, \mathcal{X})$  becomes  $ParTrap_{\mathbf{T}}(\mathbf{n}, \mathcal{X}, \mathcal{U})$ .

While the extension of WS1S with dyadic predicates is no longer decidable, we show that the problem of checking if  $SafetyCheck_{\mathcal{T}}$  is true can be reduced to the validity problem of first-order logic when  $SafetyCheck_{\mathcal{T}}$  is a universal formula, i.e., a formula in prenex normal form in which a block of universal second-order quantifiers is followed by a first-order formula. In this case it is easy to construct a first-order formula FO( $SafetyCheck_{\mathcal{T}}$ ) such that  $SafetyCheck_{\mathcal{T}}$  is true iff FO( $SafetyCheck_{\mathcal{T}}$ ) is valid. This allows us to use an automatic first-order theorem prover to check validity of FO( $SafetyCheck_{\mathcal{T}}$ ), and, therefore, the truth of  $SafetyCheck_{\mathcal{T}}$ . The price to pay is that, if  $SafetyCheck_{\mathcal{T}}$  does not hold, then the checker no longer returns a model of  $PReach_{\mathcal{T}}(n, \mathcal{M}) \wedge$ 

 $\neg Safe(n, \mathcal{M})$ ; instead, the checker just does not terminate (in practice, it reaches a timeout). For this reason, we replace the CEGAR loop by the following one, consisting of two communicating processes:

- Process 1 iteratively constructs the Petri nets N(1), N(2), N(3), ..., and uses the CEGAR loop for ordinary Petri nets (see Section 5.1.1) to compute sets T<sub>1</sub>, T<sub>2</sub>, T<sub>3</sub>, ... of traps proving that N(1), N(2), N(3), ..., satisfy the safety property. Whenever this process computes a new trap, it passes it to Process 2.
- Process 2 maintains a set *T* of traps, initially empty. First, for every *T* ∈ *T* it constructs the formula *ParTrap*<sub>T</sub>(*n*, *X*, *U*). It then constructs the formula FO(*SafetyCheck*<sub>T</sub>), and passes to a first-order theorem prover. If the prover returns that FO(*SafetyCheck*<sub>T</sub>) is valid, then *SafetyCheck*<sub>T</sub> is true, and the safety property holds for all instances. If the prover reaches a timeout, then Process 2 waits for Process 1 to send a new trap *T*, adds *T* to *T*, and iterates.

In order to complete the description of this procedure we must explain

- How to construct FO(SafetyCheck<sub>T</sub>). For universal formulas SafetyCheck<sub>T</sub> this is a standard syntax-guided procedure, that we sketch below.
- How to construct  $ParTrap_{\mathbf{T}}(n, \mathcal{X}, \mathcal{U})$ . In the next section we do this for *inspection programs*, a topology in which we can model Dijkstra's algorithm and other mutual exclusion algorithms.

The formula FO(SafetyCheck<sub> $\mathcal{T}$ </sub>). A formula of WS1S with dyadic predicates is *universal* if it is in prenex normal form and has the form  $\varphi := \forall X_1 \dots X_n \ \forall U_1 \dots U_m \ \hat{\varphi}$ , where the  $X_i$  and  $U_j$  are monadic and dyadic predicates, respectively, and  $\hat{\varphi}$  does not contain any second-order quantifiers, i.e.,  $\hat{\varphi}$  is a formula over the syntax

$$\begin{array}{rcl} t &:= & \boldsymbol{x} \mid 0 \mid \operatorname{succ}(t) \\ \varphi &:= & t_1 \leq t_2 \mid \boldsymbol{x} \in \boldsymbol{X} \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \boldsymbol{U} \mid \varphi_1 \land \varphi_2 \mid \neg \varphi_1 \mid \exists \boldsymbol{x} \colon \varphi \end{array}$$

We describe the folklore result that for any universal sentence  $\varphi$  there is a formula FO( $\varphi$ ) of first-order logic that is valid iff  $\varphi$  holds. Applying the result to  $SafetyCheck_{\mathcal{T}}$  we obtain the desired formula FO( $SafetyCheck_{\mathcal{T}}$ ). The signature of FO( $\varphi$ ) replicates the syntax above: it contains two constant symbols 0 and N, a unary function symbol succ, a binary predicate  $\leq$ , a monadic predicate  $InX_i(x)$  for every monadic second-order variable  $X_i$ , and a dyadic predicate  $InU_j(x, y)$  for every dyadic second-order variable  $U_j$ . FO( $\varphi$ ) is of the form  $\psi_0 \to \psi[\widehat{\varphi}]$ . The sentence  $\psi_0$  ensures that  $\leq$  is a discrete linear order with minimal element 0 and maximal element N, that succ is irreflexive, injective, and respects  $\leq$ , i.e.,  $\forall x \forall y : x \leq y \to succ(x) \leq succ(y)$ . The formula  $\psi[\widehat{\varphi}]$  is defined inductively on the structure of  $\widehat{\varphi}$  as follows:

- if  $\widehat{\varphi} = \boldsymbol{x}, 0, \operatorname{succ}(t), t_1 \leq t_2$ , then  $\psi[\widehat{\varphi}] = \widehat{\varphi}$ .
- if  $\widehat{\varphi} = \boldsymbol{x} \in \boldsymbol{X}, \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \boldsymbol{U}$  then  $\psi[\widehat{\varphi}] = InX(\boldsymbol{x}), InU(\boldsymbol{x}, \boldsymbol{y})$ , respectively.
- if  $\widehat{\varphi} = \neg \widehat{\varphi}_1, \widehat{\varphi}_1 \wedge \widehat{\varphi}_2$ , then  $\psi[\widehat{\varphi}] = \neg \psi[\widehat{\varphi}_1], \psi[\widehat{\varphi}_1] \wedge \psi[\widehat{\varphi}_2]$ , respectively.

Intuitively,  $\psi[\widehat{\varphi}]$  is the result of dropping all universal second-order quantifiers from  $\varphi$ , and interpreting the set membership symbol  $\in$  as a unary or binary predicate. For example, we have

$$\begin{array}{lll} \varphi &= & \forall \boldsymbol{U} \,\forall \boldsymbol{X} \,\forall \boldsymbol{x} \,\exists \boldsymbol{y} \colon (\boldsymbol{x} \in \boldsymbol{X} \land \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \boldsymbol{U}) \to \boldsymbol{y} \in \boldsymbol{X} \\ \mathrm{FO}(\varphi) &= & \psi_0 \to & \forall \boldsymbol{x} \,\exists \boldsymbol{y} \colon (\mathit{InX}(\boldsymbol{x}) \land \mathit{InU}(\boldsymbol{x}, \boldsymbol{y})) \to \mathit{InX}(\boldsymbol{y}) \ . \end{array}$$

It is easy to see that if FO( $\varphi$ ) is valid, then  $\varphi$  holds. For the other direction one observes that if  $\varphi$  holds then  $\psi[\widehat{\varphi}]$  holds in every model satisfying  $\psi_0$ .

In our implementation of this procedure we construct the formula  $FO(SafetyCheck_{\mathcal{T}})$  directly using the specific topology of looping programs without taking the detour through this translation. This also allows us to represent  $FO(SafetyCheck_{\mathcal{T}})$  in a form more suitable for a first-order theorem prover. This direct construction is presented in Appendix A.

### 8.2. Inspection programs

The loop of Dijkstra's algorithm in which an agent inspects the local state of all other agents is a quite general construction at the core of many other distributed algorithms [43, 44]. We introduce *inspection programs*, a topology tailored for describing these algorithms. In inspection programs, agents maintain a local copy of a set of variables with finite domain. For this, we assume that  $\mathcal{P}$  is partitioned into a number of sets  $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_k$ , each set representing the range of one variable. This means means that  $\mathcal{P}_j \times \{i\}$  is a 1BB-set in every  $\mathcal{N}(n)$  for all  $1 \leq j \leq k$  and  $i \in [n]$ , since the copy of every variable for each agent holds exactly *one* value at every moment in time. Further, we assume that there exists one distinct set of *states*, which we call Q in the following. As in the previous topologies, transitions are generated by transition patterns, in this case two different ones: *local* and *loop* patterns.

Local patterns. A local pattern is of the form

$$\langle origin, \langle P_1, P_2 \rangle, target \rangle$$

where  $origin, target \in Q$  and  $P_1, P_2 \subseteq \mathcal{P}$ . Roughly speaking, the pattern specifies that an agent can change its state from origin to target, simultaneously, changing its copies of the places of  $P_1$  to  $P_2$ . More formally, in the instance  $\mathcal{N}(n)$  the pattern generates for every  $i \in [n]$  a transition  $\langle Q_1, Q_2 \rangle$  with

$$Q_1 = (\{origin\} \cup P_1) \times \{i\} \text{ and } Q_2 = (\{target\} \cup P_2) \times \{i\}.$$

The pattern is called *local* because it involves only the places of one particular index.

