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A Note on Calculi for Non-deterministic Many-valued Logics

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Abstract. We present two deductively equivalent calculi for non-deterministic many-valued logics. One is defined by axioms and the other – by rules of inference. The two calculi are obtained from the truth tables of the logic under consideration in a straightforward manner. We prove soundness and strong completeness theorems for both calculi and also prove the cut elimination theorem for the calculi defined by rules of inference.

1. Introduction

Non-deterministic many-valued logics [1, 2, 3] are a generalization of "ordinary" many-valued logics and, in this note, we extend two of the "deterministic" calculi introduced in [9] to non-deterministic ones. Like in [9], the logics under considerations are presented semantically, based on the connectives' truth tables. The non-deterministic semantics of an ℓ-ary connective $*$ is given by the connective truth table that is a function from the set of truth values $V = \{v_1, \ldots, v_n\}$, $n \geq 2$, into the set of the non-empty subsets of V: $* : V^\ell \to P(V) \setminus \{\emptyset\}$.

Similarly to [9], we construct proof systems for non-deterministic many-valued logics out of the truth tables for the connectives, cf. [4, 6, 7, 10, 11, 12, 13]. Our construction is general, transparent, and uniform.

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This note is organized as follows. In Section 2 we introduce the many-valued logic $\boldsymbol{NMVL_A}$[1] that is based on an axiomatic approach and prove the strong soundness and completeness (i.e., with respect to the consequence relation) theorem for that logic. In Section 3 we introduce the logic $\boldsymbol{NMVL_R}$ by, equivalently, replacing some axioms of $\boldsymbol{NMVL_A}$ with rules of inference and prove the cut elimination theorem. We conclude the paper with the appendix containing a list of calculi dual to $\boldsymbol{NMVL_A}$ and $\boldsymbol{NMVL_R}$. The proofs of the properties of these dual calculi are very similar to their counterparts in [9] and are omitted.

2. Translating truth tables to axioms

The semantics of non-deterministic many-valued logic is as follows.

A valuation v is a function from the set of formulas $\mathcal{F}m$ into the set of truth values $V = \{v_1, \ldots, v_n\}$, $n \geq 2$, such that for each connective $*$,

$$v(*(\varphi_1, \ldots, \varphi_\ell)) \in *(v(\varphi_1), \ldots, v(\varphi_\ell)))$$

The logic $\boldsymbol{NMVL_A}$ considered in this section has only structural rules of inference and axioms instead of logical rules, cf. [4, 10]. We use the notion of a *labelled* formula that is a pair (φ, k), where φ is a formula and $k = 1, \ldots, n$, introduced in [5, 7]. The intended meaning of such a labelled formula is that v_k is the truth value associated with φ.

Sequents are expressions of the form $\Gamma \to \Delta$, where Γ and Δ are finite (possibly empty) sets of labelled formulas and $\to$ is not a symbol of the underlying language. As we shall see in the sequel, such sequents are more appropriate for meta-reasoning about labelled formulas than those from [4, 6, 10, 11, 12, 13].

The axioms of $\boldsymbol{NMVL_A}$ are sequents of the form

$$(\varphi, k) \to (\varphi, k) \tag{1}$$

$k = 1, \ldots, n$, or of the form

$$(\varphi_1, k_1), \ldots, (\varphi_\ell, k_\ell) \to \{(*(\varphi_1, \ldots, \varphi_\ell), k) : v_k \in *(v_{k_1}, \ldots, v_{k_\ell})\} \tag{2}$$

for each table entry $*(v_{k_1}, \ldots, v_{k_\ell})$. The latter axiom will be referred to as a *table* axiom.

The rules of inference of $\boldsymbol{NMVL_A}$ are the structural rules below.

k-L-shift, $k = 1, \ldots, n$,

$$\frac{\Gamma, (\varphi, k) \to \Delta}{\Gamma \to \Delta, \{\varphi\} \times \overline{\{k\}}}\,{}^{2} \tag{3}$$

[1]The subscript "A" indicates the axiom description of the logic.

