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Structural Liveness of Immediate Observation Petri Nets

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Abstract. We look in detail at the structural liveness problem (SLP) for subclasses of Petri nets, namely immediate observation nets (IO nets) and their generalized variant called branching immediate multi-observation nets (BIMO nets), that were recently introduced by Esparza, Raskin, and Weil-Kennedy. We show that SLP is PSPACE-hard for IO nets and in PSPACE for BIMO nets. In particular, we discuss the (small) bounds on the token numbers in net places that are decisive for a marking to be (non)live.

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1. Introduction

Petri nets are an established model of concurrent systems, and a natural part of related research aims to clarify computational complexity of verifying basic behavioural properties for various (sub)classes of this model. A famous example is the reachability problem for standard place/transition Petri nets, which was recently shown to be Ackermann-complete ([1, 2, 3]).

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Here we are interested in the *structural liveness problem* (SLP): given a net N, is there an initial marking M_0 such that the marked net (N, M_0) is live? We recall that a marked net (N, M_0) is live if no transition can become disabled forever, in markings reachable from M_0. The *liveness problem* (LP), asking if a marked net (N, M_0) is live, has been long known to be tightly related to the reachability problem [4]; hence LP has turned out to be Ackermann-complete as well. Somewhat surprisingly, for SLP even the decidability status was open until recently (see, e.g., [5]); the currently known status is that the problem is EXPSPACE-hard and decidable [6].

Here we look at the complexity of SLP for a subclass of place/transition Petri nets, namely for the class of *immediate observation Petri nets* (IO nets), and their generalized variant called *branching immediate multi-observation Petri nets* (BIMO nets). These models were introduced and studied recently, in [7, 8, 9], motivated by a study of population protocols and chemical reaction networks. *Population protocols* [10] are a model of computation where an arbitrary number of indistinguishable finite-state agents interact in pairs; an interaction of two agents, being in states q_1 and q_2, consists in changing their states to q_3 and q_4, respectively, according to a transition function. A global state of a protocol is just a function assigning to each (local) state the number of agents in this state. It is natural to represent population protocols by Petri nets where places represent (local) states, and markings represent global states. The above mentioned IO nets represent a special subclass, so called *immediate observation protocols* (IO protocols), that was introduced in [11]. Here an agent can change its state q_1 to q_2 when observing that another agent is in state q_3. We also note that BIMO nets can be viewed as a generalization of Petri nets related to basic parallel processes (BPP nets), studied, e.g., in [12]. In the BPP nets each transition t has precisely one input place p, the edge (p, t) having the weight 1. In the BIMO nets, for performing such a transition t it is not only necessary that the place p has at least one token but there is also a context-condition, requiring that some places have sufficient amounts of tokens. (The relation of BPP nets and BIMO nets resembles the relation of context-free and context-sensitive grammars, where the word-concatenation is viewed as commutative.)

Among the results of [7] is the PSPACE-completeness of the liveness problem (LP) for IO nets. The paper [7] does not deal with the structural liveness problem (SLP) directly but it can be derived from its results that SLP is in PSPACE for IO nets.

Our contribution. We first show that also SLP is PSPACE-hard for IO nets. Here we proceed similarly as [7] where the hardness proof for LP is given; we show a modification of a standard simulation of linear bounded automata by 1-safe nets, but the construction for SLP is more subtle than for LP. The PSPACE membership is straightforward for LP on IO nets, since IO nets are a special case of conservative nets, but it is not so straightforward for SLP. We show that for a BIMO net, where P is the set of places and w the maximum edge-weight, the fact whether or not a marking M is live ($M : P \to \mathbb{N}$ assigns the number of tokens to each place) is determined by the values $M(p)$ that are less than $2 \cdot w \cdot |P|$. This result allows us to give a simple proof that SLP is in PSPACE (and thus PSPACE-complete) for BIMO nets.

The organization of the paper. In Section 2 we give the basic definitions related to Petri nets, and define the subclasses (BIMO, BIO, IMO, IO nets) in which we are interested; in part 2.3 we summarize our results. Section 3 shows the PSPACE-hardness of the structural liveness problem (SLP) for ordinary IO nets (where “ordinary” means that all edge-weights are 1). In Sections 4, 5 and 6 we prove the announced results for ordinary BIMO, BIO, IMO and IO nets. Section 7 extends these results to all BIMO, BIO, IMO, IO nets, by a simple construction. In Section 8 we use the achieved results to show that SLP for BIMO nets is in PSPACE. Some additional remarks are given in Section 9.

2. Preliminaries, and results

By $\mathbb{N}$ we denote the set $\{0, 1, 2, \dots\}$ of nonnegative integers, and we put $[i, j] = \{i, i+1, \dots, j\}$ for $i, j \in \mathbb{N}$.

Multisets. Given a set U, called the *universe*, by a *multiset* M *over* U we mean a function $M : U \to \mathbb{N}$; for $x \in U$ we write $x \in M$ if $M(x) \geq 1$. We use the notation $M = \wr x_1, x_2, \dots, x_n \wr$ for finite multisets; here we have $M(x) = |\{i \in [1, n]\,; x_i = x\}|$, and we put $|M| = n$. A set $X \subseteq U$ is naturally viewed as a multiset $X : U \to \{0, 1\}$. By $\emptyset$ we denote the empty (multi)set ($\emptyset(x) = 0$ for all $x \in U$).

Given two multisets M, M' over U, we define the multisets $M'' = M + M'$ and $M''' = M - M'$ so that $M''(x) = M(x) + M'(x)$ and $M'''(x) = \max\{M(x) - M'(x), 0\}$ for all $x \in U$. By $M \leq M'$ we denote that $M(x) \leq M'(x)$ for all $x \in U$. We also use the intersection of multisets: for $M'' = M \cap M'$ we have $M''(x) = \min\{M(x), M'(x)\}$ for all $x \in U$.

2.1. Standard Petri net definitions

Nets, subnets (ordinary, and weighted). A *net* N is a triple (P, T, F) where P and T are finite disjoint sets of *places* and *transitions*, respectively, and $F : (P \times T) \cup (T \times P) \longrightarrow \mathbb{N}$ is a *flow function*. A pair $(x, y) \in (P \times T) \cup (T \times P)$ where $F(x, y) \geq 1$ is also called an *edge* in N, and $F(x, y)$ is viewed as its *weight*. A *net* $N = (P, T, F)$ is *ordinary* if F is of the type $(P \times T) \cup (T \times P) \longrightarrow \{0, 1\}$ (hence the weights of edges are 1).

We use a standard depiction of nets; for instance, the net in Figure 1 has 6 places (circles), 6 transitions (boxes), and the depicted edges; the edge weights 1 are implicit. If the weight is larger than 1, then it is depicted explicitly, like, e.g., the weight 3 in Figure 9 (in Section 7).

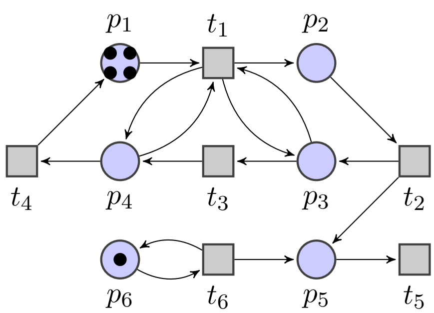

Figure 1. Example of a marked ord-BIMO net.

Given a net $N = (P, T, F)$ and sets $P' \subseteq P$, $T' \subseteq T$, by $N_{\downarrow(P',T')}$ we denote the (sub)net (P', T', F') where F' arises from F by restricting its domain to $(P' \times T') \cup (T' \times P')$. Sometimes we also deal with subnets arising by removing some edges.

Pre-msets, post-msets, siphons. Let $N = (P, T, F)$ be a fixed net. For each transition $t \in T$ we define its *pre-mset* ${}^\bullet t$ and its *post-mset* $t^\bullet$ as multisets over P where ${}^\bullet t(p) = F(p, t)$ and $t^\bullet(p) = F(t, p)$, for each place $p \in P$.

A set of places $S \subseteq P$ is a *siphon* if for each $t \in T$ we have that $t^\bullet \cap S \neq \emptyset$ entails ${}^\bullet t \cap S \neq \emptyset$. For instance, the set $S = \{p_2, p_3, p_4\}$ in Figure 1 is a siphon, since $\{t \in T \mid t^\bullet \cap S \neq \emptyset\} = \{t_1, t_2, t_3\}$ and $\{t \in T \mid {}^\bullet t \cap S \neq \emptyset\} = \{t_1, t_2, t_3, t_4\}$.

Markings, marked nets. Given a net $N = (P, T, F)$, a *marking* M of N is a multiset over P (hence $M : P \to \mathbb{N}$), where $M(p)$ is viewed as the number of *tokens* on the place p (alternatively we also say "in the place p"). For $P' \subseteq P$, $M_{\downarrow P'}$ denotes the restriction of M to P'. A *place* $p \in P$ is *marked at* M if $M(p) \geq 1$; a *set of places* $P' \subseteq P$ is *marked at* M if $|M_{\downarrow P'}| \geq 1$.

When the places in P are ordered, we can also naturally view markings as vectors; e.g., the marking M depicted in Figure 1 can be given as $(4, 0, 0, 0, 0, 1)$. By $\mathbf{0}$ we denote the zero vector (with the dimension clear from context). E.g., for the siphon $S = \{p_2, p_3, p_4\}$ in Figure 1 we have $M_{\downarrow S} = \mathbf{0}$, i.e., the siphon S is unmarked at M.

A *marked net* (or a *Petri net*) is a tuple (N, M_0) where N is a net and M_0 is a marking of N, called the *initial marking*.

Executions, reachability. Given a net $N = (P, T, F)$, a *transition* t is *enabled at a marking* M, which is denoted by $M \xrightarrow{t}$, if $M \geq {}^\bullet t$ (i.e., $M(p) \geq F(p, t)$ for all $p \in P$). If t is enabled at M, it can *fire* (or *be performed*, or *be executed*), which yields the marking $M' = (M - {}^\bullet t) + t^\bullet$ (hence $M'(p) = M(p) - F(p, t) + F(t, p)$ for all $p \in P$); this is denoted by $M \xrightarrow{t} M'$.

A sequence $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_k} M_k$ is called an *execution*, *from* M_0 *to* M_k, which is also presented as $M_0 \xrightarrow{\sigma} M_k$ where $\sigma = t_1 t_2 \cdots t_k$. A marking M' is *reachable from* M if there is an execution $M \xrightarrow{\sigma} M'$. By $[M\rangle$ we denote the set of all markings that are reachable from M; we also write $M \xrightarrow{*} M'$ instead of $M' \in [M\rangle$.

