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On Finding Hamiltonian Cycles in Barnette Graphs

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Abstract. In this paper we deal with hamiltonicity in planar cubic graphs G having a facial $2-$factor $\mathcal{Q}$ via (quasi) spanning trees of faces in $G/\mathcal{Q}$ and study the algorithmic complexity of finding such (quasi) spanning trees of faces. Moreover, we show that if Barnette's Conjecture is false, then hamiltonicity in $3-$connected planar cubic bipartite graphs is an NP-complete problem.

Keywords: Barnette's Conjecture; eulerian plane graph; hamiltonian cycle; spanning tree of faces; $A-$trail.

1. Introduction

Our joint paper [1] can be considered as the point of departure for the subsequent discussion and results of this paper. We start with a few historical remarks. In 1884, Tait conjectured that every cubic

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$3-$connected planar graph is hamiltonian [2]. And Tait knew that the validity of his conjecture would yield a simple proof of the Four Color Conjecture. On the other hand, the Petersen graph is the smallest non-planar $3-$connected cubic graph which is not hamiltonian, [3]. Tait's Conjecture was disproved by Tutte in 1946, [4]. However, none of the known counterexamples of Tait's Conjecture is bipartite. Tutte himself conjectured that every cubic $3-$connected bipartite graph is hamiltonian [5], but this was shown to be false by the construction of a counterexample, the Horton graph [6]. Barnette proposed a combination of Tait's and Tutte's Conjectures implying that every counterexample to Tait's conjecture is non-bipartite. However, it is a well-known fact that hamiltonicity in planar cubic graphs is an NP-complete problem. This implies that the existence of an $A-$trail in plane eulerian graphs is also an NP-complete problem even if restricted to planar $3-$connected eulerian graphs (see Definition 2.1.(i) below).

Barnette's Conjecture [7] *Every* $3-$*connected cubic planar bipartite graph is hamiltonian.*

We denote $3-$connected cubic planar bipartite graphs as Barnette graphs. Holton, Manvel and McKay showed in [8] that Barnette graphs with up to 64 vertices are hamiltonian. The conjecture also holds for the infinite family of Barnette graphs where all faces are either quadrilaterals or hexagons, as shown by Goodey [9]. However, it is NP-complete to decide whether a $2-$connected cubic planar bipartite graph is hamiltonian [10].

We note that [11] can essentially be viewed in part as a preliminary version of [1] and the current paper, with [1] focusing on graph theoretical results in [11], whereas the current paper's focus are mainly algorithmic and complexity considerations as developed in [11]. However, additional results were developed to put [11] in a more general perspective.

We outline the structure of this paper as follows.

Section 2 of this paper starts with listing several known results from [1] and other papers, and several definitions; they can be viewed as the basis for this paper.

In Section 3 we first prove some structural results leading to a decision in polynomial time whether a Barnette graph with certain properties has a hamiltonian cycle of a special type (Corollary 3.3).

In Section 4 we first prove NP-completeness of the existence of certain types of spanning trees of faces (see Definition 2.3), and subsequent corollaries. Finally it is shown that if Barnette's Conjecture is false, then hamiltonicity in Barnette graphs is an NP-complete problem.

2. Preliminary discussion

As for the terminology used in this paper we follow [12] unless stated explicitly otherwise. In particular, the subset $E(v) \subseteq E(G)$ denotes the set of edges incident to $v \in V(G)$. For definitions we refer to [1] but for completeness' sake we repeat some of them. Moreover, we present several known results (Theorems A-E) which can be viewed as the frame in which the results of this paper take place.

Definition 2.1.

(i) Let H be a $2-$connected eulerian plane graph. An eulerian trail L in H is an $A-$trail if any two consecutive edges of L belong to a face boundary.

(ii) An $A-$trail L in an eulerian triangulation of the plane is called non-separating if for every face boundary T at least two edges of $E(T)$ are consecutive in L.

(iii) An $A-$partition of H is a vertex partition $V_L(H) = \{V_1, V_2\}$ induced by an $A-$trail $L = e_1 e_2 \dots e_m$ as follows. Consider a $2-$face-coloring of H with colors 1 and 2. For every vertex v of H, $v \in V_i$ if and only if there is $j \in \{1, \dots, m-1\}$ such that $v \in V(e_j) \cap V(e_{j+1})$ and the face containing e_j and e_{j+1} in its boundary is colored $3 - i$, $i = 1, 2$, where $V(e)$ is the set of vertices incident to the edge e.

There is a close conection between hamiltonian cycles in Barnette graphs and A-trails in their dual.

Theorem A. ([13, Theorem VI.71]) A $2-$connected plane cubic bipartite graph has a hamiltonian cycle if and only if its dual graph has a non-separating $A-$trail.

Definition 2.2.

(i) Suppose H is a $2-$connected plane graph. Let $\mathcal{F}(H)$ be the set of faces of H. The radial graph of H denoted by $\mathcal{R}(H)$ is a bipartite graph with the vertex bipartition $\{V(H), \mathcal{F}(H)\}$ such that $xf \in E(\mathcal{R}(H))$ if and only if x is a vertex in the boundary of $F \in \mathcal{F}(H)$ corresponding to $f \in V(\mathcal{R}(H))$.

(ii) Let $U \subseteq V(H)$ and let $\mathcal{T} \subset \mathcal{F}(H)$ be a set of bounded faces of H. The restricted radial graph $\mathcal{R}(U, \mathcal{T}) \subset \mathcal{R}(H)$ is defined by $\mathcal{R}(U, \mathcal{T}) = \langle U \cup \mathcal{T} \rangle_{\mathcal{R}(H)}$.