**Loop patterns.** Loop patterns contain four loop states, called origin,  $q_o$ ,  $target_{succ}$ , and  $target_{fail}$ . Intuitively, an agent initiates the loop by moving from origin to  $q_o$ , and exits it by moving from  $q_o$  to  $target_{succ}$  or  $target_{fail}$ . The agent moves to  $target_{fail}$  whenever the agent being currently inspected fails the inspection, and to  $target_{succ}$  if all agents pass the inspection. Further, a relation  $r_o \in \mathcal{R}$  maintains the agent that is being currently inspected. Finally, the condition being inspected is modeled by two sets  $P, \overline{P} \subseteq \mathcal{P}$ : if the inspected agent is currently occupying a state of P resp.  $\overline{P}$ , then the agent passes resp. fails the inspection. Additionally, we enforce that  $q_{\circ}$  and  $r_{\circ}$  occur nowhere else in any pattern. Formally, a loop pattern is of the form

$$\langle origin, q_{\circ}, r_{\circ}, P, \overline{P}, target_{succ}, target_{fail} \rangle$$

where origin,  $q_{\circ}$ ,  $target_{succ}$ ,  $target_{fail} \in Q$  while  $P, \overline{P} \subseteq \mathcal{P}$  and  $r_{\circ} \in \mathcal{R}$ . For every agent  $i \in [n]$ , the pattern generates several transitions within the instance  $\mathcal{N}(n)$ :

• A transition  $\langle Q_1, Q_2 \rangle$ , modeling the start of the loop, given by:

$$Q_1 = \{ \langle origin, i \rangle, \langle r_{\circ}, i, \bot \rangle \} \text{ and } Q_2 = \{ \langle q_{\circ}, i \rangle, \langle r_{\circ}, i, 0 \rangle \}.$$

• A transition  $\langle Q_1, Q_2 \rangle$ , modeling that the agent does not inspect itself, given by

$$Q_1 = \{ \langle q_{\circ}, i \rangle, \langle r_{\circ}, i, i \rangle \}$$
 and  $Q_2 = \{ \langle q_{\circ}, i \rangle, \langle r_{\circ}, i, i+1 \rangle \}$ 

for every  $i \in [n-1]$ , and

$$Q_1 = \{\langle q_\circ, i \rangle, \langle r_\circ, i, i, \rangle\} \text{ and } Q_2 = \{\langle target_{succ}, i \rangle, \langle r_\circ, i, \perp \rangle\}$$

if i = n - 1.

• A transition  $\langle Q_{1j}, Q_{2j} \rangle$  for every agent j, modeling a successful inspection of agent j, given by:

$$Q_{1j} = \left\{ \left\langle q_{\circ}, i \right\rangle, \left\langle r_{\circ}, i, j \right\rangle \right\} \cup P \times \left\{ j \right\} \text{ and } Q_{2j} = \left\{ \left\langle q_{\circ}, i \right\rangle, \left\langle r_{\circ}, i, j+1 \right\rangle \right\} \cup P \times \left\{ j \right\}$$

if  $j \in [n-1]$ , and

$$Q_{1j} = \{ \langle q_{\circ}, i \rangle, \langle r_{\circ}, i, j \rangle \} \cup P \times \{j\} \text{ and } Q_{2j} = \{ \langle target_{succ}, i \rangle, \langle r_{\circ}, i, \bot \rangle \} \cup P \times \{j\}$$

- if j = n 1.
- A transition (Q<sub>1j</sub>, Q<sub>2j</sub>) for every agent j, modeling an unsuccessful inspection of agent j, given by: P:

$$Q_{1j} = \{ \langle q_{\circ}, i \rangle, \langle r_{\circ}, i, j \rangle \} \cup \overline{P} \times \{j\} \text{ and } Q_{2j} = \{ \langle target_{fail}, i \rangle, \langle r_{\circ}, i, \bot \rangle \} \cup \overline{P} \times \{j\} \text{ for every } j \in [n].$$

**Example 8.5.** The transitions of Dijkstra's mutual exclusion algorithm (Example 8.4) are generated by the four local transition patterns

$\langle initial,$	$\{idle\},$	$\{trying\},\$	$loop\rangle$
$\langle break,$	$\{trying\},\$	$\left\{ idle ight\} ,$	$initial \rangle$
$\langle done,$	$\{trying\},\$	$\left\{ idle ight\} ,$	$initial \rangle$
$\langle crit,$	Ø,	Ø,	$done \rangle$

and the single loop transition pattern

 $\langle loop, looping, ptr, \{idle\}, \{trying\}, crit, break \rangle$ .

We depict parts of the instance  $\mathcal{N}(3)$  in Figure 4.

It is straightforward to see that every inspection program can be modeled by a generalized Petri net. Notice that we already gave a part of Tr in Example 8.4 such that, when instantiated in some instance  $\mathcal{N}(n)$ , these transitions coincide with the transitions induced by the loop pattern above when the loop is advanced.

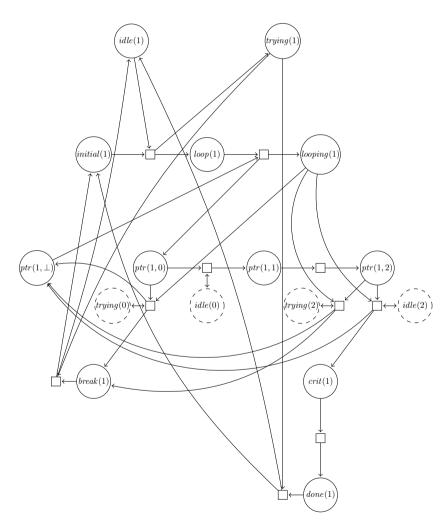


Figure 4: We illustrate here parts of  $\mathcal{N}(3)$  for the generalized Petri net  $\mathcal{N}$  of Example 8.5. Specifically, we include all places that model the state of the first agent and the transitions that change the state of the first agent. Some of these transitions "observe" places of the state of the zeroth and second agent. These places are added with dashed lines.

## 8.3. Parametrizing traps of inspection programs

We introduce parametrization results for inspection programs that, given a trap for an instance, produce a set of traps for all instances. The essential observation is that every transition involves a finite amount of indices: for local transitions there is exactly one index involved, while for loop transitions there are at most 3 involved indices; the index i of the agent that is executing the loop transition, the index jof the agent being currently inspected by agent i and (potentially) the index j + 1 to which agent iadvances its pointer.

Indexed traps of inspection programs. We denote an indexed trap as a triple  $\langle n, Q, R \rangle$ , where  $Q \subseteq \mathcal{P} \times [n]$  and  $R \subseteq \mathcal{R} \times [n] \times ([n] \cup \{\bot\})$  are sets of places such that  $Q \cup R$  is a trap of  $\mathcal{N}(n)$ . Given an indexed trap  $\langle n, Q, R \rangle$ , we introduce the following notations, where  $i, j \in [n]$ :

$$Q[i] = \{ p \in \mathcal{P} \mid p(i) \in Q \}$$
  

$$R[i, j] = \{ r \in \mathcal{R} \mid r(i, j) \in R \}$$
  

$$R[-, j] = \{ r(i) \in \mathcal{R} \times [n] \mid r(i, j) \in R \}$$
  

$$L = \{ i \in [n] \mid \text{ there exists } j \in ([n] \cup \{ \bot \}) \text{ with } R[i, j] \neq \emptyset \}$$

Q[i] is used in the same way as before, while R[i, j] and R[-, j] are generalizations of this concept to the relation symbols of the generalized net. L is the set of agents such that the trap  $Q \cup R$  contains at least one place for which there is some  $r \in R$  such that r(i, j) is in the trap for some j. We call Lthe set of *looping indices* of the indexed trap. This naming convention is inspired by the considered topology: any place r(i, j) corresponds to the fact that agent i executes some loop and currently points to j. Let us illustrate these various notions with an example.

**Example 8.6.** Consider the parameterized net N from Example 8.4. In the Petri net N(7), the following set of places constitutes a trap:

$$\left\{ \begin{array}{l} break(3), loop(3), false(3), \\ break(5), loop(5), false(5), \\ ptr(3,0), ptr(3,1), ptr(3,2), \\ ptr(3,3), ptr(3,4), ptr(3,5) \\ ptr(5,0), ptr(5,1), ptr(5,2), ptr(5,3), \end{array} \right\}$$

The corresponding indexed trap is

$$\langle n, Q, R \rangle := \left\langle 7, \left\{ \begin{array}{l} break(3), loop(3), false(3), \\ break(5), loop(5), false(5), \end{array} \right\}, \left\{ \begin{array}{l} ptr(3, 0), ptr(3, 1), ptr(3, 2), \\ ptr(3, 3), ptr(3, 4), ptr(3, 5), \\ ptr(5, 0), ptr(5, 1), ptr(5, 2), ptr(5, 3) \end{array} \right\} \right\rangle.$$

and we have

**Parametrizing indexed traps.** Let  $C_i$  denote the column of the table of Example 8.6 for index *i*. The table itself is then determined by the sequence  $C_0 C_1 \cdots C_6$ : the *column representation* of the indexed trap.