[2]As usual, $\overline{K}$ denotes the complement $\{1, \ldots, n\} \setminus K$ of K.

k', k''-R-shift, $k', k'' = 1, \ldots, n$, $k' \neq k''$,

$$\frac{\Gamma \to \Delta, (\varphi, k')}{\Gamma, (\varphi, k'') \to \Delta} \tag{4}$$

k-L-weakening, $k = 1, \ldots, n$,

$$\frac{\Gamma \to \Delta}{\Gamma, (\varphi, k) \to \Delta} \tag{5}$$

k-R-weakening, $k = 1, \ldots, n$,

$$\frac{\Gamma \to \Delta}{\Gamma \to \Delta, (\varphi, k)} \tag{6}$$

k-cut, $k = 1, \ldots, n$,

$$\frac{\Gamma \to \Delta, (\varphi, k) \qquad \Gamma, (\varphi, k) \to \Delta}{\Gamma \to \Delta} \tag{7}$$

and

k', k''-resolution, $k', k'' = 1, \ldots, n$, $k' \neq k''$

$$\frac{\Gamma \to \Delta', (\varphi, k') \qquad \Gamma \to \Delta'', (\varphi, k'')}{\Gamma \to \Delta', \Delta''} \tag{8}$$

In fact, by [9, Proposition 3.3], rules (7) and (8) are derivable from each other.

Remark 2.1. The axioms (1) belong to all the calculi considered in this paper and all the calculi in [9]. Also, the structural rules of all the calculi considered in this paper and in [9] are rules (3)–(8). Thus, when the proofs for "deterministic" many valued logics in [9] rely on axioms (1) and rules (3)–(8), only, they apply to the non-deterministic ones as well.

Proposition 2.2. ([9, Proposition 3.4]) The sequent

$$\to \{\varphi\} \times \{1, \ldots, n\} \tag{9}$$

is derivable in $\boldsymbol{NMVL_A}$.

Next, we prove the strong completeness theorem for $\boldsymbol{NMVL_A}$.

Definition 2.3. A valuation v *satisfies* a sequent $\Gamma \to \Delta$ if the following holds.

- If for each $(\varphi, k) \in \Gamma$, $v(\varphi) = v_k$, then for some $(\varphi, k) \in \Delta$, $v(\varphi) = v_k$.[3]

Definition 2.4. A set of sequents $\boldsymbol{\Sigma}$ *semantically entails* a sequent Σ, denoted $\boldsymbol{\Sigma} \models \Sigma$, if each valuation satisfying all sequents from $\boldsymbol{\Sigma}$ also satisfies Σ.

[3] That is, v satisfies a sequent $\Gamma \to \Delta$, if the meta-value of the classical meta-sequent $\{v(\varphi) = v_k : (\varphi, k) \in \Gamma\} \to \{v(\varphi) = v_k : (\varphi, k) \in \Delta\}$ is "true."

Theorem 2.5. (Soundness and completeness of $\boldsymbol{NMVL_A}$) Let $\mathbf{\Sigma}$ be a set of sequents. Then $\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_A}} \Gamma \to \Delta$ if and only if $\mathbf{\Sigma} \models \Gamma \to \Delta$.

An immediate corollary to Theorem 2.5 is that $\boldsymbol{NMVL_A}$ is (strongly) decidable.

Regarding Theorem 2.5 itself, soundness is easy to verify and, for the proof of completeness, we proceed as follows.

Lemma 2.6. ([9, Lemma 3.12]) If $\not\vdash_{\boldsymbol{NMVL_A}} \Gamma \to \Delta$, then for no formula φ and no k', k'' such that $k' \neq k''$, $(\varphi, k'), (\varphi, k'') \in \Gamma$.