For the next definition let H be a $2-$connected plane graph, let $U \subseteq V(H)$ and let $\mathcal{T} \subset \mathcal{F}(H)$ be a set of bounded faces whose boundaries are pairwise edge-disjoint and such that every vertex of H is contained in some element of $\mathcal{T}$. We define a subgraph $H_{\mathcal{T}}$ of H by $H_{\mathcal{T}} = H[\cup_{F \in \mathcal{T}} E(F)]$.

Definition 2.3. If $\left|\{F \in \mathcal{T} \,:\, x \in V(F)\}\right| = \frac{1}{2}\deg_H(x)$ for every $x \in V(H) \setminus U$, and if $\mathcal{R}(U, \mathcal{T})$ is a tree, then we call $H_{\mathcal{T}}$ a quasi spanning tree of faces of H, and the vertices in U $(V(H) \setminus U)$ are called proper (quasi) vertices. If $U = V(H)$, then $H_{\mathcal{T}}$ is called a spanning tree of faces.

In other words, a spanning tree of faces is a spanning bridgeless cactus whose cycles are face boundaries.

Definition 2.4. The leapfrog extension $Lf(G)$ of a $2-$connected cubic plane graph G is the cubic plane graph obtained from G by replacing every $v \in V(G)$ by a hexagon $C_6(v)$, with $C_6(v)$ and $C_6(w)$ sharing an edge if and only if $vw \in E(G)$; and these hexagons are faces of $Lf(G)$.

We note in passing that we call leapfrog extension what is called in other papers vertex envelope, or leapfrog construction, or leapfrog operation, or leapfrog transformation (see e.g. [14, 15, 16, 17]).

If a plane graph has a face-coloring with color set X, the faces of color $x \in X$ will be called $x-$faces.

Observation 1. We observe that if H is a plane eulerian graph with $\delta(H) \geq 4$ having an $A-$trail L, then L defines uniquely a quasi spanning tree of faces. Conversely, a (quasi) spanning tree of faces $H_{\mathcal{T}}$ defines uniquely an $A-$trail in $H_{\mathcal{T}}$ (where $\mathcal{T}$ is defined as in the paragraph preceding Definition 2.3).

Proof:
Start with a $2-$face-coloring of H with colors 1 and 2, suppose the outer face of H is colored 1. To show the first part of Observation 1, let $V_L(H) = \{V_1, V_2\}$ be the $A-$partition of $V(H)$ induced by L (see Definition 2.1.(iii)). Now, the set of all $2-$faces defines a quasi spanning tree of faces $H_{\mathcal{T}}$ with V_1 being the set of all quasi vertices of $H_{\mathcal{T}}$. The second part of Observation 1 follows similarly. □

The aforementioned relation between the concepts of $A-$trail and (quasi) spanning tree of faces is not a coincidence. In fact, it had been shown ([13, pp. $VI.112 - VI.113$] – see also Theorem A) that

• *Barnette's Conjecture is true if and only if every simple $3-$connected eulerian triangulation of the plane admits an $A-$trail.*

We point out, however, that the concept of (quasi) spanning tree of faces is a somewhat more general tool to deal with hamiltonian cycles in plane cubic graphs, than the concept of $A-$trails. We are thus focusing our considerations below on the complexity of the existence of $A-$trails and (quasi) spanning trees of faces in plane (eulerian) graphs.

Parts of this paper are the result of extracting some results and their proofs of [11]; they have not been published in a refereed journal yet. Moreover, we relate some of the results of this paper to the theory of $A-$trails, as developed in [13].

Next we list some results of the preceding joint paper [1] which will be essential for the current paper. In general, when we say that F is an $\mathcal{X}-$face ($\mathcal{X}^c-$face), we mean that $F \in \mathcal{X}$ ($F \notin \mathcal{X}$).

Consider a $3-$connected cubic plane graph G with a facial $2-$factor $\mathcal{Q}$. Given a quasi spanning tree of faces $H_{\mathcal{T}}$ in the reduced graph $H = G/\mathcal{Q}$, we assume the outer face is not in $\mathcal{T}$, and traverse the $A-$trail of $H_{\mathcal{T}}$ (see the second part of Observation 1), to obtain a hamiltonian cycle C in G such that the faces of $\mathcal{Q}$ corresponding to the proper vertices of $H_{\mathcal{T}}$ lie inside of C whereas the faces of $\mathcal{Q}$ corresponding to quasi vertices of $H_{\mathcal{T}}$ lie outside of C. In fact the following is true.

Theorem B. ([1, Proposition 1]) Let G be a $3-$connected cubic plane graph G with a facial $2-$factor $\mathcal{Q}$. Then, the reduced graph $H = G/\mathcal{Q}$ has a quasi spanning tree of faces, $H_{\mathcal{T}}$, with the outer face not in $\mathcal{T}$, if and only if G has a hamiltonian cycle C with the outer $\mathcal{Q}^c-$face outside of C, with all $\mathcal{Q}-$faces corresponding to proper vertices of $H_{\mathcal{T}}$ inside of C, with all $\mathcal{Q}-$faces corresponding to quasi vertices of $H_{\mathcal{T}}$ outside of C, and such that no two $\mathcal{Q}^c-$faces sharing an edge are both inside of C.

Theorem C. ([1, Corollary 7]) Every simple $4-$connected eulerian triangulation of the plane has a quasi spanning tree of faces.

Note, however, that it is an open problem whether such triangulations have an A–trail.

Additionally, two old results are listed below.

Theorem D. ([18, Corollary 3]) The problem of deciding whether a planar eulerian graph admits an $A-$trail remains NP-complete for $3-$connected graphs having only $3-$cycles and $4-$cycles as face boundaries, and for which all faces with $4-$cycles as boundaries have the same color in the $2-$face-coloring.