For instance, an execution of the net in Figure 1 is $(1, 1, 1, 1, 1, 1) \xrightarrow{t_2} (1, 0, 2, 1, 2, 1) \xrightarrow{t_3} (1, 0, 1, 2, 2, 1) \xrightarrow{t_3} (1, 0, 0, 3, 2, 1) \xrightarrow{t_4} (2, 0, 0, 2, 2, 1) \xrightarrow{t_4} (3, 0, 0, 1, 2, 1) \xrightarrow{t_4} (4, 0, 0, 0, 2, 1) \xrightarrow{t_5} (4, 0, 0, 0, 1, 1) \xrightarrow{t_5} (4, 0, 0, 0, 0, 1)$. We might also note that generally any unmarked siphon S cannot get marked; i.e., $M_{\downarrow S} = \mathbf{0}$ entails $M'_{\downarrow S} = \mathbf{0}$ for all $M' \in [M\rangle$.

Dead and live transitions, liveness and structural liveness. Given $N = (P, T, F)$, a *transition* t is *dead at* a *marking* M if there is no $M' \in [M\rangle$ such that $M' \xrightarrow{t}$ (hence t is disabled in all markings reachable from M). A *transition* t is *live at* M if it is non-dead at each $M' \in [M\rangle$. We note that a transition can be both non-live and non-dead at M.

A *marking* M of N is *live* if all transitions are live at M. A *marked net* (N, M_0) is *live* if M_0 is live (for N). A net N is *structurally live* if there is M_0 such that (N, M_0) is live.

For instance, the net in Figure 1 is clearly not structurally live (if $M(p_6) = 0$, then M is clearly non-live, and otherwise there is $M' \in [M\rangle$ such that the siphon $S = \{p_2, p_3, p_4\}$ is unmarked at M', i.e. $M'_{\downarrow S} = \mathbf{0}$). If we removed the edges $(p_3, t_1), (t_1, p_3)$ and $(p_4, t_1), (t_1, p_4)$, then the net would become structurally live.

The *liveness problem* (LP) asks if a given marked net (N, M_0) is live. The *structural liveness problem* (SLP) asks if a given net N has a marking M_0 for which (N, M_0) is live.

Conservative nets, and the (structural) liveness problem. We call a *net* $N = (P, T, F)$ *conservative* if $|{}^\bullet t| = |t^\bullet|$ for each $t \in T$; in this case $M \xrightarrow{t} M'$ entails $|M| = |M'|$ (hence in every execution the number of tokens is constant). We remark that the definition of conservative nets in the literature is sometimes more general, in which case our notion corresponds to $\mathbf{1}$-conservative nets.

For the results of this paper, it is particularly useful to recall the well-known fact:

Proposition 2.1. The liveness problem (LP) for conservative nets is PSPACE-complete.

The PSPACE-hardness follows by a standard reduction from the acceptance problem for linear bounded automata (LBA), even for ordinary conservative nets, as we also recall later in detail. For the membership in PSPACE we can refer to [13]; this holds even for the case with general edge-weights that are given in binary.

Remark 2.2. It is useful to recall the idea of the PSPACE-membership: Given a conservative net N, a transition t and a marking M, deciding if t is non-dead at M is obviously in NPSPACE (we just perform a nondeterministically chosen execution from M until covering ${}^\bullet t$, i.e., until reaching M' such that $M' \geq {}^\bullet t$). Since NPSPACE=PSPACE, we deduce that deciding if t is dead at M is in PSPACE. Given a conservative net N and a marking M, deciding if there is $M' \in [M\rangle$ and a transition t that is dead at M' is thus in NPSPACE, hence in PSPACE, as well.

Remark 2.3. The liveness problem (LP) for general nets is well-known to be tightly related to the reachability problem; the recent break-through results [3, 2, 1] thus show its huge computational complexity, namely the Ackermann-completeness. The structural liveness problem (SLP) is more unclear so far. For general nets SLP is known to be EXPSPACE-hard and decidable [6]; for conservative nets we can show that SLP is EXPSPACE-hard and elementary [14].

2.2. Immediate observation nets, and their (more general) variants

We recall the notions of IO (immediate observation) nets and BIO (branching IO) nets, including the multi-observer versions: IMO and BIMO nets. These nets were introduced in [7, 9], being originally motivated by (special types of) population protocols. They have restricted types of transitions; we start with defining the most general case.

Branching immediate multiple-observation (BIMO) transitions. Given a net $N = (P, T, F)$, we say that a transition $t \in T$ is a *BIMO transition* if $|{}^\bullet t - t^\bullet| \leq 1$ (hence there is at most one place p for which ${}^\bullet t(p) > t^\bullet(p)$, in which case we have ${}^\bullet t(p) - t^\bullet(p) = 1$).

Convention. In our considerations, for each BIMO transition t we will assume that ${}^\bullet t \neq \emptyset$. I.e., in the case ${}^\bullet t = \emptyset$ we tacitly assume an additional "dummy place" $\textsc{d}$ such that ${}^\bullet t(\textsc{d}) = t^\bullet(\textsc{d}) = 1$ and $\textsc{d}$ is marked in all considered markings.

Having the convention in mind, to each BIMO transition t we fix a presentation

$$t : p_s \xrightarrow{\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr} \wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr$$

(for some $k, \ell \in \mathbb{N}$) where

$${}^\bullet t = \wr p_s \wr + \wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr, \text{ and } t^\bullet = \wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr + \wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr.$$

We note that the multisets $\wr p_s \wr$, $\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr$, and $\wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr$ are not necessarily disjoint (their pairwise intersections might be nonempty).

The place p_s is called the *source place of* t. If $|{}^\bullet t - t^\bullet| = 1$, then p_s is the unique place satisfying ${}^\bullet t(p_s) = 1 + t^\bullet(p_s)$; if $|{}^\bullet t - t^\bullet| = 0$ (hence ${}^\bullet t(p) \leq t^\bullet(p)$ for all $p \in P$), then we fix p_s as one of the places p for which ${}^\bullet t(p) \geq 1$.

For instance, all transitions in Figure 9 (in Section 7) are BIMO-transitions; the only presentation of t in Figure 9(left) is $t : p_1 \xrightarrow{\wr p_1 \wr} \wr p_1, p_1, p_2 \wr$, t' in Figure 9(right) can be presented as $t' : p_{\langle 1,1 \rangle} \xrightarrow{\wr p_{\langle 1,2 \rangle} \wr} \wr p_{\langle 1,1 \rangle}, p_{\langle 1,3 \rangle}, p_{\langle 2,1 \rangle} \wr$ or as $t' : p_{\langle 1,2 \rangle} \xrightarrow{\wr p_{\langle 1,1 \rangle} \wr} \wr p_{\langle 1,2 \rangle}, p_{\langle 1,3 \rangle}, p_{\langle 2,1 \rangle} \wr$, and $t_{\langle 1,1 \rangle}$ only as $t_{\langle 1,1 \rangle} : p_{\langle 1,1 \rangle} \xrightarrow{\emptyset} \wr p_{\langle 1,2 \rangle} \wr$ (which is also written as $t_{\langle 1,1 \rangle} : p_{\langle 1,1 \rangle} \to \wr p_{\langle 1,2 \rangle} \wr$).

We observe that performing a BIMO-transition $t : p_s \xrightarrow{\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr} \wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr$ (i.e., executing a step $M \xrightarrow{t} M'$) can be viewed so that a "source" token from p_s has "branched" into new tokens in the *destination places* constituting the set $\{p_{d_1}, p_{d_2}, \ldots, p_{d_k}\}$; the new tokens are created in the destination places with the multiplicities determined by the multiset $\wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr$. We note that it is not excluded that $k = 0$ (in which case the source token disappears since there are no destination places) or that $p_{d_i} = p_s$ for some i (which is the case of the transition on the left of Figure 9). Performing t is conditioned not only on the presence of a token in the source place p_s but also on the presence of enough tokens in the *observation places* constituting the set $\{p_{o_1}, p_{o_2}, \ldots, p_{o_\ell}\}$; this "enough tokens" is determined by the multiset $\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr$.

BIMO transitions of the type BIO, IMO, IO. Given a BIMO transition

$$t : p_s \xrightarrow{\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr} \wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr$$

we say that t is:

- a *BIO transition* if the multiset $\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr$ is a singleton set $\{p_o\}$ or the empty set (performing t is conditioned on at most one observation-token, which is the case for all transitions in Figure 8); in this case we also write

$$t : p_s \xrightarrow{p_o} \wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr \text{ or } t : p_s \to \wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr;$$

- an *IMO transition* if the multiset $\wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr$ is a singleton set $\{p_d\}$ (the source token does not branch, nor disappears, it just moves from p_s to p_d); in this case we also write

$$t : p_s \xrightarrow{\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr} p_d;$$

- an *IO transition* if it is a BIO and IMO transition; in this case we also write

$$t : p_s \xrightarrow{p_o} p_d \text{ or } t : p_s \rightarrow p_d.$$

Nets of the type BIMO, ord-BIMO, BIO, ord-BIO, IMO, ord-IMO, IO, ord-IO. For a type X $\in$ {BIMO, BIO, IMO, IO} we say that a net N is an *X net* if all its transitions are X transitions; moreover, N is an *ord-X net* if N is an X net that is ordinary (the edge weights are just 1).

We note that for any BIMO transition $t : p_s \xrightarrow{\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr} \wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr$ in an ordinary net it holds that $\wr p_{o_1}, p_{o_2}, \ldots, p_{o_\ell} \wr$ and $\wr p_{d_1}, p_{d_2}, \ldots, p_{d_k} \wr$ are two disjoint *sets*, and $p_s \notin \{p_{o_1}, p_{o_2}, \ldots, p_{o_\ell}\}$ (while we still can have $p_s \in \{p_{d_1}, p_{d_2}, \ldots, p_{d_k}\}$).

We observe that IMO nets are conservative, hence Proposition 2.1 entails:

Proposition 2.4. The liveness problem (LP) for IMO nets is in PSPACE.

We recall that this also holds when the edge-weights (i.e., the multiplicities of observation places) are given in binary. Moreover, the PSPACE-hardness proof for conservative nets has been enhanced in [7] to show that LP is PSPACE-hard also for IO nets.

2.3. Results

Below we summarize the results of this paper.

a) By Theorem 3.3 and its proof we show that a modification of the hardness proof for the liveness problem (LP) for IO nets in [7] can be enhanced to demonstrate the PSPACE-hardness of the structural liveness problem (SLP) for ord-IO nets. (We remark that SLP is EXPSPACE-hard for conservative nets [14].)

Table 1. Given a structurally live net with maximum edge-weight w, there is a live marking in which each component is bounded by the 1st upper bound; moreover, the (non)liveness status of any marking does not change if each component greater than the 2nd upper bound is replaced with this bound.