Proof of the completeness part of Theorem 2.5:
Assume to the contrary that $\mathbf{\Sigma} \not\vdash_{\boldsymbol{NMVL_A}} \Gamma \to \Delta$. Then, by Zorn's lemma, there is a maximal (with respect to inclusion) set of labelled formulas $\mathbf{\Gamma}$ including Γ such that for no finite subset Γ' of $\mathbf{\Gamma}$, $\mathbf{\Sigma} \not\vdash_{\boldsymbol{NMVL_A}} \Gamma' \to \Delta$.

We observe that for each formula φ there is a $k \in \{1, \ldots, n\}$ such that $(\varphi, k) \in \mathbf{\Gamma}$.[4] For the proof, assume to the contrary that for each $k \in \{1, \ldots, n\}$ there is a finite subset Γ_k of $\mathbf{\Gamma}$ such that

$$\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_A}} \Gamma_k, (\varphi, k) \to \Delta \tag{10}$$

Then, from (9) and (10), by n cuts we obtain

$$\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_A}} \bigcup_{k=1}^{n} \Gamma_k \to \Delta$$

which contradicts the definition of $\mathbf{\Gamma}$.

Let the valuation $v : \mathcal{F}m \to \{v_1, \ldots, v_n\}$ be defined by

$$v(\varphi) = v_k, \text{ if } (\varphi, k) \in \mathbf{\Gamma} \tag{11}$$

We contend that v is well-defined.

First we show that v is a function. For the proof, assume to the contrary that for some formula φ and some k' and k'' such that $k' \neq k''$ both (φ, k') and (φ, k'') are in $\mathbf{\Gamma}$. Then, by (the contraposition of) Lemma 2.6,

$$\vdash_{\boldsymbol{NMVL_A}} (\varphi, k'), (\varphi, k'') \to \Delta$$

which contradicts the definition of $\mathbf{\Gamma}$.

Next, we are going to show that the function $v : \mathcal{F}m \to V$ defined by (11) is indeed a valuation.

The proof is by induction on the complexity of φ. The basis (in which φ is an atomic formula) is by the definition of v, see (11), and, for the induction step assume that φ is of the form $*(\varphi_1, \ldots, \varphi_\ell)$.

Let $(\varphi, k) \in \mathbf{\Gamma}$ and let $v(\varphi_j) = v_{k_j}$, $j = 1, \ldots, \ell$. By the induction hypothesis, $(\varphi_j, k_j) \in \mathbf{\Gamma}$, $j = 1, \ldots, \ell$. In addition, from the table axiom (2), by a number of weakenings, we obtain

$$(\varphi_1, k_1), \ldots, (\varphi_\ell, k_\ell) \to \Delta, \{(\varphi, k') : v_{k'} \in *(v_{k_1}, \ldots, v_{k_\ell})\} \tag{12}$$

[4] In other words, $\mathbf{\Gamma}$ is *complete*, cf. [7, paragraph 3.63] and the definition of the "classical" negation completeness.

Now, assume to the contrary that $v(\varphi) \notin *(v_{k_1}, \ldots, v_{k_j})$. Then, from (12) and the axiom $(\varphi, k) \to (\varphi, k)$, by a number of k, k'-resolutions,

$$(\varphi, k), (\varphi_1, k_1), \ldots, (\varphi_\ell, k_\ell) \to \Delta$$

However, the latter contradicts the definition of $\mathbf{\Gamma}$.

We note next that for no labelled formula $(\varphi, k) \in \Delta$, $v(\varphi) = v_k$. Indeed, if for some $(\varphi, k) \in \Delta$, $v(\varphi) = v_k$, then, by the definition of v, see (11), $(\varphi, k) \in \mathbf{\Gamma}$, which contradicts the definition of $\mathbf{\Gamma}$.

It remains to prove that v satisfies each sequent in $\mathbf{\Sigma}$. Let $\Gamma' \to \Delta' \in \mathbf{\Sigma}$ be such that v satisfies each labelled formula in Γ', which, by (11), is equivalent to $\Gamma' \subseteq \mathbf{\Gamma}$. We have to show that v satisfies some labelled formula in Δ', which, by (11), is equivalent to $\Delta' \cap \mathbf{\Gamma} \neq \emptyset$.