In contrast, Andersen et al. in [19] gave a polynomial algorithm for finding $A-$trails in simple $2-$connected outerplane eulerian graphs.

Theorem E. ([14, Theorem 23]) The decision problem of whether the leapfrog extension of a plane cubic graph with multiple edges is hamiltonian is NP-complete.

3. Graph theoretical results and polynomially solvable problems

In proving Propositions 3 and 6 and Theorem 11 in [1], we used implicitly some algorithms to construct a (quasi) spanning tree of faces. In all of them, we find some triangular face such that the graph resulting from the contraction of this face, still satisfies the hypothesis of the respective result. By repeating this process, finally the contracted faces together with a special face form a (quasi) spanning tree of faces. Note that it is possible to identify the contractible faces in linear time, since every simple plane graph has $O(n)$ faces, where n is the order of a graph under consideration. Therefore, our algorithms for finding a quasi spanning tree of faces in [1] are polynomial.

We show next that one can decide in polynomial time whether the reduced graph H with a set $\mathcal{D}$ of edge-disjoint faces in H has a spanning tree of faces in $\mathcal{D}$ provided every face boundary of H is a digon or a triangle.

The Spanning Tree Parity Problem: *Given a graph G and a collection of disjoint pairs of edges, $\{\{e_i, f_i\} \mid i = 1, \ldots, k\}$.* The Spanning Tree Parity Problem (STPP) *asks whether G has a spanning tree T satisfying $|\{e_i, f_i\} \cap E(T)| \in \{0, 2\}$, for each $i = 1, \ldots, k$.*

Note that the STPP is solvable in polynomial time (see [20, 21]).

Theorem 3.1. Let G be a $3-$connected cubic plane graph having a facial $2-$factor $\mathcal{Q}$ and $H = G/\mathcal{Q}$. Let $\mathcal{D}$ be a set of edge-disjoint faces in H such that $\mathcal{D}$ covers all of $V(H)$ and such that all faces in $\mathcal{D}$ are either digons or triangles. Then we can decide in polynomial time whether H has a spanning tree of faces in $\mathcal{D}$, yielding a hamiltonian cycle for G, by a spanning tree parity algorithm.

Proof:
Construct a graph $H^{'}$ related to H as follows. $V(H^{'}) = V(H)$. If xyx is a digon in $\mathcal{D}$, then let xy be an edge in $H^{'}$. If $xyzx$ is a triangle in $\mathcal{D}$, then put edges xy and yz in $H^{'}$ (the naming of the vertices of the triangle with the symbols x, y, z is arbitrary but fixed). A spanning tree of faces in $\mathcal{D}$ for H then corresponds to a spanning tree T in $H^{'}$ satisfying $|\{xy, yz\} \cap E(T)| \in \{0, 2\}$, for each triangle $xyzx$ in $\mathcal{D}$. Thus, these conditions on pairs of edges in $H^{'}$ transform the problem of finding a spanning tree of faces in $\mathcal{D}$ for H, yielding a hamiltonian cycle for G by Theorem B, equivalently in polynomial time into an STPP in $H^{'}$. □

If $\mathcal{D}$ contains faces with four or more sides, say a face $xyztx$, then we could include three edges linking these four vertices, say $xy,\ yz$, and zt, and require that a spanning tree must contain either all three or none of these three edges. Such a Spanning Tree Triarity Problem (STTP), as we shall see later in Theorem 4.2, turns out to be NP-complete.

The following result demonstrates the close relationship between $A-$trails and spanning trees of faces vis-a-vis hamiltonian cycles.

Theorem 3.2. Let G be a Barnette graph whose faces are $3-$colored with color set $\{1, 2, 3\}$ and suppose without loss of generality that the outer face of G is a $3-$face. The following statements are equivalent.

(i) G has a hamiltonian cycle C with the $2-$faces lying inside of C, the $3-$faces lying outside of C, and $1-$faces on either side;

(ii) the reduced graph H obtained by contracting the $1-$faces has an $A-$trail;

(iii) the reduced graph $H^{'}$ obtained by contracting the $2-$faces has a spanning tree of $1-$faces;

(iv) the reduced graph $H^{''}$ obtained by contracting the $3-$faces has a spanning tree of $1-$faces.

Proof:
$(i) \Rightarrow (ii)$: Let T_C be a closed trail in H induced by hamiltonian cycle C of G having the properties described in (i). T_C is an eulerian trail, otherwise there are two faces of G with two different colors 2 and 3 lying on one side of C. This obvious contradiction to (i) guaranties that T_C is an eulerian trail. Since all $2-$faces ($3-$faces) of G are lying inside (outside) of C, for every $1-$face F_1 of G we conclude that $E(F_1) \cap E(C)$ is a matching. Therefore, every pair of consecutive edges of T_C corresponds to a path of length three in C such that the central edge of this path from one side belongs to a $1-$face boundary, and from the other side all three edges belong to an $i-$face boundary, $i \in \{2, 3\}$. Thus, any two consecutive edges of T_C belong to a face boundary in H and so T_C is an $A-$trail.

$(ii) \Rightarrow (i)$: The $3-$face-coloring of G induces a $2-$face-coloring in H using colors 2 and 3 and such that the outer face of H is a $3-$face. Now it is easy to see that any $A-$trail of H can be transformed into a hamiltonian cycle C of G with the $2-$faces lying inside of C, the $3-$faces lying outside of C, and $1-$faces lying on either side.