Class of nets	**1st upper bound**	**2nd upper bound**
ord-IO and ord-IMO	1	$2 \cdot \lvert P\rvert$
IO	2	$4 \cdot \lvert P\rvert$
IMO	w	$2 \cdot w \cdot \lvert P\rvert$
ord-BIO and ord-BIMO	$\lvert P\rvert$	$2 \cdot \lvert P\rvert$
BIO and BIMO	$w \cdot \lvert P\rvert$	$2 \cdot w \cdot \lvert P\rvert$

b) Table 1 summarizes our results concerning the sizes of live markings in the mentioned net classes, as stated by the following theorem.

Theorem 2.5. Given a structurally live net $N = (P, T, F)$ of a type in the first column of Table 1, with the maximum edge-weight w (where $w = 1$ if N is ordinary), then

1. there is a live marking M of N in which $M(p)$ is no bigger than the respective 1st upper bound in Table 1, for each $p \in P$;
2. whether or not a marking M of N is live is determined by the restriction $M_{\downarrow P'}$ where P' consists of the places p for which the values $M(p)$ are smaller than the respective 2nd upper bound in Table 1.

For instance, if an ord-IMO net $N = (P, T, F)$ is structurally live, then there is $M_0 : P \to \{0, 1\}$ such that (N, M_0) is live; moreover, (N, M) is live iff (N, M') is live where $M'(p) = M(p)$ if $M(p) < 2 \cdot |P|$ and $M'(p) = 2 \cdot |P|$ otherwise (for all $p \in P$).

c) By Theorem 8.1 we show that the structural liveness problem (SLP) for BIMO nets is in PSPACE (and thus PSPACE-complete).

Regarding the result c), we note that our bounds from b) for IMO nets (already the 1st upper bound, in fact) immediately show that SLP for IMO nets is in PSPACE (and thus PSPACE-complete), by recalling Proposition 2.4. For general BIMO nets we show the PSPACE-membership of SLP by using further ideas (one of them being captured by Lemma 4.2 in particular).

Remark 2.6. The papers [7, 9], and in particular the PhD thesis by Chana Weil-Kennedy in preparation, contain a detailed study of subclasses of BIMO nets, concentrating mainly on the general analysis questions like reachability and coverability. The PSPACE-membership of structural liveness could be also derived from the published results for IO nets, while for BIMO nets this will follow from the mentioned PhD thesis in preparation. Nevertheless, our direct handling of structural liveness for BIMO nets yields stronger bounds captured by Table 1 and a more specific insight into this problem (that remains so far a bit elusive for more general nets).

3. PSPACE-hardness of structural liveness for ord-IO nets

In this section, we show that the structural liveness problem (SLP) for ord-IO nets is PSPACE-hard; this is achieved by an enhancement of (a modification of) the construction showing PSPACE-hardness of the liveness problem (LP) from [7]. (This lower bound is later matched by a PSPACE upper bound that holds for the most general of the considered classes, i.e., for BIMO nets, in which the edge-weights are given in binary.)

We first introduce the notion of carriers of markings, and an observation captured by Proposition 3.2: Informally speaking, if we add a token onto a place p in a marking M where $M(p) \geq 1$ (in an ord-IO net N), then this additional token could be imagined as pasted down to an original token with which it can then be moving together (by repeating the transitions moving the original token), whereas the resulting marking carrier remains the same.

Remark 3.1. Later we will look at this observation in more detail for ord-BIMO nets, but now it suffices to deal with ord-IO nets.

Carriers of markings. Given a net $N = (P, T, F)$ and a marking M of N, by $car(M)$ we denote the *carrier of* M, i.e. the set $\{p \in P \mid M(p) \geq 1\}$.

Proposition 3.2. (Added tokens can be pasted down to original tokens in ord-IO nets)
We assume an ord-IO net $N = (P, T, F)$ and its execution $M \xrightarrow{\sigma} M'$. Then for any $\bar{M} \geq M$ such that $car(\bar{M}) = car(M)$ there are $\bar{\sigma}$ and $\bar{M}' \geq M'$ for which $\bar{M} \xrightarrow{\bar{\sigma}} \bar{M}'$ and $car(\bar{M}') = car(M')$.

Proof:
We prove the claim by induction on the value $|\sigma|$ (the length of σ). We thus assume that $M \xrightarrow{\sigma} M'$, $\bar{M} \geq M$, $car(\bar{M}) = car(M)$, and that the claim holds for all σ' shorter than σ (by the induction hypothesis). In the case $|\sigma| = 0$ the claim is trivial, so we now assume that $\sigma = t\,\sigma'$, where $t : p_s \xrightarrow{p_o} p_d$ (or $t : p_s \to p_d$) and $M \xrightarrow{t} M'' \xrightarrow{\sigma'} M'$. We thus have $\bar{M}(p_s) \geq M(p_s) \geq 1$, and $\bar{M} \xrightarrow{t^d} \bar{M}''$ where $d = 1 + (\bar{M}(p_s) - M(p_s))$. It is clear that $\bar{M}'' \geq M''$ and $car(\bar{M}'') = car(M'')$; hence we can apply the induction hypothesis to $M'' \xrightarrow{\sigma'} M'$ and $\bar{M}''$, which entails that $\bar{M} \xrightarrow{t^d} \bar{M}'' \xrightarrow{\bar{\sigma}'} \bar{M}'$ where $\bar{M}' \geq M'$ and $car(\bar{M}') = car(M')$. □

Now we prove the announced PSPACE-hardness, by a reduction from the standard PSPACE-complete problem asking if a deterministic linear bounded automaton with a two-letter tape-alphabet accepts a given word.

Theorem 3.3. The structural liveness problem (SLP) for ord-IO nets is PSPACE-hard.

Proof:
We show the above announced reduction in a stepwise manner, also using informal descriptions that are formalized afterwards. We thus assume a given deterministic linear-bounded Turing machine

$$\text{LBA} = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{ACC}}, q_{\text{REJ}}\}) \text{ and a word } w = x_1 x_2 \cdots x_n, \tag{1}$$

where $\Sigma = \Gamma = \{a, b\}$, $n \geq 1$, and $x_i \in \Sigma = \{a, b\}$ for $i \in [1, n]$. W.l.o.g. we assume that the computation of LBA on w (starting in the configuration $q_0 w$) finishes in the accepting state q_{ACC} or the rejecting state q_{REJ} with the head scanning the cell 1. We also view the transition function $\delta : (Q \smallsetminus \{q_{\text{ACC}}, q_{\text{REJ}}\}) \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$ as the set of *instructions* (q, x, q', x', m), writing rather $(q, x, q', x', m) \in \delta$ instead of $\delta(q, x) = (q', x', m)$, and w.l.o.g. we assume that $q' \neq q_0$ for each instruction $(q, x, q', x', m) \in \delta$; hence each computation of LBA, starting with the initial state q_0, never returns to q_0.

It is straightforward (and standard) to simulate the computation of LBA on $w = x_1 x_2 \cdots x_n$ with a 1-safe conservative Petri net $(N_{\langle \text{LBA}, w \rangle}, M_0)$, as we now sketch (see Figure 2(left)); by "1-safe" we mean that $M(p) \in \{0, 1\}$ for all $M \in [M_0\rangle$ and all places p of $N_{\langle \text{LBA}, w \rangle}$. Given (1), we construct $N_{\langle \text{LBA}, w \rangle}$ as follows.

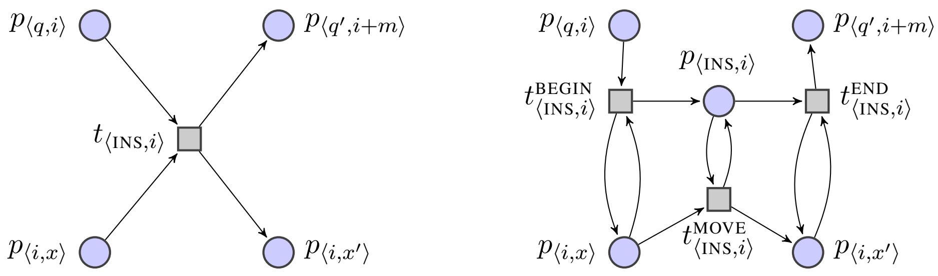

Figure 2. $\text{INS} = (q, x, q', x', m)$ mimicked in $N_{\langle \text{LBA},w\rangle}$ (left) and in $N'_{\langle \text{LBA},w\rangle}$ (right).

- For each control state $q \in Q$ and each head-position $i \in [1, n]$ we create a place $p_{\langle q,i\rangle}$.
- For each tape-cell $i \in [1, n]$ and each tape-symbol $x \in \Gamma = \{a, b\}$ we create a place $p_{\langle i,x\rangle}$.
- Each instruction $\text{INS} = (q, x, q', x', m) \in \delta$ is implemented by net transitions $t_{\langle \text{INS},i\rangle}$, $i \in [1, n]$, where ${}^\bullet t_{\langle \text{INS},i\rangle} = \{p_{\langle q,i\rangle}, p_{\langle i,x\rangle}\}$ and $t^\bullet_{\langle \text{INS},i\rangle} = \{p_{\langle q',i+m\rangle}, p_{\langle i,x'\rangle}\}$, excluding the cases where $i+m \notin [1, n]$. (The notation stresses that the multisets ${}^\bullet t_{\langle \text{INS},i\rangle}$ and $t^\bullet_{\langle \text{INS},i\rangle}$ are sets.)

A configuration (q, i, u) of LBA, where $q \in Q$, $i \in [1, n]$, and $u = y_1 y_2 \cdots y_n \in \Gamma^* = \{a, b\}^*$, is mimicked by the marking of $N_{\langle \text{LBA},w\rangle}$ with one token in $p_{\langle q,i\rangle}$ and one token in $p_{\langle j,y_j\rangle}$ for each $j \in [1, n]$. The initial configuration $q_0 w$ of LBA is mimicked by the respective *initial marking* M_0 in $N_{\langle \text{LBA},w\rangle}$, from which there exists only one execution, simulating the computation of LBA on w.