Assume to the contrary that $\Delta' \cap \mathbf{\Gamma} = \emptyset$. Let $(\varphi, k) \in \Delta'$ and let $v(\varphi) = v_{k_\varphi}$. Then, by (11), $k_\varphi \neq k$ and $(\varphi, k_\varphi) \in \mathbf{\Gamma}$. From $\Gamma' \to \Delta'$ by k, k_φ-R-shifts (for each $(\varphi, k) \in \Delta'$) we obtain

$$\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_A}} \Gamma', \{(\varphi, k_\varphi) : (\varphi, k) \in \Delta'\} \to$$

from which, by weakenings,

$$\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_A}} \Gamma', \{(\varphi, k_\varphi) : (\varphi, k) \in \Delta'\} \to \Delta \qquad (13)$$

However, since

$$\Gamma', \{(\varphi, k_\varphi) : (\varphi, k) \in \Delta'\} \subseteq \mathbf{\Gamma}$$

(13) contradicts the definition of $\mathbf{\Gamma}$. □

3. Replacing axioms with with rules of inference

The sequent calculus $\boldsymbol{NMVL_R}$ in this section is the "sequent counterpart" of the deduction system $SF^d_{\mathcal{M}}$ from [1, Section 3.1]. Namely, $\boldsymbol{NMVL_R}$ results from $\boldsymbol{NMVL_A}$ by replacing axioms (2) with the rules of inference

$$\frac{\Gamma \to \Delta, (\varphi_j, k_j), \qquad j = 1, \ldots, \ell}{\Gamma \to \Delta, \{(*(\varphi_1, \ldots, \varphi_\ell), k) : v_k \in *(v_{k_1}, \ldots, v_{k_\ell})\}} \qquad (14)$$

for each table entry $*(v_{k_1}, \ldots, v_{k_\ell})$.

Proposition 3.1. Let $\mathbf{\Sigma}$ and Σ be a set of sequents and a sequent, respectively. Then $\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_R}} \Sigma$ if and only if $\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_A}} \Sigma$.

Proof:
The proof is similar to that of [9, Proposition 4.1].

For the proof of the "only if" part of the proposition it suffices to show that axioms (2) are derivable in $\boldsymbol{NMVL_R}$. The derivation is as follows.

$$\dfrac{\dfrac{(\varphi_j,k_j)\to(\varphi_j,k_j)}{(\varphi_1,k_1),\ldots,(\varphi_\ell,k_\ell)\to(\varphi_j,k_j)}\text{L-weakenings} \qquad j=1,\ldots,\ell}{(\varphi_1,k_1),\ldots,(\varphi_\ell,k_\ell)\to\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k_1},\ldots,v_{k_\ell})\}} \quad (14)$$

Conversely, for the proof of the "if" part of the proposition it suffices to show that rules (14) are derivable in $\boldsymbol{NMVL_A}$. The derivation is by ℓ cuts:

$$\dfrac{\{\Gamma\to\Delta,(\varphi_j,k_j): j=1,\ldots,\ell\},\ \ (\varphi_1,k_1),\ldots,(\varphi_\ell,k_\ell)\to\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k_1},\ldots,v_{k_\ell})\}}{\Gamma\to\Delta,\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k_1},\ldots,v_{k_\ell})\}}$$

□

Corollary 3.2. (Soundness and completeness of $\boldsymbol{NMVL_R}$) Let $\boldsymbol{\Sigma}$ and Σ be a set of sequents and a sequent, respectively. Then $\boldsymbol{\Sigma}\vdash_{\boldsymbol{NMVL_R}}\Sigma$ if and only if $\boldsymbol{\Sigma}\models\Sigma$.

Proof:
The corollary follows from Proposition 3.1 and Theorem 2.5. □

Theorem 3.3. (Cut/resolution elimination) Each $\boldsymbol{NMVL_R}$-derivable sequent is derivable without cut or resolution.