$(i) \Rightarrow (iii)$: Now consider the $2-$face-coloring of $H^{'}$ induced by the $3-$face-coloring of G. Let $U = V(H^{'})$ be the vertex set corresponding to the $2-$faces. Also, let $\mathcal{T}$ be the set of $1-$faces of $H^{'}$ corresponding to the $1-$faces in $int(C)$.

Observe that $G_{int} := C \cup int(C)$ is a spanning outerplane subgraph of G, and that the weak dual (the subgraph of the dual graph whose vertices correspond to the bounded faces) of G_{int} is a tree (see [13]). Therefore, $H^{'}_{int} \subset H^{'}$ being the reduced graph of G_{int} after contracting the $2-$faces, is a spanning tree of faces in $H^{'}$.

$(iii) \Rightarrow (i)$: Suppose $H^{'}$ has a spanning tree of $1-$faces $H^{'}_{\mathcal{T}}$. Then $H^{'}_{\mathcal{T}}$ has a unique $A-$trail which can be transformed into a hamiltonian cycle C of G such that the $2-$faces (corresponding to $V(H^{'})$) lie in $int(C)$ and the corresponding $3-$faces lie in $ext(C)$.

The equivalence of (i) and (iv) is established analogously by looking at $G_{ext} := C \cup ext(C)$ which is also an outerplanar graph. □

An application of Theorems 3.1 and 3.2 yields the following.

Corollary 3.3. Let G be a Barnette graph with a $3-$face-coloring with color set $\{1, 2, 3\}$, and let H be the reduced graph obtained by contracting the $1-$faces. Suppose all vertices of H have degree 4 or 6. Then one can decide in polynomial time whether H has an $A-$trail which in turn yields a hamiltonian cycle in G.

Proof:
Let H be the reduced graph of G obtained by contracting the $1-$faces, and let $H^{'}$ be the reduced graph of G obtained by contracting the $2-$faces instead of the $1-$faces. Then each $1-$face of G yields a digon or triangle in $H^{'}$. By Theorem 3.2, an $A-$trail in H corresponds to a spanning tree of $1-$faces in $H^{'}$ and vice versa. Since all $1-$faces of $H^{'}$ are either digons or triangles, one can decide in polynomial time by Theorem 3.1 whether such a spanning tree of $1-$faces exists in $H^{'}$. □

By Observation 1, we have the following theorem in which we make use of the fact that an (eulerian) triangulation of the plane admits two interpretations, namely: as the dual of a plane cubic (bipartite) graph, and as the contraction of a facial (even) $2-$factor Q in G whose faces in Q^c are hexagons.

Theorem 3.4. Let G be a Barnette graph and let $\mathcal{F}$ be the set of its faces. Let $\mathcal{Q}_{\mathcal{F}}$ be the facial $2-$factor of $Lf(G)$ corresponding to $\mathcal{F}$ and let the color classes of the $3-$face-coloring of $Lf(G)$ be denoted by F_1, F_2, and F_3 such that $F_3 = \mathcal{Q}_{\mathcal{F}}$, and thus F_1, F_2 translates into a $2-$face-coloring of $Lf(G)/\mathcal{Q}_{\mathcal{F}}$ denoting the corresponding sets of faces by $F1$, $F2$ and whose vertex set (corresponding to F_3) be denoted by V_3. G^* denotes the dual of G. Then the following is true.

(1) $G^* = Lf(G)/\mathcal{Q}_{\mathcal{F}}$.

(2) G is hamiltonian if and only if $Lf(G)$ has a hamiltonian cycle C such that $int(C) = F_1 \cup F_3^{'}$ and $ext(C) = F_2 \cup F_3^{''}$ where $F_3 = F_3^{'} \dot{\cup} F_3^{''}$.

Statement (2) is equivalent to

(3) G^* has a non-separating $A-$trail if and only if there is a partition $V_3 = V_3^{'} \dot{\cup} V_3^{''}$ such that $Lf(G)/\mathcal{Q}_{\mathcal{F}}$ has a quasi spanning tree of faces containing all of $F1$ and where $V_3^{'}$ is its set of proper vertices and $V_3^{''}$ is its set of quasi vertices. ($V_3^{'}$ and $V_3^{''}$ are the vertex sets in $Lf(G)/\mathcal{Q}_{\mathcal{F}}$ corresponding to $F_3^{'}$ and $F_3^{''}$, respectively, - see (2) above).

Proof:
By Definition 2.4 and definition of the dual graph of a plane graph, statement (1) is true.

Next we show that (2) is true. Assume G has a hamiltonian cycle $C_0 = e_1 e_2 \dots e_n$ such that $e_i = v_i v_{i+1}$ for $i = 1, \dots, n-1, e_n = v_n v_1$. Let $e = v_i v_j \in E(G)$ be the edge corresponding to $e^{'} \in E(C_6(v_i)) \cap E(C_6(v_j)) \subset E(Lf(G))$, for $1 \leq i \neq j \leq n$ (see Definition 2.4 concerning $C_6(v_i)$).

Now we construct a hamiltonian cycle C in $Lf(G)$ corresponding to C_0 as follows. Consider $C^\circ = \{e_i^{'} \mid i = 1, \ldots, n\}$. If $C_6(v_{i+1}) \in F_1$ ($C_6(v_{i+1}) \in F_2$) where $\{e_i^{'}, e_{i+1}^{'}\} \subset E(C_6(v_{i+1}))$, add the path in $C_6(v_{i+1})$ connecting the endvertices of $e_i^{'}$ and $e_{i+1}^{'}$ outside (inside, respectively) C_0 to C°, for $i = 1, \ldots, n-1$. Then add the path in $C_6(v_1)$ connecting the endvertices of $e_1^{'}$ and $e_n^{'}$ inside (outside, respectively) C_0 to C°; call the final set thus constructed C. By the construction of C, $int(C) = F_1 \cup F_3^{'}$ and $ext(C) = F_2 \cup F_3^{''}$ where $F_3 = F_3^{'} \dot\cup F_3^{''}$. Since every vertex $v \in V(Lf(G))$ is incident to an adge e which belongs to an $i-$face boundary, $i \in \{1, 2\}$, by $F_1 \subset int(C)$ and $F_2 \subset ext(C)$, we have C traverses the edge e and then $v \in V(C)$. Therefore, C is hamiltonian. Conversely, it is straightforward to see that a hamiltonian cycle in $Lf(G)$ as described yields a hamiltonian cycle in G. Thus, (2) is true.