Since $N_{\langle \text{LBA},w\rangle}$ is not an ord-IO net in general, we transform $N_{\langle \text{LBA},w\rangle}$ to an ord-IO net $N'_{\langle \text{LBA},w\rangle}$ as depicted in Figure 2: for each $\text{INS} = (q, x, q', x', m) \in \delta$ and $i \in [1, n]$,

- we add a place $p_{\langle \text{INS},i\rangle}$ and
- replace the transition $t_{\langle \text{INS},i\rangle}$ with three IO transitions, namely

$$t^{\text{BEGIN}}_{\langle \text{INS},i\rangle} : p_{\langle q,i\rangle} \xrightarrow{p_{\langle i,x\rangle}} p_{\langle \text{INS},i\rangle},\ t^{\text{MOVE}}_{\langle \text{INS},i\rangle} : p_{\langle i,x\rangle} \xrightarrow{p_{\langle \text{INS},i\rangle}} p_{\langle i,x'\rangle},\ t^{\text{END}}_{\langle \text{INS},i\rangle} : p_{\langle \text{INS},i\rangle} \xrightarrow{p_{\langle i,x'\rangle}} p_{\langle q',i+m\rangle};$$

  in fact, we omit $t^{\text{MOVE}}_{\langle \text{INS},i\rangle}$ if $x = x'$.

It is clear that $N'_{\langle \text{LBA},w\rangle}$ starting from M_0 also simulates the computation of LBA on w.

Since the computation of LBA always finishes with the head scanning the cell 1, we get that LBA accepts w iff the "state-position token" from $p_{\langle q_0,1\rangle}$ in M_0 eventually moves to the place $p_{\langle q_{\text{ACC}},1\rangle}$. Hence the *reachability* (and *coverability*) problem for ord-IO nets is PSPACE-hard (in fact, PSPACE-complete).

For the *liveness* problem, we construct the ord-IO net $N''_{\langle \text{LBA},w\rangle}$ arising from $N'_{\langle \text{LBA},w\rangle}$ as depicted in Figure 3:

- we add a place p_{RUN}, with a token in the initial marking M_0, and a place p_{FREE}, unmarked in M_0, and transitions $t_{\text{A}} : p_{\text{RUN}} \xrightarrow{p_{\langle q_{\text{ACC}},1\rangle}} p_{\text{FREE}}$ and $t'_{\text{A}} : p_{\text{FREE}} \xrightarrow{p_{\langle q_{\text{ACC}},1\rangle}} p_{\text{RUN}}$;

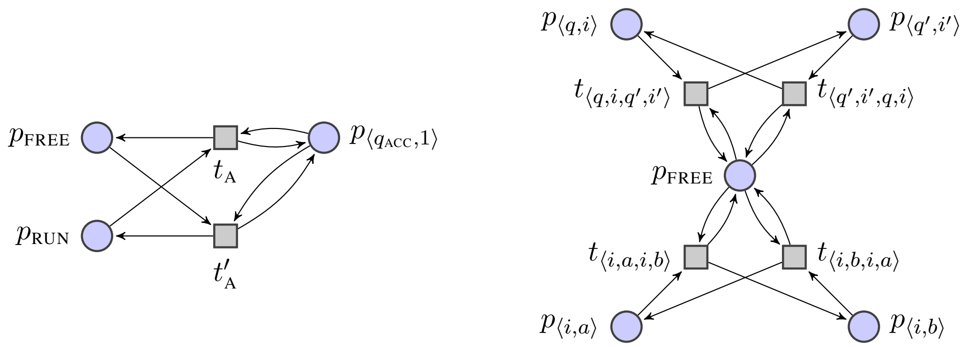

Figure 3. Marked p_{FREE} allows to freely change the mimicked configurations of LBA in $N''_{\langle \text{LBA},w\rangle}$.

- for each pair $(\langle q,i\rangle,\langle q',i'\rangle)$ where $\langle q,i\rangle \neq \langle q',i'\rangle$ we add a transition

$$t_{\langle q,i,q',i'\rangle} : p_{\langle q,i\rangle} \xrightarrow{p_{\text{FREE}}} p_{\langle q',i'\rangle},$$

and for each (i,x,y) where $i \in [1,n]$, $x,y \in \Gamma = \{a,b\}$ and $x \neq y$ we add a transition

$$t_{\langle i,x,i,y\rangle} : p_{\langle i,x\rangle} \xrightarrow{p_{\text{FREE}}} p_{\langle i,y\rangle}.$$

We can easily verify that LBA accepts w iff the initial marking M_0 of $N''_{\langle \text{LBA},w\rangle}$ (mimicking q_0w and having a token in p_{RUN}) is live: A crucial fact is that t_{A} gets enabled iff LBA accepts w, after which any configuration (i.e., any marking mimicking a configuration of LBA) can be repeatedly installed, which makes each transition live, including all $t^{\text{BEGIN}}_{\langle \text{INS},i\rangle}$, $t^{\text{MOVE}}_{\langle \text{INS},i\rangle}$, $t^{\text{END}}_{\langle \text{INS},i\rangle}$ (from Figure 2) and also t_{A} due to t'_{A} (from Figure 3). Hence the liveness problem is PSPACE-hard (and PSPACE-complete) as well.

If LBA accepts w, then the ord-IO net $N''_{\langle \text{LBA},w\rangle}$ is structurally live since M_0 is a live marking. But to show the PSPACE-hardness of the *structural liveness* problem (SLP) we also need to guarantee

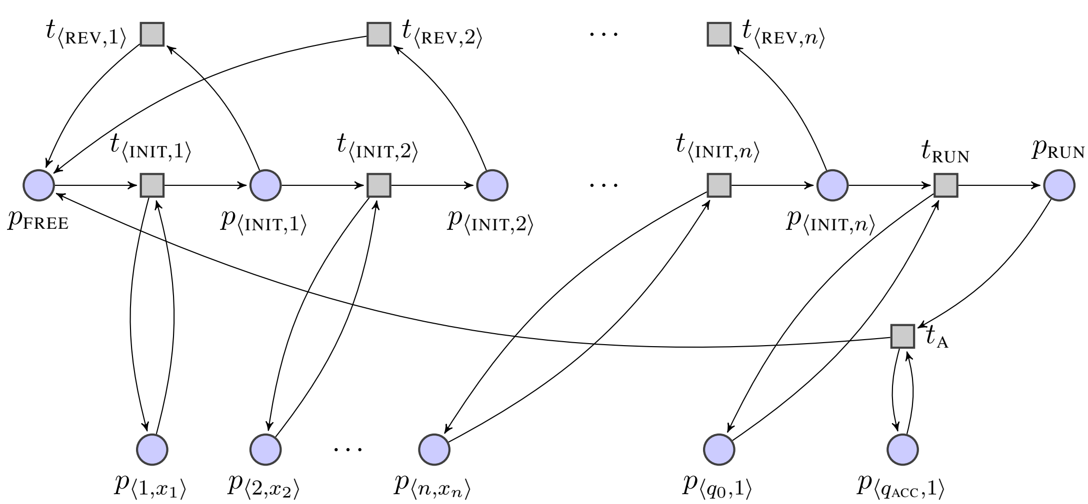

Figure 4. $\bar{N}_{\langle \text{LBA},w\rangle}$ arises from $N''_{\langle \text{LBA},w\rangle}$ by replacing t'_{A} from Figure 3 by "hard-wiring" M_0.

that all markings are non-live in the respective net if LBA rejects w. This leads us to a final step of our construction: the net $N''_{\langle \text{LBA},w\rangle}$ is modified by replacing the transition t'_{A} (recall Figure 3) with a collection of places and transitions that essentially "hard-wire" the initial marking M_0, as depicted in Figure 4; thus the final ord-IO net $\bar{N}_{\langle \text{LBA},w\rangle}$ arises.

More concretely, the net $\bar{N}_{\langle \text{LBA},w\rangle}$ arises from $N''_{\langle \text{LBA},w\rangle}$ by removing t'_{A} (with its adjacent edges) and adding the following objects (we recall that $w = x_1 x_2 \cdots x_n$):

- places $p_{\langle \text{INIT},i\rangle}$ for all $i \in [1,n]$, with zero tokens in M_0;
- a transition $t_{\langle \text{INIT},1\rangle} : p_{\text{FREE}} \xrightarrow{p_{\langle 1,x_1\rangle}} p_{\langle \text{INIT},1\rangle}$;
- for each $i \in [2,n]$, transitions

$$t_{\langle \text{INIT},i\rangle} : p_{\langle \text{INIT},i-1\rangle} \xrightarrow{p_{\langle i,x_i\rangle}} p_{\langle \text{INIT},i\rangle} \text{ and } t_{\langle \text{REV},i\rangle} : p_{\langle \text{INIT},i\rangle} \rightarrow p_{\text{FREE}};$$

- a transition $t_{\text{RUN}} : p_{\langle \text{INIT},n\rangle} \xrightarrow{p_{\langle q_0,1\rangle}} p_{\text{RUN}}$.

(We do not need observation-places for transitions $t_{\langle \text{REV},i\rangle}$.)

It is easy to check that we also have

$$\text{LBA accepts } w \text{ iff } M_0 \text{ is live in } \bar{N}_{\langle \text{LBA},w\rangle}.$$

Indeed: We recall that LBA accepts w iff $p_{\langle q_{\text{ACC}},1\rangle}$ gets eventually marked when $\bar{N}_{\langle \text{LBA},w\rangle}$ starts with M_0. Then the token from p_{RUN} can move to p_{FREE} by t_{A}, which allows to freely perform all transitions that are changing the mimicked configurations of LBA and that are simulating the instructions $\text{INS} = (q, x, q', x', m)$ from δ. The token in p_{FREE} can go back to p_{RUN} only via performing the whole sequence

$$t_{\langle \text{INIT},1\rangle} t_{\langle \text{INIT},2\rangle} \cdots t_{\langle \text{INIT},n\rangle} t_{\text{RUN}}. \tag{2}$$

Performing this sequence can be always "aborted" by performing one of the transitions $t_{\langle \text{REV},i\rangle}$; on the other hand, performing the whole sequence guarantees that the initial M_0 (corresponding to the initial configuration $q_0 w$) had been installed before the sequence (2) started; here we use the assumption that in all $\text{INS} = (q, x, q', x', m) \in \delta$ we have $q' \neq q_0$, which entails that during performing the whole sequence (2) no other transition could be performed (since the single "state-position token" must be in $p_{\langle q_0,1\rangle}$ during the whole sequence).