**Definition 8.7.** Let  $\langle n, Q, R \rangle$  be an indexed trap. We call the sequence  $C_0 \ C_1 \ \cdots \ C_{n-1}$ , where  $C_i := (Q[i], R[-, i])$ , the *column representation* of  $\langle n, Q, R \rangle$ .

Since  $\langle n, Q, R \rangle$  and  $C_0 C_1 \cdots C_{n-1}$  are different representations of the same object, we abuse language and speak of the indexed trap  $C_0 C_1 \cdots C_{n-1}$ .

Observe that the indexed trap of Example 8.6 satisfies  $C_1 = C_2 = C_3$ . We are going to prove that, if an indexed trap  $C_0 C_1 \cdots C_{n-1}$  of  $\mathcal{N}(n)$  satisfies  $C_{i-1} = C_i = C_{i+1}$  for some 0 < i < n-1, then one can "insert another  $C_i$ " to get an indexed trap for n + 1. We introduce for this  $shift_m^i(C_j)$ which is the result of shifting indices in R[-, j] that are larger than i by m steps; i.e., applying the simultaneous substitution  $[i + m \leftarrow i, i + m + 1 \leftarrow i + 1, \ldots, n + m \leftarrow n]$  to R[-, j]. Then every sequence of the form

$$shift_m^i(C_0) \cdots shift_m^i(C_{i-1}) shift_m^i(C_i) (shift_m^i(C_i))^m shift_m^i(C_{i+1}) \cdots shift_m^i(C_{n-1})$$

for every  $m \ge 0$  is an indexed trap of  $\mathcal{N}(n+m)$ .

**Example 8.8.** For Example 8.6, this observation allows us to find a trap in  $\mathcal{N}(8)$ :

_	i	0	1	2	3	4	5	6	7
_	Q[i]	Ø	Ø	Ø	Ø	$\{break, loop, false\}$	Ø	$\{break, loop, false\}$	Ø
	R[-,i]	$\{r(4),r(6)\}$	$\{r(4),r(6)\}$	$\{r(4),r(6)\}$	$\{r(4),r(6)\}$	$\{r(4), r(6)\}$	$\{r(4),r(6)\}$	${r(4)}$	Ø

Note that the looping indices 3 and 5 increased consistently in all R[-, j] sets. This is a consequence of the  $shift_1^1$  operation, since 2 < 3 < 5.

**Lemma 8.9.** Let  $T = C_0 C_1 \ldots C_{n-1}$  be an indexed trap (in column representation) for some inspection program  $\mathcal{N}$  with looping indices L. If there exists 0 < i < n-1 such that  $C_{i-1} = C_i = C_{i+1}$  and  $i-1, i, i+1 \notin L$ , then for every  $m \ge 0$  the sequence

$$T_{i,m} := shift_m^i(C_0) \ldots shift_m^i(C_{i-1}) \left(shift_m^i(C_i)\right)^{m+1} shift_m^i(C_{i+1}) \ldots C_{n-1}$$

is an indexed trap of  $\mathcal{N}(n+m)$ .

#### **Proof:**

We only consider the special case m = 1, since the general case follows by applying the special case m times. We have

$$T_{i,1} := shift_1^i(C_0) \dots shift_1^i(C_{i-1}) shift_1^i(C_i) shift_1^i(C_i) shift_1^i(C_{i+1}) \dots C_{n-1}$$
(11)

Let  $rev_1^i$  be the inverse operation of  $shift_1^i$ ; that is,  $rev_1^i$  applies simultaneously the substitution  $[i \leftarrow i + 1, i + 1 \leftarrow i + 2, ..., n \leftarrow n + 1]$  to all R[-, j] sets. One can easily verify that for every j = i - 1, i, i + 1, i + 2 the following holds:

$$rev_{1}^{i}(shift_{1}^{i}(C_{0})) \dots rev_{1}^{i}(shift_{1}^{i}(C_{j-1}))rev_{1}^{i}(shift_{1}^{i}(C_{j+1})) \dots rev_{1}^{i}(shift_{1}^{i}(C_{n-1})) = C_{0} \dots C_{j-1} C_{j+1} \dots C_{n-1} = T.$$

This is a consequence of  $shift_1^i(C_{i-1}) = shift_1^i(C_i) = shift_1^i(C_{i+1}) = shift_1^i(C_{i+2}), C_{i-1} = C_i = C_{i+1}$ , and  $rev_1^i(shift_1^i(C_i)) = C_i$ .

For the sake of contradiction, assume that  $T_{i,1}$  is not an indexed trap of  $\mathcal{N}(n+1)$ . Hence, there exists a transition  $\langle Q_1, Q_2 \rangle$  in  $\mathcal{N}(n+1)$  such that  $T_{i,1} \cap Q_1 \neq \emptyset$  but  $T_{i,2} \cap Q_2 = \emptyset$ .

As mentioned before, at most 3 indices are involved in any instance of any transition pattern of an inspection program. So there are indices  $j_1, j_2, j_3$  such that  $Q_1, Q_2 \subseteq \mathcal{P} \times \{j_1, j_2, j_3\} \cup \mathcal{R} \times \{j_1, j_2, j_3\} \times \{j_1, j_2, j_3, \bot\}$ . Let  $A_0 \ldots A_n$  and  $B_0 \ldots B_n$  be the column representations of  $Q_1$ and  $Q_2$  respectively. Pick now some  $k \in \{i - 1, i, i + 1, i + 2\} \setminus \{j_1, j_2, j_3\}$ . We have  $A_k = \langle \emptyset, \emptyset \rangle$ and  $B_k = \langle \emptyset, \emptyset \rangle$ . Informally speaking, we are going to remove agent k from the system and obtain a transition that contradicts T being a trap. The rest of the proof is the implementation of this idea.

Define  $\langle Q_1',Q_2'\rangle$  as the transition of  $\mathcal{N}(n)$  given by (in column representation)

$$Q'_{1} := rev_{1}^{k}(A_{0}) \dots rev_{1}^{k}(A_{k-1}) rev_{1}^{k}(A_{k+1}) \dots rev_{1}^{k}(A_{n+1})$$
(12)

$$Q'_{2} := rev_{1}^{k}(B_{0}) \dots rev_{1}^{k}(B_{k-1}) rev_{1}^{k}(B_{k+1}) \dots rev_{1}^{k}(B_{n+1})$$
(13)

respectively.  $(\langle Q'_1, Q'_2 \rangle)$  is indeed a transition of  $\mathcal{N}(n)$ , obtained from the same transition pattern as  $\langle Q_1, Q_2 \rangle$  with appropriate indices.)

We prove now that  $Q'_1 \cap T \neq \emptyset$  but  $Q'_2 \cap T = \emptyset$ , contradicting the assumption that T is an indexed trap of  $\mathcal{N}(n)$ . Since  $k \in \{i, i+1, i+2, i+3\}$ , and none of these indices is a looping index of  $T_{i,1}$ , we have

$$T = rev_{1}^{i}(shift_{1}^{i}(C_{0})) \dots rev_{1}^{i}(shift_{1}^{i}(C_{j-1}))rev_{1}^{i}(shift_{1}^{i}(C_{j+1})) \dots rev_{1}^{i}(shift_{1}^{i}(C_{n-1}))$$
  
$$= rev_{1}^{k}(shift_{1}^{i}(C_{0})) \dots rev_{1}^{k}(shift_{1}^{i}(C_{j-1}))rev_{1}^{k}(shift_{1}^{i}(C_{j+1})) \dots rev_{1}^{k}(shift_{1}^{i}(C_{n-1}))$$
(14)

Let us prove  $Q'_1 \cap T \neq \emptyset$ . Recall that  $Q_1 = A_0 \dots A_n$ . Since  $Q_1 \cap T_{i,1} \neq \emptyset$  holds by assumption, from (11) we obtain an index  $a \in \{j_1, j_2, j_3\}$  such that  $shift_1^i(C_a) \cap A_a \neq \emptyset$ ; that is,  $shift_1^i(C_a) = \langle Q^{C_a}, R^{C_a} \rangle$  and  $A_a = \langle Q^{A_a}, R^{A_a} \rangle$  such that either  $Q^{C_a} \cap Q^{A_a} \neq \emptyset$  or  $R^{C_a} \cap R^{A_a} \neq \emptyset$ . Then, however,  $rev_1^k(shift_1(C_a)) \cap rev_1^k(A_a) \neq \emptyset$ .