Theorem A emplies that (2) is equivalent to the left side of (3).

And finally we show that (2) is equivalent to the right side of (3). Again consider a hamiltonian cycle in G. By (2), there is a hamiltonian cycle C in $Lf(G)$ such that $int(C) = F_1 \cup F_3^{'}$ and $ext(C) = F_2 \cup F_3^{''}$ where $F_3 = F_3^{'} \dot\cup F_3^{''}$. Now, let $V_3^{'}$ be the set of vertices in $Lf(G)/\mathcal{Q}_{\mathcal{F}}$ corresponding to $F_3^{'}$. Then, it can be seen easily that $\mathcal{R}(V_3^{'}, F1)$ is a tree; and therefore, $Lf(G)/\mathcal{Q}_{\mathcal{F}}$ has a quasi spanning tree of faces containing all of $F1$ and where $V_3^{'}$ is its set of proper vertices. Thus, (3) is true. The converse is true by Theorem 3.2. □

Theorem 3.4 puts hamiltonicity in G in a qualitative perspective of the algorithmic complexity regarding quasi spanning trees of faces of a special type in the reduced graph of the leapfrog extension of G. In fact, if $\mathcal{G}$ is a class of Barnette graphs where hamiltonicity can be decided in polynomial time, then the same can be said regarding special types of quasi spanning trees of faces in the reduced graphs of the leapfrog extensions of the elements of $\mathcal{G}$ (as stated in the theorem). For, given a hamiltonian cycle C_0 in $G \in \mathcal{G}$, a non-separating $A-$trail L_{C_0} in G^* can be found in polynomial time which in turn yields a quasi spanning tree of faces in $Lf(G)/\mathcal{Q}_{\mathcal{F}}$ as described in (3), also in polynomial time. Compare this with Theorem E and Theorem C.

The following proposition shows that if Barnette's Conjecture is false then there is a particular edge e in some hamiltonian Barnette graph such that every hamiltonian cycle of that graph contains e.

Proposition 3.5. If there exists a non-hamiltonian Barnette graph, then there exists a hamiltonian Barnette graph G_1 with a particular edge $e = uv$ such that $e \in E(C)$ for every hamiltonian cycle C of G_1. Furthermore, if e_1 and e_2 are the two edges other than e incident to u in G_1, then G_1 has a hamiltonian cycle C_i traversing e and e_i, for $i = 1, 2$.

Proof:
Suppose G_0 is a smallest counterexample to Barnette's Conjecture.

First we construct a hamiltonian Barnette graph G_1 with a particular edge $e_0 = u_0v_0$ such that $e_0 \in E(C)$ for every hamiltonian cycle C of G_1.

Let $Q = wxyzw$ be a facial quadrilateral in G_0 and let a_1 be the third neighbour of a in G_0, for $a \in \{w, x, y, z\}$.

Set $G_0^{'} = (G_0 \setminus \{w, x, y, z\}) \cup \{w_1x_1, y_1z_1\}$ and $G_0^{''} = (G_0 \setminus \{w, x, y, z\}) \cup \{w_1z_1, x_1y_1\}$. Both $G_0^{'}$ and $G_0^{''}$ are planar, cubic and bipartite.

Suppose that $G_0^{'}$ is $3-$connected. By minimality of G_0, the graph $G_0^{'}$ has a hamiltonian cycle. Furthermore, no hamiltonian cycle of $G_0^{'}$ goes through either the edge w_1x_1 or the edge y_1z_1; otherwise, we can extend this cycle to a hamiltonian cycle in G_0, a contradiction.

We have thus guaranteed that no hamiltonian cycle in $G_1 = G_0^{'}$ traverses a particular edge w_1x_1, and thus every hamiltonian cycle traverses an edge e_0 adjacent to w_1x_1, as desired. The same conclusions can be drawn if $G_0^{''}$ is $3-$connected.

Suppose now that $G_0^{'}$ and $G_0^{''}$ are both $2-$connected only. Then there are two edge cuts of size four, T_1 and T_2, in G_0 such that $\{wx, yz\} \subset T_1$ and $\{wz, xy\} \subset T_2$.

Removing the vertices w, x, y, z and the remainder of the two edge cuts T_1 and T_2 separates G_0 into four components R_1, R_2, R_3, R_4, with the removed edges of G_0 including an edge from R_i to R_{i+1}, for $i = 1, 2, 3$, and an edge from R_4 to R_1, plus the four edges from the four R_i's incident to a vertex of Q. That is, each R_i is incident to three edges whose endvertices not in R_i can be identified to a single vertex r_i to obtain a bipartite $R_i^{'}$, since their three endvertices in R_i are at even distance from each other. For, in the $2-$vertex-coloring of G_0, the three vertices of degree 2 of $R_i,\ 1 \leq i \leq 4$, must have the same color; otherwise, two copies of such R_i could be used to construct a cubic bipartite graph having a bridge. Clearly, $R_i^{'}$ is $3-$connected, cubic, planar, and bipartite, for each $i = 1, \ldots, 4$.