Proof:
By double induction, the outer on the derivation length and the inner on the complexity of the principal formula, we eliminate the first cut/resolution in the derivation.

The outer induction and the basis of the inner induction do not involve rules (14). Thus, they are like in the corresponding proofs in [9, Section 4.2].

The the induction step of the inner induction is treated as follows.

Since $*(\varphi_1,\ldots,\varphi_\ell)$ may be introduced into the succeedent, only, this is the case of k',k''-resolution (8), with φ being $*(\varphi_1,\ldots,\varphi_\ell)$. Namely, we have

$$\dfrac{\dfrac{\Gamma\to\Delta,(\varphi_j,k'_j),\quad j=1,\ldots,\ell}{\Gamma\to\Delta,\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k'_1},\ldots,v_{k'_\ell})\}} \quad \dfrac{\Gamma\to\Delta,(\varphi_j,k''_j),\quad j=1,\ldots,\ell}{\Gamma\to\Delta,\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k''_1},\ldots,v_{k''_\ell})\}}}{\Gamma\to\Delta,\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k'_1},\ldots,v_{k'_\ell}),\ k\neq k'\}\cup\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k''_1},\ldots,v_{k''_\ell}),\ k\neq k''\}} \quad (15)$$

If $(k'_1,\ldots,k'_\ell)\neq(k''_1,\ldots,k''_\ell)$ then, for some $j=1,\ldots,\ell$, $k'_j\neq k''_j$ and we may apply k'_j,k''_j-resolution from which we proceed by R-weakenings:

$$\dfrac{\dfrac{\Gamma\to\Delta,(\varphi_j,k'_j)\quad \Gamma\to\Delta,(\varphi_j,k''_j)}{\Gamma\to\Delta}}{\Gamma\to\Delta,\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k'_1},\ldots,v_{k'_\ell}),\ k\neq k'\}\cup\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k''_1},\ldots,v_{k''_\ell}),\ k\neq k''\}}$$

Otherwise, i.e., if $(k'_1,\ldots,k'_\ell)=(k''_1,\ldots,k''_\ell)$ then the conclusion of (15) is

$$\Gamma\to\Delta,\{(*(\varphi_1,\ldots,\varphi_\ell),k): v_k\in *(v_{k'_1},\ldots,v_{k'_\ell})\}$$

and we can replace (15) with any of its premises,

$$\frac{\Gamma \to \Delta, (\varphi_j, k'_j), \quad j = 1, \ldots, \ell}{\Gamma \to \Delta, \{(*(\varphi_1, \ldots, \varphi_\ell), k) : v_k \in *(v_{k'_1}, \ldots, v_{k'_\ell})\}}$$

say.[5] □

A. Duality

In the appendix, we consider two kinds of duality with respect to $\boldsymbol{NMVL_A}$ and $\boldsymbol{NMVL_R}$. One is the "distributive" duality resulting from the duality between metaconjunction and metadisjunction and the other is the "sequent" duality resulting from the duality between the succeedent and the antecedent of a sequent.

Accordingly, Section A.1 deals with the "di-stri-bu-ti-vely" dual systems $\boldsymbol{NMVL_{ADD}}$ and $\boldsymbol{NMVL_{RDD}}$ of $\boldsymbol{NMVL_A}$ and $\boldsymbol{NMVL_R}$, respectively; and Section A.2 deals with the "sequent" duality of $\boldsymbol{NMVL_R}$ and $\boldsymbol{NMVL_{RDD}}$. The results in Section A.2 are the respective "antecedent" counterparts of those in Sections 3 and A.1.

A.1. Distributive duality

In this section, we present the calculi $\boldsymbol{NMVL_{ADD}}$ and $\boldsymbol{NMVL_{RDD}}$ which are *distributively dual* and deductively equivalent to $\boldsymbol{NMVL_A}$ and $\boldsymbol{NMVL_R}$, respectively, cf. [10]. These calculi are presented in Sections A.1.1 and A.1.2. Both employ the following notation.