Similarly, we observe that  $shift_1^i(C_a) \cap B_a = \emptyset$  for all a. It follows, now, that  $rev_1^k(shift_1^i(C_a)) \cap rev_1^k(B_a) = \emptyset$  for all a. Consequently, by (12), (13) and (14) we obtain the contradiction that T is not a trap. The result follows.

Lemma 8.9 paves the way for a generalization result for inspection programs which, given a trap T of an instance, constructs a parametrization  $ParTrap_{T}(n, \mathcal{X}, \mathcal{U})$ .

Note that Lemma 8.9 essentially allows to insert  $C_i^*$ ; that is, an arbitrary repetition of the letter  $C_i$  at the appropriate location, to obtain a family of traps. One has to still account for the correct m in the  $shift_m$  operation but this observation might help to understand the following generalization theorem. Let us introduce a bit of simplifying notation first. We write

$$Q_{=P}(t)$$
 for  $\bigwedge_{p\in P} t\in \mathcal{X}_p \land \bigwedge_{p\in \mathcal{P}\setminus P} t\notin \mathcal{X}_p.$ 

That means that "Q[t]" is P; i.e., this formula is satisfied by some interpretation  $\mu$  if  $\{\mu(t)\} \times P \subseteq \bigcup_{p \in \mathcal{P}} \mu(\mathcal{X}_p) \times \{p\}$  (the set of places  $\mu$  encodes in the placeset variables  $\mathcal{X}$ ). We introduce a similar notion to represent that R[-,t] is of a certain shape for some term t. Since R[-,t] contains elements of the form  $r(\ell)$  for some  $r \in \mathcal{R}$  and some looping index  $\ell \in L$ , assume that the looping indices are  $\ell_1 < \ell_2 < \ldots < \ell_k < u_1 < u_2 < \ldots < u_m$  and there exists a corresponding first-order variable  $\ell_i$  for every  $1 \le i \le k$  and  $u_i$  for  $1 \le i \le m$ . Now we write

$$R_{=R}(t) \text{ for } \begin{bmatrix} \bigwedge_{\substack{1 \le v \le k \\ r(\ell_v) \in R}} \langle \ell_v, t \rangle \in \mathcal{U}_r \land & \bigwedge_{\substack{1 \le v \le k \\ r(\ell_v) \notin R}} \langle \ell_v, t \rangle \notin \mathcal{U}_r \\ \land \bigwedge_{\substack{1 \le v \le m \\ r(u_v) \in R}} \langle u_v, t \rangle \in \mathcal{U}_r \land & \bigwedge_{\substack{1 \le v \le m \\ r(u_v) \notin R}} \langle u_v, t \rangle \notin \mathcal{U}_r \end{bmatrix}$$

This allows us to formulate our generalization result in a compact way:

**Theorem 8.10.** Let  $T = C_0 C_1 \ldots C_{n-1}$  be an indexed trap (in column representation) in some looping program  $\mathcal{N}$  with looping indices  $L = \{\ell_1, \ldots, \ell_k, u_1, \ldots, u_m\}$ . If there exists  $i \in [n]$  such that  $C_{i-1} = C_i = C_{i+1}$  and  $\ell_1 < \ell_2 < \ldots < \ell_k < i-1 < i < i+1 < u_1 < u_2 < \ldots < u_m$ , then every model of

$$ParTrap_{\mathbf{T}}(\boldsymbol{n}, \mathcal{X}, \mathcal{U}) \coloneqq \boldsymbol{n} \leq \boldsymbol{n} \wedge \exists \boldsymbol{y} \cdot \boldsymbol{y} + (\boldsymbol{n} - (i+2)) = \boldsymbol{n} \wedge i + 1 \leq \boldsymbol{y}$$

$$\wedge \bigwedge_{j < i-1} Q_{=Q[j]}(j) \wedge \bigwedge_{i+1 < j < \boldsymbol{n}} Q_{=Q[j]}(\boldsymbol{y} + (j - (i+2)))$$

$$\wedge \forall \boldsymbol{j} \cdot i - 1 \leq \boldsymbol{j} \leq \boldsymbol{y} \rightarrow Q_{=Q[i]}(\boldsymbol{j})$$

$$\wedge \forall \boldsymbol{\ell}_{1}, \dots, \boldsymbol{\ell}_{k}, \boldsymbol{u}_{1}, \dots, \boldsymbol{u}_{m} \cdot \bigwedge_{1 \leq v \leq k} \boldsymbol{\ell}_{v} = \boldsymbol{\ell}_{v} \wedge \bigwedge_{1 \leq v \leq m} \boldsymbol{u}_{v} = \boldsymbol{y} + (u_{v} - (i+2))$$

$$\rightarrow \left( \bigwedge_{j < i-1} R_{=R[*,j]}(j) \wedge \bigwedge_{i < j} R_{=R[*,j]}(\boldsymbol{y} + (j - (i+2))) \right)$$

$$\wedge \forall i - 1 \leq \boldsymbol{j} \leq \boldsymbol{y} \rightarrow R_{=R[*,i]}(\boldsymbol{j})$$

corresponds to an indexed trap.

#### **Proof:**

Let  $\mu$  be a model of  $ParTrap_{\mathbf{T}}(\boldsymbol{n}, \mathcal{X}, \mathcal{U})$ . Consider the triple  $\langle n', Q', R' \rangle$  such that  $n' = \mu(\boldsymbol{n})$ ,  $Q' = \bigcup_{p \in \mathcal{P}} \{p\} \times \mu(\mathcal{X}_p)$ , and  $R' = \bigcup_{r \in \mathcal{R}} \{r\} \times \mu(\mathcal{U}_r)$ . Examining  $ParTrap_{\mathbf{T}}(\boldsymbol{n}, \mathcal{X}, \mathcal{U})$  closely reveals that its column representation is

$$shift_m^i(C_0) shift_m^i(C_1) \ldots shift_m^i(C_{i-1}) shift_m^i(C_i)^{m+1} shift_m^i(C_{i+1}) \ldots C_{n-1}$$

for some  $m \ge 0$ . The result follows now immediately from Lemma 8.9.

## 9. Experiments

## 9.1. ostrich

We implemented the CEGAR loop and the parameterization techniques of Sections 6 and 7 in our tool ostrich. ostrich heavily relies on MONA as a WS1S-solver. The results of our experiments are presented in Figure 5. In the first two columns the table reports the topology and the name of the system to be verified. The array topology is a linear topology where agents can refer existentially or universally to agents with smaller or larger indices. Analogously to the other topologies we derive a sound parameterization technique for traps, 1-BB sets, and siphons. The rings are Dijsktra's token ring for mutual exclusion [46] and a model of the dining philosophers in which philosophers pick both forks simultaneously. For headed rings we consider Example 3.2 and a model of a message passing leader

Topology	Example	Init. (ms)	Property	Check (ms)	1BB-sets	Traps	Siphons	Semi-automatic invariants	
	Dijkstra ring	558	deadlock	40	1 (1)	0 (0)	0 (0)	2	
ring			mutual exclusion	125	1 (1)	1(1)	0 (0)	2	
	atomic phil.	409	deadlock	79	1 (1)	0 (0)	0 (0)	4	
	lefty phil.	495	deadlock	294	7 (4)	0 (0)	0 (0)	3	
headed ring	leader election	670	not 0 and $n - 1$ leader not two leaders	965	1 (0)	0 (0)	2 (1)	1	
				-	-	-	-		
array	Burns	501	deadlock	16	0 (0)	0(0)	0 (0)	1	
			mutual exclusion	379	0 (0)	8 (7)	0 (0)		
	Dijkstra	1830	deadlock	88	2 (1)	0 (0)	0 (0)	3	
			mutual exclusion	1866	0 (0)	3 (1)	0 (0)		
	Berkeley	414	deadlock	12	0 (0)	0 (0)	0 (0)	1	
			consistency (3/3)	361	0 (0)	9 (1)	0 (0)		
crowd	Dragon	538	deadlock	19	0 (0)	0 (0)	0 (0)	7	
			consistency (7/7)	2334	52 (7)	0 (0)	0 (0)		
	Firefly	511	deadlock	14	0 (0)	0 (0)	0 (0)	2	
			consistency (0/4)	232	0 (0)	2 (0)	0 (0)		
	Illinois	468	deadlock	13	0 (0)	0 (0)	0 (0)	1	
			consistency (0/2)	180	0 (0)	3 (0)	0 (0)	1	
	MESI	422	deadlock	12	0 (0)	0 (0)	0 (0)	1	
			consistency (2/2)	500	0 (0)	13 (2)	0 (0)	1	
	MOESI	446	deadlock	13	0 (0)	0 (0)	0 (0)		
			consistency (7/7)	1226	0 (0)	24 (4)	0 (0)	1	
	Synapse	420	deadlock	12	0 (0)	0 (0)	0 (0)		
			consistency (2/2)	22	0 (0)	0 (0)	0 (0)	0	

Figure 5: Experimental results of ostrich. The complete data is available at [45].

election algorithm. The array is Burns' mutual exclusion algorithm [47]. The crowds are Dijkstra's algorithm for mutual exclusion [42] and models of cache-coherence protocols taken from [41]. Note that we check inductiveness of the property; i.e., if it holds initially and there is no marking satisfying the property and the current abstraction and reaching in a single step a marking which violates the property. Additionally, we include in the specification of the parameterized Petri net a partition of the places  $\mathcal{P}$  such that the places of every index in every instance form a 1BB-set. Collectively, this ensures that all examples are 1-bounded and yields invariants similar to (1), (2) for Example 3.2. Since ostrich does not compute but only checks these invariants we do not count them in Figure 5 (leading to 3 semi-automatic invariants for Example 3.2 since we omit (1), and (2)). Moreover, these invariants already imply inductiveness of some safety properties; prominently deadlock-freedom for all considered cache-coherence protocols.