By minimality of G_0 each such $R_i^{'}$ has a hamiltonian cycle, yet it is not the case that each of the three choices of two edges incident to r_i yields a hamiltonian cycle, since otherwise we could obtain a hamiltonian cycle for G_0. Thus one of the three edges incident to r_i in $R_i^{'}$ must belong to every hamiltonian cycle, thus yielding a Barnette graph $G_1 = R_1^{'}$ with an edge $e_0 = u_0v_0$ that belongs to every hamiltonian cycle of G_1.

If G_1 has a hamiltonian cycle $C_i^{'}$ traversing e_0 and $e_i^{'} = u_0v_i^{'}$, for $i = 1, 2$, then let $e = e_0$, $C_i = C_i^{'}$, and $e_i = e_i^{'}$, for $i = 1, 2$. This would complete the proof of Proposition 3.5. Thus, suppose instead that every hamiltonian cycle in G_1 is forced to traverse $e_1^{'} = u_0v_1^{'} \in E(G_1)$ as well.

Let $C_0 = f_0, f_1, \ldots, f_{n-1}$ where $f_0 = e_0$ and $f_1 = e_1^{'}$ be a fixed hamiltonian cycle of G_1 and let k to be the largest index such that the section $f_0, \ldots, f_{k-1}$ belongs to every hamiltonian cycle of G_1, whereas f_k does not belong to every hamiltonian cycle of G_1. Such k must exist; otherwise G_1 would be a uniquely hamiltonian graph which is impossible since G_1 is cubic. Denote $f_{k-1} = e = uv$ such that f_k is incident to u. Set $e_1 = f_k$, and let e_2 be the third edge incident to u. Now, there must be a hamiltonian cycle C_1 other than C_0 not containing e_1 since $e_1 = f_k$ does not belong to all hamiltonian cycles of G_1. Thus C_1 traverses e and e_2. This finishes the proof of Proposition 3.5. □

4. NP-complete problems

We now establish several NP-completeness results.

In the proof of Theorem D, one may assume without loss of generality that the outer face and all quadrilaterals have color 2. Then by Observation 1, we have the following corollary.

Corollary 4.1. Let G be a Barnette graph with a $3-$face-coloring with color set $\{1, 2, 3\}$. Assume the outer face of G is colored 2 and H is the reduced graph obtained by contracting the $1-$faces and equipped with a $2-$face-coloring. Suppose that H has only triangles and quadrilaterals as face boundaries, and for which all quadrilaterals have color 2 in the $2-$face-coloring. Then the decision problem of whether H has a quasi spanning tree defined by the set of all (triangular) $3-$faces is NP-complete.

In Theorem 4.2, we give a similar result concerning the reduced graph H containing only digons and quadrilaterals. For such a graph, we are trying to find a spanning tree of quadrilaterals.

The decision problem of whether a $3-$connected planar cubic graph G_0 has a hamiltonian cycle is NP-complete, as shown by Garey et al. [22]. Let $e = uv \in E(G_0)$. Then the decision problem of whether G_0 has a hamiltonian cycle traversing this specified edge e, is also NP-complete. Let $G_0^{'} = G_0 \setminus \{e\}$. Thus, the decision problem of whether $G_0^{'}$ has a hamiltonian path from u to v is also NP-complete.

Theorem 4.2. Let G be a Barnette graph. Let c_f be a $3-$face-coloring of G with color set $\{1, 2, 3\}$, and let H be the reduced graph obtained by contracting the $1-$faces and equipped with a $2-$face-coloring induced by c_f. Suppose that the $2-$faces in H are quadrilaterals and the $3-$faces in H are digons. Then the decision problem of whether H has a spanning tree of $2-$faces is NP-complete.

Proof:
We want to construct G and H as stated in the theorem. To this end, consider G_0 and $G_0^{'}$ as described in the paragraph preceding the statement of Theorem 4.2 and assume $G_0^{'}$ is the plane graph resulting by edge deletion from a fixed imbedding of G_0. Let H be the plane graph resulting by replacing every edge of the radial graph $\mathcal{R}(G_0^{'})$ with a digon; H is eulerian and hence $2-$face-colorable. First color the digons corresponding to edges in $\mathcal{R}(G_0^{'})$ with color 3. The remaining faces of H are quadrilaterals $Q = xfx^{'}f^{'}x$ corresponding to $xx^{'} \in E(G_0^{'}) \cap bd(F) \cap bd(F^{'})$ with F and $F^{'}$ in $G_0^{'}$ corresponding to $f, f^{'} \in V(\mathcal{R}(G_0^{'}))$. Color these quadrilaterals with color 2. Let G be the plane cubic graph obtained from H by replacing each $w \in V(H)$ with a cycle $C_w = w_1 \dots w_{\deg_H(w)} w_1$ and replacing $e_i = u_i w \in E(H)$ with $e_i^{'} = u_i w_i$, for $i = 1, \dots, deg_H(w)$. We have $\kappa(H) \geq 2$, but $\kappa^{'}(H) > 2$. Therefore, G is $3-$connected and thus a Barnette graph whose $3-$face-coloring has color set $\{1, 2, 3\}$; the $1-$faces of G correspond to the vertices of H.

Claim 4.3. A set L of edges in $G_0^{'}$ forms a hamiltonian path from u to v in $G_0^{'}$ if and only if $H_{\mathcal{T}}$ is a spanning tree of $2-$faces in H where $\mathcal{T}$ is the set of $2-$faces (which are quadrilaterals) in H corresponding to the edges in $E(G_0^{'}) \setminus L$.