For a labelled formula $(*(\varphi_1, \ldots, \varphi_\ell), k)$, we define the set of sets of labelled formulas $(*(\varphi_1, \ldots, \varphi_\ell) \times k)^{-1}$ by

$$(*(\varphi_1, \ldots, \varphi_\ell) \times k)^{-1} = \{\{(\varphi_1, k_1), \ldots, (\varphi_\ell, k_\ell)\} : v_k \in *(v_{k_1}, \ldots, v_{k_\ell})\} \tag{16}$$

and, in what follows, we enumerate the sets in $(*(\varphi_1, \ldots, \varphi_\ell) \times k)^{-1}$ as

$$(*(\varphi_1, \ldots, \varphi_\ell), k)^{-1} = \{\{(\varphi_1, k_{1,q}), \ldots, (\varphi_\ell, k_{\ell,q})\} : q = 1, \ldots, s\}^{6} \tag{17}$$

That is,

$$(*(\varphi_1, \ldots, \varphi_\ell) \times k)^{-1} = \{\Theta_1, \ldots, \Theta_s\}$$

where

$$\Theta_q = \{(\varphi_1, k_{1,q}), \ldots, (\varphi_\ell, k_{\ell,q})\} \tag{18}$$

$q = 1, \ldots, s$.[7]

Next, for sets $\Theta_1, \ldots, \Theta_s$ of labelled formulas, Θ_q as in (18), $q = 1, \ldots, s$, we define the set $\bigvee_{q=1}^{s} \Theta_q$ of sets of labelled formulas by

[5] In fact, the case of the equality $(k'_1, \ldots, k'_\ell) = (k''_1, \ldots, k''_\ell)$ is the only modification needed for the extension of the proofs of cut and resolution elimination in [9] to the case of non-deterministic logics.

[6] Note that s depends both on $*$ and k.

[7] Note the form of Θ_q: for each $j = 1, \ldots, \ell$ it contains exactly one labelled formula with the first component φ_j.

$$\bigvee_{q=1}^{s} \Theta_q = \{\{(\varphi_{j_1}, k_{j_1,1}), \ldots, (\varphi_{j_s}, k_{j_s,s})\} : (\varphi_{j_q}, k_{j_q,q}) \in \Theta_q\,,\ q = 1, \ldots, s\}$$

That is, for each $q = 1, \ldots, s$, the elements of $\bigvee_{q=1}^{s} \Theta_q$ contain one element of Θ_q and nothing more.

Similarly, for a formula $*(\varphi_1, \ldots, \varphi_\ell))$ and a subset K of $\{1, \ldots, n\}$, we define the set of sets of labelled formulas $(*(\varphi_1, \ldots, \varphi_\ell) \times K)^{-1}$ by

$$(*(\varphi_1, \ldots, \varphi_\ell) \times K)^{-1} = \{\{(\varphi_1, k_1), \ldots, (\varphi_\ell, k_\ell)\} : *(v_{k_1}, \ldots, v_{k_\ell}) = \{v_k : k \in K\}\}$$

Remark A.1. Let $k_1, \ldots, k_\ell \in \{1, \ldots, n\}$ and let K be such that $*(v_{k_1}, \ldots, v_{k_\ell}) = \{v_k : k \in K\}$. Then, in the above notation, table axioms (2) are

$$\Theta \to \{*(\varphi_1, \ldots, \varphi_\ell)\} \times K$$

for all $\Theta \in (*(\varphi_1, \ldots, \varphi_\ell) \times K)^{-1}$.

A.1.1. Distributively dual axioms

The distributively dual counterpart of $\boldsymbol{NMVL_A}$, denoted $\boldsymbol{NMVL_{ADD}}$, is based on Proposition A.2 below.