It remains to show that

$$\text{if LBA rejects } w\text{, then each marking } M \text{ of } \bar{N}_{\langle \text{LBA},w\rangle} \text{ is non-live.}$$

For the sake of contradiction we assume that LBA rejects w and M is a live marking of $\bar{N}_{\langle \text{LBA},w\rangle}$. Since M is live, it contains at least one token on some $p_{\langle q,i\rangle}$, and for each $i \in [1,n]$ it contains at least one token on one of the places $p_{\langle i,a\rangle}$ and $p_{\langle i,b\rangle}$; moreover, from M we can reach a marking in which p_{FREE} is marked and then install a "pseudoinitial" marking $\bar{M}_0 \geq M_0$ where $car(\bar{M}_0) = car(M_0)$ (the carrier of the pseudoinitial marking coincides with the carrier of the initial marking). Since LBA rejects w, for the sequence σ of transitions of $\bar{N}_{\langle \text{LBA},w\rangle}$ that simulates the computation of LBA on w we have $M_0 \xrightarrow{\sigma} M'$ where the state-position token in M' is on $p_{\langle q_{\text{REJ}},1\rangle}$. In this marking M' all transitions are dead: for each transition t in $\bar{N}_{\langle \text{LBA},w\rangle}$ there is a place $p_t \in {}^{\bullet}t$ such that $M'(p_t) = 0$. (We note that the set $S = \{p_t \mid t \text{ is a transition in } \bar{N}_{\langle \text{LBA},w\rangle}\}$ is a non-empty set of places that is unmarked in M', and S is a siphon.) By Proposition 3.2 we also have $\bar{M}_0 \xrightarrow{\bar{\sigma}} \bar{M}'$ where $car(\bar{M}') = car(M')$, and thus in $\bar{M}'$ all transitions are dead as well (the above siphon S is also unmarked in $\bar{M}'$). Since $\bar{M}' \in [M\rangle$, we have got a contradiction with the assumption that M is live. □

4. Proof strategy for upper bounds

We first recall the monotonicity property of nets: if $M_1 \xrightarrow{\sigma} M_2$, then $(M_1 + M) \xrightarrow{\sigma} (M_2 + M)$. This entails a simple fact: if we ignore some places (that can be understood as marked with infinite amounts of tokens), then any original execution can be performed in the case with ignored places as well.

Proposition 4.1. (If a transition is dead with ignored places, then it is dead originally)
Let $N = (P, T, F)$ be a (general) net. For all M, P', t, where $M : P \to \mathbb{N}$, $P' \subseteq P$, and $t \in T$, we have: if t is dead in $(N_{\downarrow(P',T)}, M_{\downarrow P'})$, then t is dead in (N, M) as well.

Proof:
For any execution $M \xrightarrow{\sigma} M'$ in N there is obviously the execution $M_{\downarrow P'} \xrightarrow{\sigma} M'_{\downarrow P'}$ in $N_{\downarrow(P',T)}$. Hence if t is non-dead in (N, M) (we have an execution $M \xrightarrow{\sigma t} M'$ of N), then t is non-dead in $(N_{\downarrow(P',T)}, M_{\downarrow P'})$ as well. □

A crucial ingredient for proving the upper-bound results stated by Theorem 2.5 in Section 2.3 is captured by the following lemma, which will be proven in Section 5. (We use the designation "Lemma" rather than "Proposition" for the claims that we highlight as crucial for our theorems.)

Lemma 4.2. (For ord-BIMO nets, if M_0 is non-live, then there is a simple witness $M_{\text{W}} \in [M_0\rangle$)
A marking M_0 of an ord-BIMO net $N = (P, T, F)$ is non-live iff there are

- $M_{\text{W}} \in [M_0\rangle$ (a witness marking),
- $P_{\text{CR}} \subseteq P$ (a set of crucial places), and
- a nonempty set $T_{\text{D}} \subseteq T$ (a set of dead transitions)

such that

1. $|(M_{\text{W}})_{\downarrow P_{\text{CR}}}| < |P_{\text{CR}}|$ (hence in M_{W} the sum of tokens on the places from P_{CR} is at most $|P_{\text{CR}}| - 1$), and we can, moreover, require that $M_{\text{W}}(p) \in \{0, 1\}$ for each $p \in P_{\text{CR}}$;
2. $N_{\downarrow(P_{\text{CR}}, T \smallsetminus T_{\text{D}})}$ is an ord-IMO net (and is thus conservative);
3. all transitions from T_{D} are dead in $(N_{\downarrow(P_{\text{CR}},T)}, (M_{\text{W}})_{\downarrow P_{\text{CR}}})$.

We note that $N_{\downarrow(P_{\text{CR}},T)}$ can have transitions t for which ${}^\bullet t = t^\bullet = \emptyset$; they are live but have no effect. Technically we view such transitions also as IMO-transitions (we might again imagine a marked dummy place serving as both the source and the destination).

In a particular case demonstrating Lemma 4.2, P_{CR} can be a siphon of N that is unmarked at M_{W}. Recall Figure 1 with the siphon $\{p_2, p_3, p_4\}$ where $T_D = \{t_1, t_2, t_3, t_4\}$. A more general case is exemplified by the ord-BIMO net $N = (P, T, F)$ in Figure 5, with M_0 satisfying $M_0(p) = 1$ for all $p \in P$. (The "observation edges" in Figure 5 are drawn as dotted, for better lucidity.) By executing $M_0 \xrightarrow{t_3 t_4 t_2 t_1 t_1 t_6 t_6 t_6} M_{\text{W}}$ we get the marking depicted in Figure 5, where we put $P_{\text{CR}} = \{p_1, p_2, p_3, p_4, p_5\}$ and $T_{\text{D}} = \{t_1, t_6, t_7\}$.

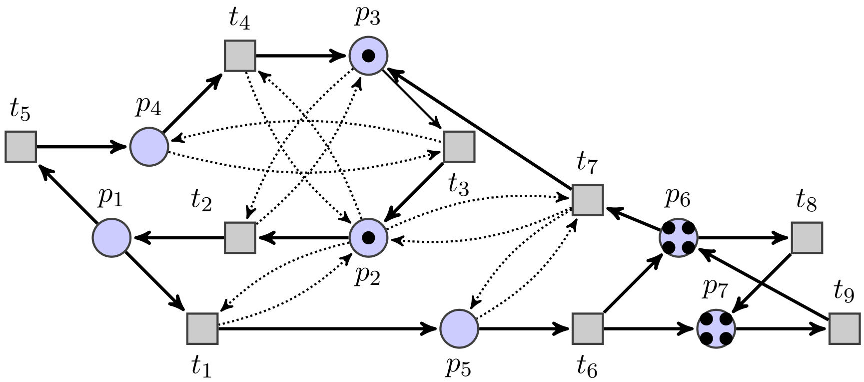

Figure 5. From $(1, 1, 1, 1, 1, 1, 1)$ we reach the depicted M_{W}, with $P_{\text{CR}} = \{p_1, p_2, p_3, p_4, p_5\}$.

We observe that the "if" direction of Lemma 4.2 is clear (by recalling Proposition 4.1). We also observe that, given an ord-BIMO net $N = (P, T, F)$, $M_{\text{W}} : P \to \mathbb{N}$, $P_{\text{CR}} \subseteq P$, and $T_{\text{D}} \subseteq T$, conditions 1 and 2 of Lemma 4.2 can be checked trivially, while deciding condition 3 is surely in PSPACE (since $N_{\downarrow(P_{\text{CR}},T \smallsetminus T_{\text{D}})}$ is conservative): we recall Remark 2.2, and note that we can verify that in $(N_{\downarrow(P_{\text{CR}},T \smallsetminus T_{\text{D}})}, (M_{\text{W}})_{\downarrow P_{\text{CR}}})$ we cannot cover ${}^\bullet t \cap P_{\text{CR}}$ for any $t \in T_{\text{D}}$.

Lemma 4.2 (whose "only if" direction is proven in Section 5) will allow us to derive the upper bounds in Table 1 for ordinary nets in Section 6, which is extended to non-ordinary nets by a simple construction in Section 7. These upper bounds also enable to give a smooth proof that reachability of a "simple witness" M_{W} of non-liveness of M_0 in BIMO nets can be verified in polynomial space as well; this is shown in Section 8.

5. Proof of Lemma 4.2

We first sketch the organization of the proof. We start by introducing the notion of optimal markings and an observation that guarantees reachability of such markings, in the case of general live Petri nets. Then we study optimal markings in the case of ord-BIMO nets; here we also use the notion of "relaxed nets", with omitted observation edges. A crucial fact is then captured by Lemma 5.3. Finally, another general notion, of DL-markings where each transition is either dead or live and at least one is dead, leads to finishing the whole proof of Lemma 4.2.

Optimal markings, i.e. carrier-maximal and self-coverable markings.
Let $N = (P, T, F)$ be a net, and M a marking of N. We define the following notions:

- M is *carrier-maximal* if for each $M' \in [M\rangle$ we have $|car(M')| \leq |car(M)|$;
- M is *self-coverable* if there is an execution $M \xrightarrow{\sigma} M'$ where $M \leq M'$ and σ is *full*, i.e., each $t \in T$ has at least one occurrence in σ;
- M is *optimal* if M is both carrier-maximal and self-coverable.

For instance, the marking M_{w} in Figure 5 is optimal for the net N' arising from the depicted net N by removing all dead transitions (t_1, t_6, t_7) with their incident edges.

Proposition 5.1. Let M_0 be a live marking of a net $N = (P, T, F)$. Then there is an optimal marking $M \in [M_0\rangle$.

Proof:
Let M_0 be a live marking of $N = (P, T, F)$; we assume $T \neq \emptyset$ (otherwise the claim is trivial). We consider an infinite execution $M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 \xrightarrow{\sigma_3} \cdots$ where

1. for each $i \geq 1$, σ_i is full (contains all $t \in T$), and
2. for each $i \geq 1$ we have $|car(M_i)| = \max\{\, |car(M)|; M_{i-1} \xrightarrow{\sigma} M$ for some full $\sigma\}$.

Such an execution obviously exists since M_0 is live (which entails that all $M \in [M_0\rangle$ are live).

We observe that M_i are carrier-maximal for all $i \geq 1$: if we had $M_i \xrightarrow{\sigma} M$ where $|car(M_i)| < |car(M)|$, then $M_{i-1} \xrightarrow{\sigma_i\sigma} M$ would violate the condition 2 (since $\sigma_i\sigma$ is full). By Dickson's lemma there are i_1, i_2 such that $1 \leq i_1 < i_2$ and $M_{i_1} \leq M_{i_2}$; hence M_{i_1} is both carrier-maximal and self-coverable (which also entails that $|car(M_{i_1})| = |car(M_{i_2})|$). ☐

The relaxed net $Relax(N)$ associated with an ord-BIMO net N. Let $N = (P, T, F)$ be an ord-BIMO net, and let $E \subseteq (P \times T) \cup (T \times P)$ be the set of its "moving" edges, i.e. the least set such that each transition $t : p_s \xrightarrow{\{p_{o_1},\dots,p_{o_\ell}\}} \{p_{d_1}, \dots, p_{d_k}\}$ entails that the edges (p_s, t) and (t, p_{d_j}) ($j \in [1, k]$) are in E. The *relaxed net related to* N is the subnet

$$Relax(N) = (P, T, F')$$

of N where for all $(x, y) \in (P \times T) \cup (T \times P)$ we have $F'(x, y) = 1$ if $(x, y) \in E$, and $F'(x, y) = 0$ otherwise. (We recall that we assume that each transition t has a fixed source place p_s, and that here we deal with *sets* $\{p_{o_1}, \dots, p_{o_\ell}\}$ and $\{p_{d_1}, \dots, p_{d_k}\}$ since the considered nets are ordinary.)