Suppose L is a hamiltonian path from u to v in $G_0^{'}$. Let $L^{'} = E(G_0^{'}) \setminus L$ (which is a perfect matching in both G_0 and $G_0^{'}$), and let $\mathcal{T}$ be the set of all quadrilaterals in H corresponding to $L^{'}$. Note that since L is a hamiltonian path, for any two edges $g, h \in L^{'}$, there is a sequence of edges $g = \ell_1, \ell_2, \dots, \ell_k = h$ in $L^{'}$ such that each pair of edges ℓ_i, ℓ_{i+1} belongs to a face boundary in $G_0^{'}$, $1 \leq i \leq k-1$. Therefore the $2-$faces in $\mathcal{T}$ induce a connected subgraph of H. Notice also that every vertex in H belongs to some face in $\mathcal{T}$ since every vertex $x \in V(G_0^{'})$ is incident to an edge in $L^{'}$, and every face F in $G_0^{'}$ contains at least one edge of $L^{'}$ in its boundary.

Finally, there is no cycle of $2-$faces in $H_{\mathcal{T}}$. Suppose to the contrary, we had a cycle $Q_1 Q_2 \dots Q_r Q_1$ of $2-$faces in $H_{\mathcal{T}}$. Since the number of $2-$faces in $H_{\mathcal{T}}$ containing x is equal to $\deg_{G_0^{'}}(x) - \deg_L(x) = 1$, for every vertex $x \in V(G_0^{'})$, so Q_i and Q_{i+1} share a vertex $f \in V(H)$ corresponding to a face

$F \in \mathcal{F}(G_0^{'})$. Thus $\{q_1, q_2, \ldots, q_r\} \subset L^{'}$, with q_i corresponding to the face Q_i in the cycle of 2−faces in $H_{\mathcal{T}}$, separates the graph $G_0^{'}$ into at least two components; so the hamiltonian path L would have to contain at least one of these edges $q_i \in L^{'}$, a contradiction. Therefore, $H_{\mathcal{T}}$ is a spanning tree of 2−faces for H.

Conversely, suppose $H_{\mathcal{T}}$ is a spanning tree of 2−faces for H. Let $L^{'}$ be the corresponding edges in $G_0^{'}$ (which appear as chords of the elements of $\mathcal{T}$ if one draws $G_0^{'}$ and H in the plane as described before); and let $L = E(G_0^{'}) \backslash L^{'}$. Each vertex $x \in V(G_0^{'})$ belongs to exactly one 2−face $Q = xfx^{'}f^{'}x$ in $\mathcal{T}$, since every other 2−face in $\mathcal{T}$ containing x also contains either f or $f^{'}$, and therefore these two 2−faces share two vertices joined by parallel edges and thus cannot both be in the spanning tree of faces $H_{\mathcal{T}}$. Therefore every vertex in $G_0^{'}$ is incident to exactly one edge in $L^{'}$, and so the two vertices u and v of degree 2 in $G_0^{'}$ are incident to exactly one edge in L, while the remaining vertices of degree 3 in $G_0^{'}$ are incident to exactly two edges in L. That is, L induces a path joining u and v in $G_0^{'}$ plus a possibly empty set of cycles in $G_0^{'}$, such that the path and the cycles are disjoint and cover all of $V(G_0^{'})$. We show that L cannot contain a cycle in $G_0^{'}$, and so L is just a hamiltonian path joining u to v.

Suppose L contains a cycle $C = h_1h_2 \ldots h_kh_1$ in $G_0^{'}$. Let F and $F^{'}$ be faces of $G_0^{'}$ inside and outside the cycle C, respectively, and let f and $f^{'}$ be the vertices in H corresponding to F and $F^{'}$, respectively. Since f and $f^{'}$ are vertices in the spanning tree $H_{\mathcal{T}}$ of 2−facces, there is a unique sequence of 2−faces $Q_1^*, Q_2^*, \ldots, Q_l^*$ in $\mathcal{T}$ such that Q_1^* contains f, Q_l^* contains $f^{'}$ and each pair Q_{i-1}^*, Q_i^* share a vertex f_i corresponding to a face in $G_0^{'}$, for $2 \leq i < l$. In particular, if we denote $f_1 = f$ and $f_l = f^{'}$, then for some pair f_i, f_{i+1}, for the corresponding face F_i in $G_0^{'}$ we must have $F_i \subseteq G_0^{'} \cap int(C)$ and for the corresponding face F_{i+1} in $G_0^{'}$ we must have $F_{i+1} \subseteq G_0^{'} \cap ext(C)$. This implies that the 2−face Q_i^* in $H_{\mathcal{T}}$ corresponds to one of the edges in L and not in $L^{'}$, a contradiction. This completes the proof of Claim 4.3.

Therefore by Claim 4.3, H has a spanning tree of 2−faces if and only if $G_0^{'}$ has a hamiltonian path from u to v, and so the decision problem of whether H has a spanning tree of 2−faces is NP-complete. □

We obtain two Corollaries from this result.

Corollary 4.4. Let G be a Barnette graph with a 3−face-coloring with color set $\{1, 2, 3\}$, and let $H^{'}$ be the reduced graph obtained by contracting the 1−faces. Suppose all vertices of $H^{'}$ have degree 8. Then the decision problem of whether $H^{'}$ has an A−trail is NP-complete.

Proof:
Consider the reduced graph H in the statement of Theorem 4.2 where all 2−faces in H are quadrilaterals, corresponding to a facial 2−factor of octagons in G. If we contract in G these octagonal 2−faces, we obtain an 8−regular reduced graph $H^{'}$. By Theorem 3.2, $H^{'}$ has an A−trail if and only if H has a spanning tree of 2−faces, and this problem is NP-complete by Theorem 4.2. □

Corollary 4.5. Let G be a Barnette graph with a 3−face-coloring with color set $\{1, 2, 3\}$, and let H_0 be the reduced graph obtained by contracting the 1−faces. Suppose that the 2−faces in H_0 are octagons and digons and the 3−faces in H_0 are triangles. Then the decision problem of whether H_0 has a spanning tree of faces is NP-complete.