The third column gives the time ostrich needs to initialize the analysis; this includes verifying that the given parameterized Petri net is covered by 1BB-sets, and that it indeed has the given topology. The fourth column gives the property being checked. The specification of the cache coherence protocols consists of a number of consistency properties, specific for each protocol. The legend "consistency (x/y)" indicates that the specification consists of y properties, of which ostrich was able to automatically prove the inductiveness of x. Column 5 gives the time need to check the inductiveness the property (or, in the case of the cache-coherence protocols, either find a marking which satisfies all constraints imposed by 1BB-sets, traps or siphons, or prove the inductiveness of the properties together). Columns 6, 7, and 8 give the number of WS1S-formulas, each corresponding to a parameterized 1BB-sets, trap, or siphon that are computed by the CEGAR loop. Some of these WS1S-formulas have only one model, i.e., they correspond to a single trap, siphon, or 1BB-set of one instance. Such "artifacts" are needed when small instances (e.g., arrays of size 2) require ad-hoc proofs that cannot be parameterized. In these cases the "real" number of parametric invariants is the result of subtracting the number of artifacts from the total number. The last column reports the number of parameterized inductive invariants obtained by the semi-automatic CEGAR loop. There the user is presented a series of counter examples to the inductiveness of the property. The user can check for traps, siphons or 1BBsets to disprove the counter example. If the user then provides an invariant which proves inductive it is used to refine the abstraction until no further counter example can be found. The response time of ostrich in this setting is immediate which provides a nice user experience. MOESI is an example which shows that the semi-automatic procedure can lead to proofs with fewer invariants. For Dragon four of the seven invariants are artifacts; thus, it also shows that a semi-automatic approach allows for proofs with fewer invariants. The last step of the automatic procedure is to remove invariants until no invariant can be removed without obtaining a counter example again.

For Example 3.2 ostrich automatically computes the following family of 1BB-sets (additionally to the invariants (1) and (2)): (For readability we omit some artifacts.)

$$2 \leq \mathbf{n} \wedge \text{taken} = \text{think} = \emptyset \wedge \text{wait} = \text{eat} = \{0, 1\} \wedge \text{free} = \{1\}$$
  

$$3 \leq \mathbf{n} \wedge \text{taken} = \text{wait} = \text{think} = \emptyset \wedge \text{eat} = \{\mathbf{n} - 1, 0\} \wedge \text{free} = \{0\}$$
  

$$4 \leq \mathbf{n} \wedge \text{taken} = \text{think} = \emptyset \wedge \text{free} = \text{wait} = \{\mathbf{n} - 1\} \wedge \text{eat} = \{\mathbf{n} - 2, \mathbf{n} - 1\}$$
  

$$2 \leq \mathbf{n} \wedge \exists \mathbf{i} : 1 < \mathbf{i} < \mathbf{n} - \mathbf{2} \wedge \left( \begin{array}{c} \text{taken} = \text{think} = \emptyset \wedge \text{free} = \text{wait} = \{\mathbf{i} \oplus_{\mathbf{n}} 1\} \\ \wedge \text{eat} = \{\mathbf{i}, \mathbf{i} \oplus_{\mathbf{n}} 1\} \end{array} \right)$$

### 9.2. heron

We implemented the approach described in Section 8 in our tool heron [48, 49]. An illustration of the general concept can be found in Figure 6.

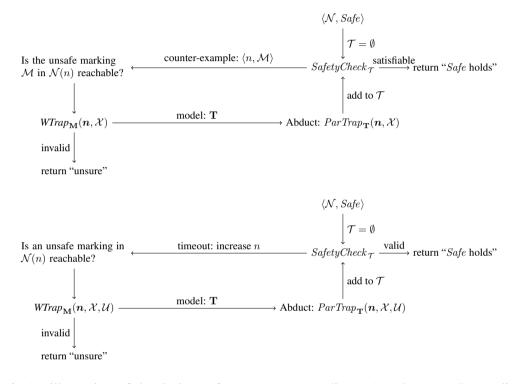


Figure 6: An illustration of the designs of ostrich (upper diagram) and heron (lower diagram). For both the input is a (generalized) parameterized Petri net  $\mathcal{N}$  and a property *Safe* that we want to check. For ostrich we can express *WTrap* in WS1S, while heron specifically relies on an appropriate embedding into SAT. Also, ostrich uses MONA to check *SafetyCheck* or get a counter-example, while heron uses first-order provers. Further, heron uses timeouts for its provers; e.g., VAMPIRE or CVC4. The experimental data suggests that these timeouts can be chosen small; in particular, 2 - 3 seconds is appropriate for our mutual exclusion benchmarks.

heron is written in Python. It uses clingo [50] as SAT solver. To solve the first-order queries heron uses VAMPIRE [51] and CVC4 [52]. As benchmarks we consider classical algorithms for mutual exclusion. These include a reduced version of Dijkstra's algorithm for mutual exclusion [42], which we presented as Example 8.4 above, a more precise formalization of Dijkstra's algorithm, an algorithm by Knuth [53], one by de Bruijn [54], and one by Eisenberg & McGuire [55]. Additionally, we have modeled Szymanski's algorithm for mutual exclusion [56] as well. For all these algorithms we consider the property that they indeed provide mutual exclusion of processes in the critical section. For most of these algorithms we need to expand the topology of inspection programs in various ways. However, all these expansions maintain that every transition involves at most a finite amount of indices

and that every re-ordering of these indices also yield a transition in an instance. Inspecting the proof of Lemma 8.9 one observes that these are the crucial observations for the stated result. Consequently, Theorem 8.10 generalizes well to all these expansions. We present positive results for all these examples but for the algorithm of Szymanski. The table in Figure 7 reports data on the positive results. The first column shows which algorithm we prove. The second column states how many seconds heron needs to compute the positive result. The third column reports the maximal n for which  $\mathcal{N}(n)$  is instantiated during this computation. In the fourth column we give the amount of traps we computed during this computation, and in the fifth column how many abducted trap families we used. The sixth column gives the maximal amount of looping indices that occur in traps for this example and, finally, we give the longest time it took for a successful query to the prover.

Algorithm	time (s)	max. N	# traps	# abducted traps	max.	max. proving
Mgonum	time (3)			" abducted traps	# indices	time (s)
Example 8.4	11	8	36	2	2	1
Dijkstra's	22	8	75	5	2	1
Knuth's	194	8	160	7	2	1
de Bruijn's	76	8	164	6	2	1
Eisenberg & McGuire's	1055	9	126	6	2	1

Figure 7: Experimental results for heron.

For Szymanski's algorithm for mutual exclusion our algorithm fails. This is because already instances with n = 2 do not allow to prove the mutual exclusion property via traps, 1BB-sets, and siphons. In fact, Szymanski's algorithm posed already a negative result for the approach of [5]. This means that instances of Szymanski's algorithm are out of reach – even when one additionally uses the marking equation for Petri nets to over-approximate reachable markings.

The data suggests that heron, as ostrich, synthesizes only a small amount of necessary invariants. Moreover, these invariants are "local"; that is, they involve at most 2 looping indices. Consequently, the proofs that heron constructs are readable and concise. The drawback, however, is the significant amount of time we need to construct and verify these proofs. Surprisingly, the queries to the theorem prover, once all the necessary invariants are synthesized, are actually very fast; most time is spent on instantiating and proving finite instances of  $\mathcal{N}$ .