For instance, in the nets depicted in Figures 5, 6, and 7 the thick edges are those remaining in the respective relaxed nets. We note that the edges between p_5 and t_7 in Figure 6 are thick, since t_7 is understood as $t_7 : p_5 \to \{p_5, p_3\}$.

Informally speaking, $Relax(N)$ represents the behaviour of N when the observation-condition is relaxed, which means that a transition can fire even if its observation places are unmarked.

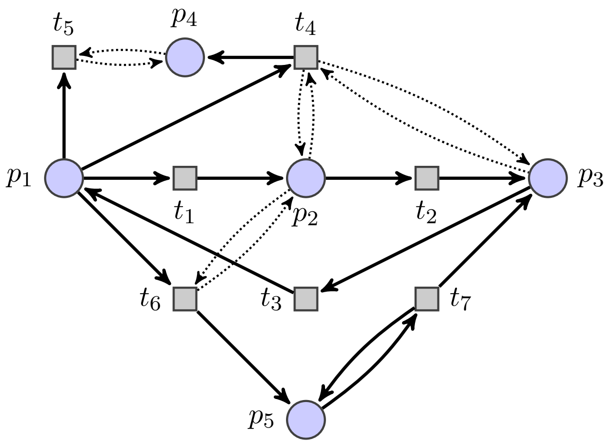

Figure 6. A structurally live ord-BIMO net.

Remark 5.2. For ord-IMO nets their relaxation nets are S-nets in terminology of, e.g., [15] (pre-msets and post-msets of all transitions are singletons). Generally, for ord-BIMO nets their relaxation nets are BPP-nets in terminology of, e.g., [12] (pre-msets of all transitions are singletons).

Nets $Relax(N)$ as directed bipartite graphs, paths, (bottom, top) components. We recall that any ordinary net $N = (P, T, F)$ (with $F : (P \times T) \cup (T \times P) \to \{0, 1\}$) can be naturally viewed as a directed bipartite graph where $P \cup T$ is the set of vertices and $\{(x, y) \in (P \times T) \cup (T \times P) \mid F(x, y) = 1\}$ is the set of edges. We thus use the standard notions like *a path in* $Relax(N)$ or *a strongly connected component (scc) of* $Relax(N)$. By a *proper successor of* an *scc* C we mean an scc C' such that $C' \neq C$ and there is a path from C to C' (in $Relax(N)$). An scc C is a *bottom component* if there is no proper successor of C; C is a *top component* if there is no C' such that C is a proper successor of C'.

For an scc C, by P_C we denote the set of places in C (which is empty if C consists just of one transition). We also say just "a component" instead of "an scc".

For instance, $Relax(N)$ related to N in Figure 5 has just one (strongly connected) component, in Figure 6 we have four components, three of them being trivial (i.e. singletons), namely $\{t_5\}, \{p_4\}, \{t_4\}$, and in Figure 7 we have two components, in this case both nontrivial.

Rich and poor components in marked ord-BIMO nets. We assume a given ord-BIMO net $N = (P, T, F)$. For a marking $M : P \to \mathbb{N}$, we say that

$$\text{an } scc\ C \text{ of } Relax(N) \text{ is } \begin{cases} \textit{rich at } M \ \dots \text{ if } M(P_C) \geq |P_C|, \\ \textit{poor at } M \ \dots \text{ if } M(P_C) < |P_C|. \end{cases}$$

If C is a rich (or poor) scc in $Relax(N)$ at M, then we also say that C is a *rich* (or *poor*) *component in* (N, M). We note that any component consisting of a single transition is always rich.

For instance, the only component of $Relax(N)$ depicted in Figure 5 is rich at both $(1,1,1,1,1,1,1)$ and M_{w}. We recall the net N' arising by removing the transitions t_1, t_6, t_7 (from the net N depicted in Figure 5), and note that all transitions of N' are live at M_{w}. We have three components in (N', M_{w}) (one of them consisting just of p_5), one rich and two poor. We also recall that M_{w} is an optimal marking in N'.

Now we show a crucial fact (called "Lemma" rather than "Proposition" in line with our remark before the statement of Lemma 4.2).

Lemma 5.3. (In optimal markings, tokens are "spread" and poor components are on top)
Let $N = (P, T, F)$ be an ord-BIMO net, and M_0 an optimal marking. Then:

1. for each rich component C in (N, M_0) we have $M_0(p) \geq 1$ for all $p \in P_C$;
2. for each poor component C in (N, M_0) we have $M_0(p) \in \{0, 1\}$ for all $p \in P_C$;
3. for each poor component C in (N, M_0) it holds that C is a top component of $Relax(N)$, and that $N_{\downarrow(P_C,T)}$ is an ord-IMO net (i.e., each transition in C has the source place in P_C and precisely one destination place in P_C).

Proof:
Let $N = (P, T, F)$, M_0 satisfy the assumptions.

For the sake of contradiction we assume that the statement does not hold, and we choose a component C_0 (an scc of $Relax(N)$) such that C_0 violates some of conditions $1-3$, while all proper successors of C_0 (if some exist) do not violate these conditions. Each proper successor C of C_0 is thus a rich component and satisfies $M_0(p) \geq 1$ for all $p \in P_C$ (which is trivial when $P_C = \emptyset$, i.e., when C consists of a single transition). By P_{SUCC} we denote the union of the sets P_C for all proper successors C of C_0 (hence $M_0(p) \geq 1$ for all $p \in P_{\text{SUCC}}$).

We fix two places $p_1, p'_1 \in P_{C_0}$ such that

$$M_0(p_1) \geq 2 \text{ and } M_0(p'_1) = 0; \tag{3}$$

this is obviously possible in the case when C_0 is rich and violates condition 1 as well as in the case when C_0 is poor and violates condition 2.

Since M_0 is self-coverable, we can fix a full sequence $\sigma \in T^*$ (containing all transitions from T) such that $M_0 \xrightarrow{\sigma} \bar{M}$ where $M_0 \leq \bar{M}$; we note that $car(M_0) = car(\bar{M})$, since M_0 is carrier-maximal. We can easily verify that we also have

$$M_0 \xrightarrow{\sigma_1} M_1 \text{ and } M_0 \leq M_1 \text{ (where } car(M_0) = car(M_1)) \tag{4}$$

for σ_1 arising from σ by omitting all occurrences of transitions contained in the proper successors of C_0. This follows from the facts that $M_0(p) \geq 1$ for all $p \in P_{\text{SUCC}}$, and that all destination places of transitions in the proper successors of C_0 are in P_{SUCC}; hence σ_1 is performable from M_0, keeping each place in P_{SUCC} marked during the whole execution $M_0 \xrightarrow{\sigma_1} M_1$.

We get the desired contradiction in both cases C1 and C2 below; the cases cover all possibilities, since each transition in C_0 has at least one destination place in P_{C_0} (which follows from the fact that C_0 does not consist of a single transition).

Case C1. *The (violating) scc C_0 is not a top component of $Relax(N)$, or there is a transition in C_0 that has at least two destination places in P_{C_0}.*

In this case there is some $p_0 \in P_{C_0}$ such that $M_0(p_0) < M_1(p_0)$, referring to (4), since no transition in σ_1 decreases the token count in C_0 while at least one transition in σ_1 increases this count. Since C_0 is a strongly connected component of $Relax(N)$, there must be a path from p_0 to p'_1 in C_0 (referring to p'_1 in (3)); let the sequence of the transitions on this path be $t'_1 t'_2 \cdots t'_m$. Now we can consider the execution

$$M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_{1,1}}\xrightarrow{\sigma_{1,2}} \cdots \xrightarrow{\sigma_{1,m}} M'$$

where $\sigma_{1,j}$ arises from σ_1 by replacing an occurrence of t'_j with $t'_j t'_j$ (the second occurrence of t'_j moves the "excessive" token on its way from p_0 towards p'_1, maybe also generating additional tokens on the way).

We clearly have $car(M') \supseteq car(M_0) \cup \{p'_1\}$, which contradicts the assumption that M_0 is carrier-maximal.

Case C2. *The (violating) scc C_0 is a top component and each transition t in C_0 has (the source place and) precisely one destination place in P_{C_0}.*

Referring to the execution $M_0 \xrightarrow{\sigma_1} M_1$ from (4), we observe that we have $M_0(p) = M_1(p)$ for all $p \in P_{C_0}$ (in our case C2). Though we have $M_0(p_1) \geq 2$, and there is a path from p_1 to p'_1 (we refer to the places from (3)), we cannot immediately say that a token on p_1 is "excessive"; to use an analogous idea as in C1, we first adjust σ_1 by a careful omission of some transition occurrences, to make clear that we actually get an excessive token after all.

In the respective "omitting" construction, for t in C_0 we use the expression $p \to t$ to denote that p is the source place of t (we necessarily have $p \in P_{C_0}$); the expression $t \to p$ denotes that p is the destination place of t that belongs to P_{C_0}.

Now we imagine constructing a sequence

$$seq = (\sigma_1, p_1), (\sigma_2, p_2), \ldots, (\sigma_k, p_k) \tag{5}$$

where $M_0 \xrightarrow{\sigma_1} M_1$ (recall (4)), and p_1 is the place from (3), hence $M_0(p_1) \geq 2$. Given (σ_i, p_i), we construct (σ_{i+1}, p_{i+1}) as follows:

- if there is no transition t in σ_i where $t \to p_i$, then halt (and put $k = i$);
- (otherwise) we write $\sigma_i = \sigma' t \sigma''$ where $t \to p_i$ and there is no t' in σ'' for which $t' \to p_i$, and put $\sigma_{i+1} = \sigma'\sigma''$ (the last transition occurrence putting a token in p_i has been omitted); the source place of t is taken as p_{i+1} (hence we have $p_{i+1} \to t \to p_i$).

We show that the construction of (5) keeps the following conditions $1 - 3$ (for all $i = 1, 2, \ldots, k$); we recall that P_{SUCC} denotes the set of places of the proper successors of C_0.