Proof:
Start with G_0 and $G_0^{'}$ as at the beginning of the proof of Theorem 4.2, and construct the reduced graph H as in the proof of Theorem 4.2, with $2-$colored quadrilaterals and $3-$colored digons. If e and f are the two parallel edges of a $3-$colored digon, subdivide e with vertex w and subdivide f with vertex x. Join w and x by two parallel edges. The $3-$colored digon splits thus into two $3-$colored triangles and a $2-$colored digon, while the $2-$colored quadrilaterals become $2-$colored octagons, in the new reduced graph H_0.

Suppose H has a spanning tree of $2-$colored quadrilaterals $H_{\mathcal{T}}$. Select the corresponding $2-$colored octagons in H_0. For a $3-$colored digon consisting of two edges e and f in H, if one of the two $2-$colored quadrilaterals containing e or f is in $\mathcal{T}$, then select the $2-$colored digon joining the subdivision vertices w and x; if neither of the two $2-$colored quadrilaterals containing e or f is in $\mathcal{T}$, then select one of the two $3-$colored triangles containing w and x. The $2-$colored and $3-$colored faces in H_0 thus selected, involving $2-$colored octagons, $2-$colored digons, and $3-$colored triangles, form a spanning tree of faces in H_0.

Conversely, suppose H_0 has a spanning tree of faces $H_{0\mathcal{T}_0}$. Let $\mathcal{T}$ be the set of $2-$colored quadrilaterals in H such that the corresponding $2-$colored octagons are in $\mathcal{T}_0$. Note that for each digon in H, at most one of the corresponding two $3-$colored triangles and $2-$colored digon in H_0 can be in $\mathcal{T}_0$. Thus $H_{\mathcal{T}}$ is a spanning tree of $2-$colored (quadrilateral) faces.

Note that the reduction process between these two decision problems can be done in polynomial time, since every simple plane graph has $O(n)$ faces, where n is the order of graph.

Thus H_0 has a spanning tree of arbitrary faces if and only if H has a spanning tree of $2-$colored faces, and NP-completeness follows from Theorem 4.2. □

Finally we show that if Barnette's Conjecture is false, then it would be NP-complete to decide whether a Barnette graph is hamiltonian.

Theorem 4.6. Assume that Barnette's Conjecture is false. Then the decision problem of whether a Barnette graph has a hamiltonian cycle, is NP-complete.

Proof:
Takanori et al. [10] showed that the decision problem of whether a $2-$connected cubic planar bipartite graph R has a hamiltonian cycle is NP-complete.

If such an R has a $2-$edge-cut $\{e_1, e_2\}$ that separates R into two components $R^{'}$ and $R^{''}$, then their endpoints in either side are at odd distance (see the above argument), so we may instead join the two endpoints of e_1 and e_2 in $R^{'}$ and $R^{''}$, separately, and ask whether $R^{'}$ and $R^{''}$ both contain a hamiltonian cycle containing the added edge joining the endpoints of e_1 and e_2.

Repeating this decomposition process, we eventually reduce the decision problem of whether R has a hamiltonian cycle to the decision problem of whether various R_i's each contain a hamiltonian cycle going through certain prespecified edges, with each R_i being $3-$connected or the cubic multigraph on 2 vertices. Thus the decision problem of whether a Barnette graph $G^{'}$ has a hamiltonian cycle going through certain prespecified edges is NP-complete.

Let a Barnette graph $G^{'}$ with certain prespecified edges $e_1^{'}, \ldots, e_k^{'}$ that a hamiltonian cycle must traverse, be given. Denote $e_i^{'} = x_i y_{i,1}$ and $N_{G'}(x_i) = \{y_{i,1}, y_{i,2}, y_{i,3}\}$, for $i = 1, \ldots, k$.

Suppose that Barnette's Conjecture is false. Then by Proposition 3.5, there exists a hamiltonian Barnette graph G_i with a vertex $u_i \in V(G_i)$ and $N_{G_i}(u_i) = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ such that every hamiltonian cycle in G_i traverses $e_i = u_i v_{i,1} \in E(G_i)$, $i = 1 \ldots, k$. Furthermore, G_i has a hamiltonian cycle traversing e_i and $u_i v_{i,j}$, for $i = 1, \ldots, k$ and $j = 2, 3$.

Construct a new Barnette graph

$$G = \left(G^{'} \setminus \{x_1, \ldots, x_k\}\right) \cup \left(\bigcup_{i=1}^{k} (G_i \setminus \{u_i\})\right) \cup \left(\bigcup_{i=1}^{k} \{v_{i,1}y_{i,1}, v_{i,2}y_{i,2}, v_{i,3}y_{i,3}\}\right).$$

Since every hamiltonian cycle in G_i traverses the edge e_i and G_i has also a hamiltonian cycle traversing e_i and $u_i v_{i,j}$, for $i = 1, \ldots, k$ and $j = 2, 3$, the resulting graph G has a hamiltonian cycle if and only if $G^{'}$ has a hamiltonian cycle traversing the edges $e^{'}_1, \ldots, e^{'}_k$. Moreover, G can be constructed from $G^{'}$ in polynomial time and its vertex set is also polynomially enlarged from $G^{'}$. Therefore, the decision problem whether the resulting Barnette graph G has a hamiltonian cycle, is NP-complete. □

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