## 10. Conclusion.

We have refined the approach to parameterized verification of systems with regular architectures presented in [9]. Instead of encoding the complete verification question into large, monolithic WS1Sformula, our approach introduces a CEGAR loop which also outputs an explanation of why the property holds in the form of a typically small set of parameterized invariants (see Example 3.2). The explanation helps to uncover false positives, where the verification succeeds only because the system or the specification are incorrectly encoded in WS1S. It has also helped to find a subtle bug in the implementation of [9] which hid unnoticed in the complexity of the monolithic formula. Additionally, our incremental approach requires to check smaller WS1S-formulas, which often decreases the verification time (cp. the verification of Dijkstra's mutual exclusion algorithm [9] in 10s to currently 2s).

On the other hand, seeing the abstraction helps one understand the analyzed system. For example, we include in [45] a leader election algorithm for which the parameterization techniques of ostrich are too coarse to establish the general safety property of having always at most one leader. However, ostrich succeeds to prove the special case that not agents 0 and n-1 can become leader at the same time. For this proof ostrich finds a family of siphons which hint to a general inductive invariant of the system. Using the semi-automatic mode of ostrich we can then verify this inductive invariant and, as a result of this, the general safety property.

We wanted to expand our methodology to models that represent actual implementations of distributed algorithms more accurately. Therefore, we discussed an expansion of the original approach that allows to model non-atomic global checks. Although one forfeits the decidability of the logical embedding – a corner stone of the CEGAR loop in the original model – we can adapt our approach to capture these expanded models as well. The resulting algorithm solves a set of non-trivial examples. Moreover, it maintains the desirable property of synthesizing concise and readable invariants for the considered examples.

**Future work.** Parameterized Petri nets rely on WS1S to specify their transitions and their initial configurations, and it is well known that the languages expressible in WS1S are exactly the regular languages. This suggests that our techniques might be extended to the regular systems analyzed in regular model-checking [57]. In this approach a finite automaton describes the language of initial configurations, and a length-preserving transducer describes the possible transitions. We think our techniques can be used to algorithmically compute a regular over-approximation of the set of reachable configurations.

The heron tool checks reachability in 1-safe nets by means of an incomplete method that tests if a marking satisfies all constraints induced by the traps, siphons, and 1-BB sets of the net. This is closely related to the approach of [5], which relies on traps, siphons, and the marking equation. Replacing the marking equation by 1-BB sets leads to a less precise test, but one that can be completely implemented on top of a SAT-solver and can be generalized to the parameterized case. We plan to study if the benchmarks of [5] can already be successfully verified using traps, siphons, and 1-BB sets, or a suitable generalization thereof.

Our method is currently restricted to looping programs. We think that it can be extended to programs with nested loops. We also plan to study stronger invariants allowing us to verify Szymanski's algorithm for mutual exclusion, for which our technique is not yet strong enough.

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The tool ostrich and associated files are available at [45]. The current version is maintained at [58]. The tool heron and associated files are available at [49]. The current version is maintained at [48].

This is an expanded version of [59]; that is, Section 8 was added. It also relies on results of [60]. We thank the anonymous reviewers of the original versions and this submission for their helpful comments.

# References

- [1] Murata T. Petri nets: Properties, analysis and applications. Proceedings of the IEEE, 1989. 77(4):541–580.
- [2] Reisig W. Understanding Petri Nets Modeling Techniques, Analysis Methods, Case Studies. Springer, 2013. ISBN:978-3-642-33277-7.
- [3] Desel J, Esparza J. Free choice Petri nets. Cambridge university press, 2005. ISBN-10:0521019451, 13:978-0521019453.
- [4] Bensalem S, Bozga M, Nguyen T, Sifakis J. D-Finder: A Tool for Compositional Deadlock Detection and Verification. In: CAV, volume 5643 of *LNCS*. 2009 pp. 614–619. doi:10.1007/978-3-642-02658-4\_45.
- [5] Esparza J, Ledesma-Garza R, Majumdar R, Meyer PJ, Niksic F. An SMT-Based Approach to Coverability Analysis. In: CAV, volume 8559 of *LNCS*. Springer, 2014 pp. 603–619. doi:10.1007/978-3-319-08867-9\_40.
- [6] Blondin M, Finkel A, Haase C, Haddad S. Approaching the Coverability Problem Continuously. In: TACAS, volume 9636 of LNCS. Springer, 2016 pp. 480–496. doi:10.1007/978-3-662-49674-9\_28.
- [7] Wimmel H, Wolf K. Applying CEGAR to the Petri Net State Equation. *Logical Methods in Computer Science*, 2012. **8**(3). doi:10.1007/978-3-642-19835-9\_19.
- [8] Bozga M, Iosif R, Sifakis J. Checking Deadlock-Freedom of Parametric Component-Based Systems. In: TACAS (2), volume 11428 of LNCS. Springer, 2019 pp. 3–20. doi:10.48550/arXiv.1805.10073.
- [9] Bozga M, Esparza J, Iosif R, Sifakis J, Welzel C. Structural Invariants for the Verification of Systems with Parameterized Architectures. In: TACAS (1), volume 12078 of *LNCS*. Springer, 2020 pp. 228–246. doi:10.1007/978-3-030-45190-5\_13.
- [10] Bloem R, Jacobs S, Khalimov A, Konnov I, Rubin S, Veith H, Widder J. Decidability of Parameterized Verification. Synthesis Lectures on Distributed Computing Theory. Morgan & Claypool Publishers, 2015. doi:10.1007/978-3-031-02011-7.
- [11] Esparza J. Parameterized Verification of Crowds of Anonymous Processes. In: Dependable Software Systems Engineering, IOS Press, 2016 pp. 59–71. doi:10.3233/978-1-61499-627-9-59.
- [12] Abdulla PA, Sistla AP, Talupur M. Model Checking Parameterized Systems. In: Handbook of Model Checking, pp. 685–725. Springer, 2018. doi:10.1007/978-3-319-10575-8.21.
- [13] The MONA Project. MONA. URL https://www.bricks.dk/mona.
- [14] Henriksen JG, Jensen JL, Jørgensen ME, Klarlund N, Paige R, Rauhe T, Sandholm A. MONA: Monadic Second-Order Logic in Practice. In: TACAS, volume 1019 of LNCS. Springer, 1995 pp. 89–110.
- [15] Apt KR, Kozen DC. Limits for automatic verification of finite-state concurrent systems. *Information Processing Letters*, 1986. 22(6):307 309.
- [16] Esparza J, Melzer S. Verification of Safety Properties Using Integer Programming: Beyond the State Equation. *Formal Methods in System Design*, 2000. 16(2):159–189. doi:10.1023/A:1008743212620.