1. $M_0 \xrightarrow{\sigma_i} M_i$ (for some M_i) where $car(M_0) = car(M_i)$ and $M_i(p_i) \geq 2$;

2. $p_i \in P_{C_0}$ (and we might have $p_i = p_{i'}$ for $i \neq i'$);
3. (a) each place from P_{SUCC} is marked during the whole execution $M_0 \xrightarrow{\sigma_i} M_i$;
   (b) for each $p \in P \smallsetminus (P_{\text{SUCC}} \cup \{p_1, p_i\})$ we have $M_i(p) = M_1(p)$ (hence $M_i(p) = M_0(p)$ for all $p \in P_{C_0} \smallsetminus \{p_1, p_i\}$);
   (c) if $p_i = p_1$, then $M_i(p_1) = M_0(p_1)$;
   (d) if $p_i \neq p_1$, then $M_i(p_1) = M_0(p_1) - 1 \geq 1$ and $M_i(p_i) = M_0(p_i) + 1$.

The conditions surely hold for $i = 1$, since $M_0 \xrightarrow{\sigma_1} M_1$ and $M_1(p) = M_0(p)$ for all $p \in P_{C_0}$; the validity of condition 3(a) was noted around (4). We also note that $p_k = p_1$ (in (5)), since in the case $p_i \neq p_1$ we have $M_0 \xrightarrow{\sigma_i} M_i$ and $M_i(p_i) = M_0(p_i) + 1$, which entails that there is t in σ_i such that $t \to p_i$.

Now we assume that the conditions hold for $i \in [1, k-1]$, and show that they hold for $i+1$ as well. We recall that $\sigma_{i+1} = \sigma'\sigma''$ where $\sigma_i = \sigma' t \sigma''$, $t \to p_i$, and there is no t' in σ'' for which $t' \to p_i$. Let us write

$$M_0 \xrightarrow{\sigma_i} M_i \text{ as } M_0 \xrightarrow{\sigma'} \bar{M}_1 \xrightarrow{t} \bar{M}_2 \xrightarrow{\sigma''} M_i.$$

Since the number of tokens on p_i never increases in the execution $\bar{M}_2 \xrightarrow{\sigma''} M_i$ and $M_i(p_i) \geq 2$, the place p_i is marked with at least two tokens during the whole execution $\bar{M}_2 \xrightarrow{\sigma''} M_i$. We verify that σ'' is enabled at $\bar{M}_1$: we recall that the destination places of t constitute a subset of $\{p_i\} \cup P_{\text{SUCC}}$, $\bar{M}_1(p) \geq 1$ for all $p \in P_{\text{SUCC}}$ (by condition 3(a)), and there is no transition in σ'' with the source place in P_{SUCC} (since σ'' is a subsequence of σ_1 defined in (4)).

In the execution $M_0 \xrightarrow{\sigma'} \bar{M}_1 \xrightarrow{\sigma''} M_{i+1}$ (which is $M_0 \xrightarrow{\sigma_{i+1}} M_{i+1}$) the place p_i is marked with at least one token during the whole segment $\bar{M}_1 \xrightarrow{\sigma''} M_{i+1}$, while each place from P_{SUCC} is marked during the whole execution $M_0 \xrightarrow{\sigma_{i+1}} M_{i+1}$.

Since $M_{i+1} = (M_i - t^\bullet) + {}^\bullet t$, we have:

- if $p_i = p_{i+1}$ (i.e., $p_i \to t \to p_i$), then $(M_{i+1})_{\downarrow(P \smallsetminus P_{\text{SUCC}})} = (M_i)_{\downarrow(P \smallsetminus P_{\text{SUCC}})}$;
- if $p_i \neq p_{i+1}$, then
  - $M_{i+1}(p) = M_i(p)$ for all $p \in P \smallsetminus (P_{\text{SUCC}} \cup \{p_i, p_{i+1}\})$,
  - $M_{i+1}(p_i) = M_i(p_i) - 1$, and
  - $M_{i+1}(p_{i+1}) = M_i(p_{i+1}) + 1$.

Since $M_i(p_i) \geq 2$, and thus $M_{i+1}(p_i) \geq 1$, we have $car(M_i) \subseteq car(M_{i+1})$. The condition $car(M_i) = car(M_0)$ and the assumption that M_0 is carrier-maximal thus entail that $car(M_{i+1}) = car(M_0)$; hence $M_i(p_{i+1}) \geq 1$ and $M_{i+1}(p_{i+1}) \geq 2$.

The validity of conditions 1–3 is thus clear for all $i \in [1, k]$. We note that the condition $car(M_i) = car(M_0)$ entails that $M_i(p'_1) = M_0(p'_1) = 0$ (referring to p'_1 from (3)), and thus $p_i \neq p'_1$.

The final element (σ_k, p_k) of the sequence (5) satisfies $p_1 = p_k$, $M_0 \xrightarrow{\sigma_k} M_k$ where $M_0 \leq M_k$, and p_1 cannot be a destination place, and neither the source place, of any transition in σ_k. Since $M_0(p_1) \geq 2$, we get that $M'_0 \xrightarrow{\sigma_k} M'_k$ where M'_0 arises from M_0, and M'_k from M_k, by removing a token on p_1. This entails that there are executions $M_i \xrightarrow{\sigma_k} \bar{M}_i$ (where $M_i \leq \bar{M}_i$) for all $i = 1, 2, \ldots, k$ (since M_i arises from M'_0 by adding a token on p_i).

Since σ_k has arisen from σ_1, in which all transitions from C_0 occur, for some $i \in [1, k]$ there must be a path from p_i to p'_1 (in C_0) such that all transitions on the path occur in σ_k; let the sequence of the transitions on this path be $t'_1 t'_2 \cdots t'_m$. Now we can consider the execution

$$M_0 \xrightarrow{\sigma_i} M_i \xrightarrow{\sigma_{k,1}}\xrightarrow{\sigma_{k,2}} \cdots \xrightarrow{\sigma_{k,m}} M'$$

where $\sigma_{k,j}$ arises from σ_k by replacing an occurrence of t'_j with $t'_j t'_j$ (the second occurrence of t'_j moves the "excessive" token on its way from p_i towards p'_1). We clearly have $car(M') \supseteq car(M_i) \cup \{p'_1\} = car(M_0) \cup \{p'_1\}$, which contradicts the assumption that M_0 is carrier-maximal. The proof is thus finished. □

Now we recall a natural notion and a simple fact (for general nets).

DL-markings. A marking M of a (general) net N is a *DL-marking* if each transition is either dead or live at M and at least one transition is dead at M.

Proposition 5.4. (From a non-live marking we can reach a DL-marking)

1. A marking M of a net N is non-live iff there is a DL-marking $M_{\text{DL}} \in [M\rangle$.
2. Given a net $N = (P, T, F)$, if M_{DL} is a DL-marking of N and $T_L \subseteq T$ is the set of transitions that are live at M_{DL}, then M_{DL} is live in the net $N_{\downarrow(P,T_L)}$.

Proof:
The claim 2 and the "if" part of the claim 1 are trivial, hence it remains to look at the "only-if" part of the claim 1. If t is a transition that is non-live at M, then there is $M' \in [M\rangle$ where t is dead; if t' is a transition that is non-live at M', then there is $M'' \in [M'\rangle$ where t' is dead, as well as t. By repeating this reasoning we arrive at some DL-marking $M_{\text{DL}} \in [M\rangle$, for any non-live M. □

Proof of Lemma 4.2 (the "only-if" direction).
Let M_0 be a non-live marking of an ord-BIMO net $N = (P, T, F)$. By Proposition 5.4(1) there is a DL-marking in $[M_0\rangle$; by Propositions 5.4(2) and 5.1 there is even a DL-marking $M_{\text{DL}} \in [M_0\rangle$ that is optimal (carrier-maximal and self-coverable) in $N_{\downarrow(P,T_L)}$ where T_L is the set of transitions of N that are live at M_{DL}. (The marking M_{W} depicted in Figure 5 is an example of such marking.)

Let us choose such an optimal M_{DL}, and let P_{CR} be the union of places from the strongly connected components of $Relax(N_{\downarrow(P,T_L)})$ that are poor at M_{DL} (hence we surely have $|(M_{\text{DL}})_{\downarrow P_{\text{CR}}}| < |P_{\text{CR}}|$). By Lemma 5.3 we derive that $M_{\text{DL}}(p) \geq 1$ for all $p \in P \smallsetminus P_{\text{CR}}$, $M_{\text{DL}}(p) \in \{0, 1\}$ for all $p \in P_{\text{CR}}$, and $N_{\downarrow(P_{\text{CR}},T_L)}$ is an ord-IMO net. We also note that each transition $t : p_s \xrightarrow{\{p_{o_1},\ldots,p_{o_\ell}\}} \{p_{d_1}, \ldots, p_{d_k}\}$

where $p_s \in P \smallsetminus P_{\text{CR}}$ satisfies $\{p_{d_1}, \ldots, p_{d_k}\} \subseteq P \smallsetminus P_{\text{CR}}$ (if the source place is outside P_{CR}, then all destination places are outside P_{CR}).

To finish proving Lemma 4.2, it suffices to show that the transitions from the set $T_{\text{D}} = T \smallsetminus T_L$ (consisting of the transitions that are dead at M_{DL} in N) are dead in $(N_{\downarrow(P_{\text{CR}},T)}, (M_{\text{DL}})_{\downarrow P_{\text{CR}}})$. For the sake of contradiction we assume that $(M_{\text{DL}})_{\downarrow P_{\text{CR}}} \xrightarrow{\sigma} M$ ($M : P_{\text{CR}} \to \mathbb{N}$) in $N_{\downarrow(P_{\text{CR}},T)}$ where $\sigma \in (T_L)^*$ and $M \geq ({}^\bullet t \cap P_{\text{CR}})$ for some $t \in T_{\text{D}}$. We recall that $M_{\text{DL}}(p) \geq 1$ for all $p \in P \smallsetminus P_{\text{CR}}$, hence for σ' arising from σ by omitting the transitions whose source place is outside P_{CR} (and whose destination places are thus outside P_{CR} as well) we get $M_{\text{DL}} \xrightarrow{\sigma'} M'$ in N. We have $M' \geq {}^\bullet t$, since M' coincides with M on P_{CR}, and $M'(p) \geq 1$ for all $p \in P \smallsetminus P_{\text{CR}}$; but this contradicts the assumption that t is dead at M_{DL} in N. □

6. Upper bounds for ordinary nets

In Section 6.1 we show an important lemma that clarifies the main bounds in Table 1 (in Section 2.3) for ordinary nets; to this aim we also have a more detailed look at executions by viewing tokens as individual objects. In Section 6.2 we give the strengthened bound for structurally live ord-IMO nets, showing that here there are live markings having at most one token in each place; in Section 6.3 we show that this does not hold for ord-BIO nets.

6.1. Live markings in ord-BIMO nets

In this section we prove the next lemma, which has a straightforward consequence for live markings in ord-BIMO nets.