Proposition A.2. (Cf. [9, Proposition 5.3].) For all k and all

$$\{(\varphi_{j_1}, k_{j_1,1}), \ldots, (\varphi_{j_s}, k_{j_s,s})\} \in \bigvee (*(\varphi_1, \ldots, \varphi_\ell) \times k)^{-1}$$

the sequent

$$(*(\varphi_1, \ldots, \varphi_\ell), k) \to (\varphi_{j_1}, k_{j_1,1}), \ldots, (\varphi_{j_s}, k_{j_s,s}) \tag{19}$$

is derivable in $\boldsymbol{NMVL_A}$.

The intuition lying behind Proposition A.2 is as follows. Using $\bigvee$, $\bigwedge$, $\Longrightarrow$, and $\Longleftrightarrow$ as meta-connectives, we see that

$$\begin{aligned} v_k \in v(*(\varphi_1, \ldots, \varphi_\ell)) &\Longrightarrow \bigvee_{v_k \in *(v_{k_1}, \ldots, v_{k_\ell})} \bigwedge_{j=1}^{\ell} v(\varphi_j) = v_{k_j} \\ &\Longleftrightarrow \bigwedge_{\Lambda \in \bigvee (*(\varphi_1, \ldots, \varphi_\ell), k)^{-1}} \bigvee_{(\varphi_j, k_j) \in \Lambda} v(\varphi_j) = v_{k_j} \end{aligned} \tag{20}$$

see (16) and (17). Now, writing (φ, k) for $v(\varphi) = v_k$, we rewrite (20) as

$$\begin{aligned} (*(\varphi_1, \ldots, \varphi_\ell), k) &\Longrightarrow \bigvee_{v_k \in *(v_{k_1}, \ldots, v_{k_\ell})} \bigwedge_{j=1}^{\ell} (\varphi_j, k_j) \\ &\Longleftrightarrow \bigwedge_{\Lambda \in \bigvee (*(\varphi_1, \ldots, \varphi_\ell), k)^{-1}} \bigvee_{(\varphi_j, k_j) \in \Lambda} (\varphi_j, k_j) \end{aligned}$$

Then, the succeedent of (19) comes from rewriting (20) as conjunction of disjunctions.

The calculus $\boldsymbol{NMVL_{ADD}}$ results from $\boldsymbol{NMVL_A}$ by replacing axioms (2) with the (distributively dual) axioms (19), cf. [10].

Theorem A.3. (Soundness and completeness of $\boldsymbol{NMVL_{ADD}}$, cf. [9, Theorem 5.5].) Let $\mathbf{\Sigma}$ and Σ be a set of sequents and a sequent, respectively. Then $\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_{ADD}}} \Sigma$ if and only if $\mathbf{\Sigma} \models \Sigma$.

A.1.2. Distributively dual rules of inference

The distributively dual calculus $\boldsymbol{NMVL_{RDD}}$ results from $\boldsymbol{NMVL_R}$ by replacing rules of inference (14) with the rule

$$\frac{\Gamma \to \Delta, \Lambda,\ \Lambda \in \bigvee(*(\varphi_1, \dots, \varphi_\ell), K)^{-1}}{\Gamma \to \Delta, \{*(\varphi_1, \dots, \varphi_\ell)\} \times K} \tag{21}$$

Theorem A.4. (Soundness and completeness of $\boldsymbol{NMVL_{RDD}}$, cf. [9, Theorem 5.7].) Let $\mathbf{\Sigma}$ and Σ be a set of sequents and a sequent, respectively. Then $\mathbf{\Sigma} \vdash_{\boldsymbol{NMVL_{RDD}}} \Sigma$ if and only if $\mathbf{\Sigma} \models \Sigma$.

Theorem A.5. (Cf. [9, Theorem 5.8].) Each $\boldsymbol{NMVL_{RDD}}$-derivable sequent is derivable without cut or resolution.

A.2. Sequent duality

One can think of $\boldsymbol{NMVL_R}$ and $\boldsymbol{NMVL_{RDD}}$ as sequent calculi with rules of introduction to the succeedent only. In this section we replace these rules with the dual rules of introduction to the antecedent and show the dual calculi posses all properties of the original ones.