- [17] Esparza J, Meyer PJ. An SMT-based Approach to Fair Termination Analysis. In: FMCAD. IEEE, 2015 pp. 49–56. doi:10.1109/FMCAD.2015.7542252.
- [18] Blondin M, Esparza J, Helfrich M, Kucera A, Meyer PJ. Checking Qualitative Liveness Properties of Replicated Systems with Stochastic Scheduling. In: CAV (2), volume 12225 of *LNCS*. Springer, 2020 pp. 372–397. doi:10.1007/978-3-030-53291-8\_20.
- [19] German SM, Sistla AP. Reasoning about Systems with Many Processes. J. ACM, 1992. 39(3):675–735.
- [20] Abdulla PA, Cerans K, Jonsson B, Tsay Y. General Decidability Theorems for Infinite-State Systems. In: LICS. IEEE Computer Society, 1996 pp. 313–321.
- [21] Finkel A, Schnoebelen P. Well-structured transition systems everywhere! *Theor. Comput. Sci.*, 2001. 256(1-2):63–92. doi:10.1016/S0304-3975(00)00102-X.
- [22] Blondin M, Finkel A, Haase C, Haddad S. Approaching the Coverability Problem Continuously. In: TACAS, volume 9636 of *LNCS*. Springer, 2016 pp. 480–496. doi:10.1007/978-3-662-49674-9\_28.
- [23] Geffroy T, Leroux J, Sutre G. Occam's Razor applied to the Petri net coverability problem. *Theor. Comput. Sci.*, 2018. **750**:38–52. doi:10.1016/j.tcs.2018.04.014.
- [24] Reynier P, Servais F. On the Computation of the Minimal Coverability Set of Petri Nets. In: RP, volume 11674 of LNCS. Springer, 2019 pp. 164–177. doi:10.1007/978-3-030-30806-3\_13.
- [25] Finkel A, Haddad S, Khmelnitsky I. Minimal Coverability Tree Construction Made Complete and Efficient. In: FoSSaCS, volume 12077 of *LNCS*. Springer, 2020 pp. 237–256. doi:10.1007/978-3-030-45231-5\_13.
- [26] Athanasiou K, Liu P, Wahl T. Unbounded-Thread Program Verification using Thread-State Equations. In: IJCAR, volume 9706 of LNCS. Springer, 2016 pp. 516–531. doi:10.1007/978-3-319-40229-1\_35.
- [27] Kesten Y, Maler O, Marcus M, Pnueli A, Shahar E. Symbolic model checking with rich assertional languages. *Theor. Comput. Sci*, 2001. 256(1-2):93 – 112. doi:10.1016/S0304-3975(00)00103-1.
- [28] Abdulla PA, Jonsson B, Nilsson M, Saksena M. A Survey of Regular Model Checking. In: CONCUR, volume 3170 of LNCS. Springer, 2004 pp. 35–48. doi:10.1007/978-3-540-28644-8\_3.
- [29] Abdulla PA, Delzanno G, Henda NB, Rezine A. Regular Model Checking Without Transducers (On Efficient Verification of Parameterized Systems). In: TACAS, volume 4424 of *LNCS*. Springer, 2007 pp. 721–736. doi:10.1007/978-3-540-71209-1\_56.
- [30] Baukus K, Bensalem S, Lakhnech Y, Stahl K. Abstracting WS1S Systems to Verify Parameterized Networks. In: TACAS, volume 1785 of LNCS. Springer, 2000 pp. 188–203. doi:10.1007/3-540-46419-0\_14.
- [31] Baukus K, Lakhnech Y, Stahl K. Parameterized Verification of a Cache Coherence Protocol: Safety and Liveness. In: VMCAI, volume 2294 of *LNCS*. Springer, 2002 pp. 317–330. doi:10.1007/3-540-47813-2\_22.
- [32] Chen Y, Hong C, Lin AW, Rümmer P. Learning to prove safety over parameterised concurrent systems. In: FMCAD. 2017 pp. 76–83. doi:10.23919/FMCAD.2017.8102244.
- [33] Browne M, Clarke E, Grumberg O. Reasoning about networks with many identical finite state processes. *Information and Computation*, 1989. **81**(1):13 31.
- [34] Emerson EA, Namjoshi KS. Reasoning about Rings. In: POPL. 1995 pp. 85-94.
- [35] Emerson EA, Kahlon V. Reducing Model Checking of the Many to the Few. In: CADE, volume 1831 of LNCS. Springer, 2000 pp. 236–254. doi:10.1007/10721959\_19.

- [36] Außerlechner S, Jacobs S, Khalimov A. Tight Cutoffs for Guarded Protocols with Fairness. In: VMCAI, volume 9583 of LNCS. Springer, 2016 pp. 476–494. doi:10.1007/978-3-662-49122-5\_23.
- [37] Jacobs S, Sakr M. Analyzing Guarded Protocols: Better Cutoffs, More Systems, More Expressivity. In: VMCAI, volume 10747 of *LNCS*. Springer, 2018 pp. 247–268. doi:10.1007/978-3-319-73721-8\_12.
- [38] Esparza J. Decidability and Complexity of Petri Net Problems An Introduction. In: Petri Nets, volume 1491 of *Lecture Notes in Computer Science*. Springer, 1996 pp. 374–428. doi:10.1007/3-540-65306-6\_20.
- [39] Esparza J, Raskin M, Welzel C. Computing Parameterized Invariants of Parameterized Petri Nets, 2021. 2103.10280, URL https://arxiv.org/abs/2103.10280.
- [40] Esparza J, Finkel A, Mayr R. On the Verification of Broadcast Protocols. In: LICS. IEEE Computer Society, 1999 pp. 352–359.
- [41] Delzanno G. Automatic Verification of Parameterized Cache Coherence Protocols. In: CAV. 2000 pp. 53–68. doi:10.1007/10722167\\_8.
- [42] Dijkstra EW. Cooperating Sequential Processes, pp. 65–138. Springer New York, New York, NY. ISBN:978-1-4757-3472-0, 2002. doi:10.1007/978-1-4757-3472-02.
- [43] Lynch NA. Distributed Algorithms. Morgan Kaufmann, 1996.
- [44] Herlihy M, Shavit N. The art of multiprocessor programming. Morgan Kaufmann, 2008.
- [45] Welzel C, Esparza J, Raskin M. ostrich, 2020. doi:10.5281/zenodo.6523828.
- [46] Fribourg L, Olsén H. Reachability sets of parameterized rings as regular languages. *Electr. Notes Theor. Comput. Sci.*, 1997. 9:40. doi:10.1016/S1571-0661(05)80427-X.
- [47] Jensen HE, Lynch NA. A Proof of Burns N-Process Mutual Exclusion Algorithm Using Abstraction. In: TACAS, volume 1384 of *Lecture Notes in Computer Science*. Springer, 1998 pp. 409–423.
- [48] Esparza J, Raskin M, Welzel C. heron, git repository. https://gitlab.lrz.de/i7/heron, 2021.
- [49] Welzel C, Esparza J, Raskin M. heron, software artifact. https://doi.org/10.5281/zenodo.5068849, 2020. doi:10.5281/zenodo.5068849.
- [50] Gebser M, Kaufmann B, Kaminski R, Ostrowski M, Schaub T, Schneider M. Potassco: The Potsdam Answer Set Solving Collection. AI Commun., 2011. 24(2):107–124. doi:10.3233/AIC-2011-0491.
- [51] Kovács L, Voronkov A. First-Order Theorem Proving and Vampire. In: CAV, volume 8044 of Lecture Notes in Computer Science. Springer, 2013 pp. 1–35. doi:10.1007/978-3-642-39799-8\_1.
- [52] Barrett CW, Conway CL, Deters M, Hadarean L, Jovanovic D, King T, Reynolds A, Tinelli C. CVC4. In: CAV, volume 6806 of *Lecture Notes in Computer Science*. Springer, 2011 pp. 171–177. doi:10.1007/978-3-642-22110-1\_14.
- [53] Knuth DE. Additional comments on a problem in concurrent programming control. *Commun. ACM*, 1966.
   9(5):321–322. doi:10.1145/363162.363167.
- [54] de Bruijn NG. Additional comments on a problem in concurrent programming control. *Commun. ACM*, 1967. 10(3):137–138. doi:10.1145/355592.365595.
- [55] Eisenberg MA, McGuire MR. Further Comments on Dijkstra's Concurrent Programming Control Problem. Commun. ACM, 1972. 15(11):999. doi:10.1145/355606.361895.

- [56] Szymanski BK. Mutual exclusion revisited. In: Next Decade in Information Technology: Proceedings of the 5th Jerusalem Conference on Information Technology 1990, Jerusalem, October 22-25, 1990. 1990 pp. 110–117. doi:10.1109/JCIT.1990.128275.
- [57] Bouajjani A, Jonsson B, Nilsson M, Touili T. Regular Model Checking. In: CAV. 2000 pp. 403–418. doi:10.1007/10722167\_31.
- [58] Esparza J, Raskin M, Welzel C. ostrich. https://gitlab.lrz.de/i7/ostrich, 2021.
- [59] Esparza J, Raskin MA, Welzel C. Computing Parameterized Invariants of Parameterized Petri Nets. In: Petri Nets, volume 12734 of *Lecture Notes in Computer Science*. Springer, 2021 pp. 141–163. doi:10.1007/978-3-030-76983-3\_8.
- [60] Esparza J, Raskin MA, Welzel C. Abduction of trap invariants in parameterized systems. In: GandALF, volume 346 of *EPTCS*. 2021 pp. 1–17. doi:10.4204/EPTCS.346.1.

# A. Constructing FO( $SafetyCheck_{T}$ ) for looping programs.

An embedding of a linear order into FO. First, we want to capture the linear topology of our agents in an FO theory. To this end, we gradually introduce an appropriate FO theory in the following. Initially, consider a relation symbol  $\leq$ . It is straightforward to give a sentence  $\varphi_{\leq}$  that ensures that  $\leq$  is a discrete linear order with a minimal element. Then, we introduce two constant symbols, 0 and N. We make sure that 0 is the minimal element w.r.t.  $\leq$ ; that is, we add  $\neg \exists x \cdot x \neq 0 \land x \leq 0$  to our theory. Furthermore, it is standard to obtain the immediate successor of some element x w.r.t.  $\leq$ . To ease presentation, we use a function symbol succ instead (which is consistent with the successor function in WS1S). In the following, we allow for constant symbols 1, 2, 3, ... in FO formulas. For this, we use the convention that the constant symbol i corresponds to the value that we obtain when applying the function succ exactly i times to 0. Moreover, we model addition of a constant value to some variable similarly by applying succ appropriately often to the variable symbol. This total order gives us now access to the linear identities of the agents.

**Representing configurations in FO.** As before, we want to capture the current configuration as models of a formula. Before we describe how we do this, we inspect the considered topology in more detail. In this way we can identify invariants which allow us to simplify the embedding into FO:

- Note that every agent maintains a local copy of variables, each of which has a finite domain: as we described before, the set *P* partitions into *P*<sub>0</sub>,...,*P*<sub>k</sub> such that {*i*} × *P*<sub>j</sub> for any *i* ∈ [*n*] and 0 ≤ *j* ≤ *k* is a 1-BB set in every instance *N*(*n*).
- Additionally, one can also deduce that {⟨r, i, j⟩ | j ∈ ([n] ∪ {⊥})} is a 1BB-set for each i ∈ [n] and r ∈ R in every instance N(n) of a looping program N.
- Moreover, we are assured that every relation symbol r<sub>o</sub> ∈ R is tied to exactly one loop transition pattern. Similarly, there is a unique q<sub>o</sub> for this transition pattern. By close inspection of the semantics of loop transition patterns, it is immediate that (r<sub>o</sub>, i, ⊥) is marked if and only if (q<sub>o</sub>, i) is not marked for all i ∈ [n] for all N(n).

Hence, we can restrict our analysis to markings that satisfy these constraints. Consequently, we call markings of  $\mathcal{N}(n)$  viable if  $\sum_{p \in \mathcal{P}_j \times \{i\}} M(p) = 1$  for all  $i \in [n]$  and  $0 \leq j \leq k$ ,  $\sum_{p \in \{r_o\} \times \{i\} \times ([n] \cup \{\bot\})} M(p) = 1$  for all  $i \in [n]$  and  $r_o \in \mathcal{R}$ , and  $M(\langle q_o, i \rangle) + M(\langle r_o, i, \bot \rangle) = 1$  for any  $q_o$  and  $r_o$  that occur in the same loop transition pattern. From now on, we refer to  $q_o^{r_o}$  for the uniquely identified state value that occurs with  $r_o$  in a loop transition pattern. Similarly, we use  $r_o^{q_o}$ .

Before we used monadic variables  $\mathcal{X}_p$  for each  $p \in \mathcal{P}$  which capture which places of  $\mathcal{P} \times [n]$ are marked in the considered instance. But by restricting our analysis to viable markings we can express the current value of this variable as a function symbol instead. For this, fix some  $0 \leq j \leq k$ and let  $val_1, \ldots, val_m$  be an enumeration of  $\mathcal{P}_j$ . We introduce now a function symbol  $var_j$  and constant symbols  $val_1, \ldots, val_m$ . Then, we can express with  $var_j(\mathbf{x}) = val_\ell$  that the agent with index  $\mathbf{x}$  currently sets its j-th variable to the value  $val_\ell$ . It is straightforward to restrict the domain of  $var_j$  only to these constant symbols for all agents:  $\forall \mathbf{x} \cdot \bigvee_{1 \leq \ell \leq m} var_j(\mathbf{x}) = val_\ell$ . In this way we translate the topological restriction of viable markings implicitly to our representation: we use a function symbol  $var_j$  instead of  $\ell$  many monadic variables. In the following, we refer to  $var_{val}$  to  $var_j$  such that  $val \in \mathcal{P}_j$ .

Similarly, we use that every agent executes at most one loop transition pattern at a time by representing *all* relations simultaneously by *one single* function symbol f: we need f to map from agents to agents or some value that represents  $\bot$ . Therefore, we add  $\forall x \cdot f(x) \leq N$  to the restricting theory; here N will be used to represent  $\bot$ .

Consider now any viable marking M in some instance  $\mathcal{N}(n)$ . Then, M induces a model  $\mu$ . Namely, we set the universe of  $\mu$  to  $\{0, 1, 2, 3, \ldots\}$ ,  $\mu(0) = 0$ ,  $\mu(N) = n$ , and  $\mu(\leq)$  to be the natural order. For every j we choose some arbitrary enumeration  $val_1, \ldots, val_m$  of  $\mathcal{P}_j$  and set  $\mu(val_k) = k$  for  $1 \leq k \leq m$ . Moreover, we set  $\mu(var_j)$  to any function such that  $\mu(var_j)(i) = val_k$  if and only if  $M(\langle val_k, i \rangle) = 1$ . The last definition uses that M is viable since there is exactly one such tuple for every  $i \in [n]$ . Since M is viable there is at most one  $r_0$  for every  $i \in [n]$  such that  $M(\langle r_0, i, j \rangle) = 1$  for any  $j \in [n]$ . If this is the case, let  $\mu(f)(i) = j$ . Otherwise, set  $\mu(f)(i) = n$ . In this way, we obtain an interpretation  $\mu$  for every viable marking M.

**Traps in FO.** The general idea of our approach is to obtain traps via Theorem 8.10. Then, we use the induced invariants of these traps to refine the FO theory of interpretations which we consider. More precisely, traps induce an abstraction of all reachable markings. We need to restrict our theory in such a way that it still contains an interpretation that represents any viable marking that satisfies the constraints of all found traps. To this end, let us introduce an FO formula which coincides with the invariant the models of  $ParTrap_{T}(n, \mathcal{X}, \mathcal{U})$  from Theorem 8.10 induce:

$$\begin{array}{l} (n \leq \mathbf{n} \land \exists \mathbf{y} . \mathbf{y} + (n - (i + 2)) = \mathbf{n}) \land i + 1 \leq \mathbf{y} \\ \land \bigvee_{\substack{j \leq i-1 \\ p \in Q[j]}} \forall ar_p(j) = p \\ \forall \exists \mathbf{j} . i - 1 \leq \mathbf{j} \leq \mathbf{y} \land \bigvee_{p \in Q[i]} \forall ar_p(\mathbf{j}) = p \\ \forall \bigvee_{\substack{i+1 < j < n \\ p \in Q[j]}} \forall ar_p(\mathbf{y} + (j - (i + 2)))) = p \\ \forall \exists \ell_1, \dots, \ell_k, \mathbf{u}_1, \dots, \mathbf{u}_m . \bigwedge_{1 \leq v \leq k} \ell_v = \ell_v \land \bigwedge_{1 \leq v \leq m} \mathbf{u}_v = \mathbf{y} + (u_v - (i + 2)) \\ \begin{pmatrix} \bigvee_{r_o \in \mathcal{R}} \bigvee_{j < i-1} \begin{bmatrix} \bigvee_{\substack{1 \leq v \leq k \\ r(\ell_v) \in R[*,j]}} \forall ar_{q_o^{r_o}}(\ell_v) = q_o^{r_o} \land f(\ell_v) = \mathbf{j} \\ \forall \bigvee_{r_o \in \mathcal{R}} \bigvee_{i < v \leq m} \forall ar_{q_o^{r_o}}(\ell_v) = q_o^{r_o} \land f(\ell_v) = \mathbf{j} \end{bmatrix} \\ \land \qquad \forall \exists \mathbf{j} . i - 1 \leq \mathbf{j} \leq \mathbf{y} \land \bigvee_{r_o \in \mathcal{R}} \begin{bmatrix} \bigvee_{\substack{1 \leq v \leq m \\ r(\ell_v) \in R[*,i]}} \forall ar_{q_o^{r_o}}(\ell_v) = q_o^{r_o} \land f(\ell_v) = \mathbf{j} \\ \forall \bigvee_{r_o \in \mathcal{R}} \bigvee_{i < v < r_{q_o^{r_o}}(\ell_v) = q_o^{r_o} \land f(\ell_v) = \mathbf{j} \end{bmatrix} \\ \bigvee_{\substack{1 \leq v \leq m \\ r(\ell_v) \in R[*,i]}} \forall ar_{q_o^{r_o}}(\ell_v) = q_o^{r_o} \land f(\ell_v) = \mathbf{j} + (j - (i + 2)) \\ \forall \bigvee_{\substack{1 \leq v \leq m \\ r(\ell_v) \in R[*,j]}} \forall ar_{q_o^{r_o}}(u_v) = q_o^{r_o} \land f(u_v) = \mathbf{y} + (j - (i + 2)) \end{bmatrix} \end{bmatrix}$$

Note the conversion of relationset variables to the logical representation of a single function symbol. This conversion is driven by the observation that viable markings M ensure that  $M(\langle q_o, i \rangle) + M(\langle r_o, i, \bot \rangle) = 1$  for  $q_o$  and  $r_o$  occurring in the same loop transition pattern. To this end, we use that  $M(\langle q_o, i \rangle) = 1$  for some  $q_o$  necessarily implies  $M(\langle q'_o, i \rangle) = 0$  for all other states attached to some loop transition pattern since the state values form a 1BB-set for every agent. This, in turn, ensures that  $M(\langle r_o^{q'_o}, i, \bot \rangle) = 1$  and, by the appropriate 1BB-cover,  $M(\langle r_o^{q'_o}, i, j \rangle) = 0$  for all  $j \in [n]$ .