For cut/resolution elimination in the calculi in this section we need the following rule of inference.

K-L-multi-shift, $K \subset \{1, \dots, n\}$,

$$\frac{\Gamma, (\varphi, k) \to \Delta,\ k \in K}{\Gamma \to \Delta, \{\varphi\} \times \overline{K}} \tag{22}$$

Remark A.6. This rule is derivable from its premises and (9) by cuts. However, we do not call it "cut," because it does not affect the subformula property[8]: since K is a proper subset of $\{1, \dots, n\}$, $\overline{K} \neq \emptyset$.

A.2.1. The sequent dual of $\boldsymbol{NMVL_R}$

The (dual) rules for introduction of $*$ to the antecedent are

$$\frac{\Gamma, (\varphi_1, k_1), \dots, (\varphi_\ell, k_\ell) \to \Delta,\ v_k \in *(v_{k_1}, \dots, v_{k_\ell})}{\Gamma, (*(\varphi_1, \dots, \varphi_\ell), k) \to \Delta} \tag{23}$$

for each $k = 1, \dots, n$, cf. [8, Proof of Theorem 8].

The proofs of Propositions A.7 and A.8 and Corollary A.9 below are like the corresponding proofs of [9, Proposition 5.11], [9, Proposition 5.12]), and [9, Corollary 5.13].

[8] We say that a labelled formula (φ, k) is a subformula of a labelled formula (φ', k'), if φ is a subformula of φ'.

Proposition A.7. Rules (23) are derivable in $\boldsymbol{NMVL_R}$.

Proposition A.8. Axioms (19) are derivable by *(23)*.

The sequent calculus $\boldsymbol{NMVL_{RSD}}$ results from $\boldsymbol{NMVL_R}$ by replacing rules of introduction to succeedent (14) with rules of introduction to antecedent (23) and adding to it the K-L-multi-shift (22).

Corollary A.9. Calculi $\boldsymbol{NMVL_{RSD}}$ and $\boldsymbol{NMVL_R}$ are deductively equivalent.

Theorem A.10. (Cf. [9, Theorem 5.14].) Each $\boldsymbol{NMVL_{RSD}}$-derivable sequent is derivable without cut or resolution.

A.2.2. The sequent dual of $\boldsymbol{NMVL_{RDD}}$

In the case of $\boldsymbol{NMVL_{RDD}}$, the (dual) rules for introduction of $*$ to the antecedent are

$$\frac{\Gamma, (\varphi_j, k_j) \to \Delta,\ (\varphi_j, k_j) \in \Lambda}{\Gamma, (*(\varphi_1, \ldots, \varphi_\ell), k) \to \Delta} \tag{24}$$

for each

$$\Lambda = (\varphi_{j_1}, k_{j_1,1}), \ldots, (\varphi_{j_s}, k_{j_s,s}) \in \bigvee(*(\varphi_1, \ldots, \varphi_\ell), k)^{-1}$$

The proofs of Propositions A.11 and A.12 and Corollary A.13 below are like in the corresponding proofs of [9, Proposition 5.17], [9, Proposition 5.18]) and [9, Corollary 5.19].

Proposition A.11. Rules (24) are derivable in $\boldsymbol{NMVL_{RDD}}$.

Proposition A.12. Axioms (19) are derivable from *(24)*.

The sequent calculus $\boldsymbol{NMVL_{RDDSD}}$ results from $\boldsymbol{NMVL_{RDD}}$ by replacing rules of introduction to succeedent (21) with rules of introduction to antecedent (24) and adding to it multi-shifts (22).

Corollary A.13. Calculi $\boldsymbol{NMVL_{RDDSD}}$ and $\boldsymbol{NMVL_{RDD}}$ are deductively equivalent.

Theorem A.14. (Cf. [9, Theorem 5.20].) Each $\boldsymbol{NMVL_{RDDSD}}$-derivable sequent is derivable without cut or resolution.

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