Lemma 6.1. Let $N = (P, T, F)$ be an ord-BIMO net, and let M, M' be two markings of N that coincide except of one place p_0 for which we have $2|P| - 1 \leq M(p_0) = M'(p_0) - 1$. Then M is live iff M' is live.

Corollary 6.2. Let $N = (P, T, F)$ be an ord-BIMO net. The set $\mathcal{L}_N$ of live markings of N is determined by its finite (basic) subset

$$\mathcal{B}_N = \{M \in \mathcal{L}_N \mid M(p) \leq B \text{ for all } p \in P\} \text{ where } B = 2|P|-1,$$

since

$$\mathcal{L}_N = \{M \in \mathbb{N}^P \mid M' \in \mathcal{B}_N \text{ where } M'(p) = M(p) \text{ if } M(p) \leq B, \text{ and } M'(p) = B \text{ otherwise}\}.$$

Remark 6.3. We note that liveness is not generally monotonic, even for ord-IO nets. Figure 7 shows an example of an ord-IO net with a live marking (as can be easily verified); adding a token to p_2 yields a non-live marking, since in this case firing $t_5t_2t_6$ makes all transitions dead. Another example is the ord-BIMO net in Figure 6; we can check that the markings $(1,0,0,0,0)$ and $(3,0,0,0,0)$ are non-live, while $(2,0,0,0,0)$ and $(3,0,0,0,1)$ are live.

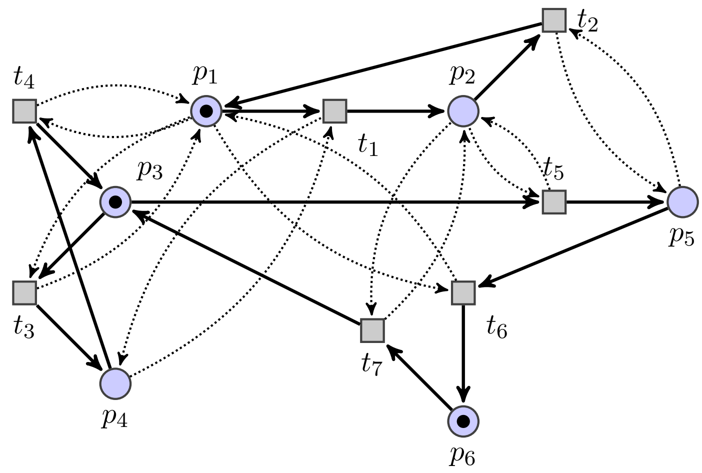

Figure 7. A live marking of an ord-IO net N; adding a token on p_2 makes it non-live.

Executions with individual tokens, ID-valuations, relations $\overset{t}{\rightsquigarrow}$, $\rightsquigarrow$, $\rightsquigarrow^*$.

Let us have a more detailed look at a fixed execution

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_m} M_m \tag{6}$$

for an ord-BIMO net $N = (P, T, F)$. Now we view tokens as individual objects; each token is identified with its unique ID from an infinite domain IDSET (e.g., we can have IDSET $= \mathbb{N}$). A marking $M : P \to \mathbb{N}$ thus has a related *id-marking* $\bar{M} : P \to \mathcal{P}_{\text{FIN}}(\text{IDSET})$, where the finite sets $\bar{M}(p)$ and $\bar{M}(p')$ of IDs are disjoint when $p \neq p'$ (and $|\bar{M}(p)| = |M(p)|$ for all $p \in P$). We also define the notion of *an ID-valuation of the execution* (6), as a sequence

$$\bar{M}_0 \xrightarrow{t_1} \bar{M}_1 \xrightarrow{t_2} \bar{M}_2 \cdots \xrightarrow{t_m} \bar{M}_m$$

that is stepwise created as follows: We start with attaching a unique ID to each token in the initial marking M_0, getting $\bar{M}_0 : P \to \mathcal{P}_{\text{FIN}}(\text{IDSET})$. For $i \in [1, m]$, $\bar{M}_i$ arises from $\bar{M}_{i-1}$ by the following change, based on the transition $t_i : p_s \xrightarrow{\{p_{o_1}, \ldots, p_{o_\ell}\}} \{p_{d_1}, \ldots, p_{d_k}\}$:

- one (arbitrarily chosen) ID $\text{I}_0 \in \bar{M}_{i-1}(p_s)$ is removed (from the source place p_s), and it is not used anymore (i.e., I_0 will not occur in $\bar{M}_j(p)$ for any $j \in [i, m]$ and $p \in P$);
- for each $j \in [1, k]$, a fresh ID I_j (so far not occurring in the created ID-valuation) is added to the destination place p_{d_j};
- moreover, we say that each of the "destination IDs" I_j is an *immediate successor of* the "source ID" I_0, which is denoted by $\text{I}_0 \rightsquigarrow \text{I}_{d_j}$; in more detail we also write $\text{I}_0 \overset{t_i}{\rightsquigarrow} \text{I}_{d_j}$.

(The "observation IDs" on the places p_{o_j} used by t_i are not changed, being the same in $\bar{M}_i$ as in $\bar{M}_{i-1}$.) We note that if the source token from p_s returns, i.e. $p_s \in \{p_{d_1}, \ldots, p_{d_k}\}$, then the returning token has a new ID, which is an immediate successor of the original one.

We have thus defined what we mean by an ID-valuation $\bar{M}_0 \xrightarrow{t_1} \bar{M}_1 \xrightarrow{t_2} \bar{M}_2 \cdots \xrightarrow{t_m} \bar{M}_m$ of the execution (6), while we have also introduced the *immediate-successor relation* $\rightsquigarrow$ on the set of IDs used in this ID-valuation. We also define the *successor relation* $\rightsquigarrow^*$, as the reflexive and transitive closure of $\rightsquigarrow$.

The relation $\rightsquigarrow^*$ is clearly a partial order that can be presented by a set of (disjoint) trees where the roots constitute the set $\bigcup_{p \in P} \bar{M}_0(p)$; this is captured by the next observation.

Fact 6.4. (Each used ID is a successor of a unique ID from $\bar{M}_0$)
Given an ID-valuation $\bar{M}_0 \xrightarrow{t_1} \bar{M}_1 \xrightarrow{t_2} \bar{M}_2 \cdots \xrightarrow{t_m} \bar{M}_m$ (of an execution $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_m} M_m$), for each triple $i \in [0, m]$, $p \in P$, and $\text{I} \in \bar{M}_i(p)$ there is a unique place $p' \in P$ and a unique ID $\text{I}' \in \bar{M}_0(p')$ such that $\text{I}' \rightsquigarrow^* \text{I}$.

The next two propositions entail Lemma 6.1.

Proposition 6.5. Let $N = (P, T, F)$ be an ord-BIMO net, and let M_0, M'_0 be two markings of N that coincide except of one place p_0 for which we have $|P| \leq M_0(p_0) = M'_0(p_0) - 1$. If (N, M_0) is non-live then (N, M'_0) is non-live as well.

Proof:
Let N, M_0, M'_0, p_0 be as in the statement, and let us assume that (N, M_0) is non-live; we aim to show that (N, M'_0) is non-live as well.

By Lemma 4.2 there are an execution $M_0 \xrightarrow{\sigma} M_\text{W}$, $P_\text{CR} \subseteq P$, and a nonempty set $T_\text{D} \subseteq T$ such that $|(M_\text{W})_{\downarrow P_\text{CR}}| < |P_\text{CR}|$, and all transitions from T_D are dead in $(N_{\downarrow(P_\text{CR},T)}, (M_\text{W})_{\downarrow P_\text{CR}})$ (and thus also in (N, M_W) by Proposition 4.1). We call the *places in P_CR crucial*, while the places in $P \smallsetminus P_\text{CR}$ are *don't care places*.

Let the execution

$$M_0 \xrightarrow{\sigma} M_\text{W} \text{ be of the form } M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_m} M_m = M_\text{W},$$

and let us fix an ID-valuation

$$\bar{M}_0 \xrightarrow{t_1} \bar{M}_1 \xrightarrow{t_2} \bar{M}_2 \cdots \xrightarrow{t_m} \bar{M}_m \tag{7}$$

of this execution. We aim to show an execution $M'_0 \xrightarrow{\sigma'} M'$ where $(M')_{\downarrow P_\text{CR}} = (M_\text{W})_{\downarrow P_\text{CR}}$, which demonstrates that (N, M'_0) is non-live.

We say that an ID I in $\bar{M}_0(p)$ is *black* (or *crucial*) if there is a place $p' \in P_\text{CR}$ and some $\text{I}' \in \bar{M}_m(p')$ such that $\text{I} \rightsquigarrow^* \text{I}'$ (hence if the initial individual token I has a successor on a crucial place in M_W). Since there are at most $|P| - 1$ tokens on the crucial places in M_W, by Fact 6.4 we deduce that there are at most $|P| - 1$ black IDs in $\bigcup_{p \in P} \bar{M}_0(p)$. Since $M_0(p_0) \geq |P|$, at least one ID in $\bar{M}_0(p_0)$ is not black; we fix one non-black ID $\text{I}_0 \in \bar{M}_0(p_0)$, and we further view I_0 as *red*. Moreover, if $\text{I} \overset{t_i}{\rightsquigarrow} \text{I}'$ and I is viewed as red, then we also view I' as red, and we also call the *transition occurrence t_i red*. Hence the red IDs are precisely the successors of I_0, and

$$\sigma = t_1 t_2 \cdots t_m \text{ can be written as } \sigma_0 t'_1 \sigma_1 t'_2 \sigma_2 \cdots t'_r \sigma_r$$

where $\sigma_i \in T^*$ (for $i \in [0, r]$) and t'_i, $i \in [1, r]$, are precisely the red transition occurrences.

We put

$$\sigma' = \sigma_0 t'_1 t'_1 \sigma_1 t'_2 t'_2 \sigma_2 \cdots t'_r t'_r \sigma_r \tag{8}$$

(each red transition occurrence in σ has been doubled), and we finish the overall proof by the following claim, which entails that (N, M'_0) is non-live (as we have aimed to prove).

Claim. There is an execution $M'_0 \xrightarrow{\sigma'} M'$ (for σ' in (8)), and $(M')_{\downarrow P_{\text{CR}}} = (M_{\text{W}})_{\downarrow P_{\text{CR}}}$.

The claim thus entails that all $t \in T_{\text{D}}$ are dead in (N, M'), by another use of Proposition 4.1. To prove the claim, we recall that M'_0 differs from M_0 just by an additional token on p_0, and we construct an ID-valuation

$$\bar{M}'_0 \xrightarrow{\sigma_0 t'_1 t'_1 \sigma_1 t'_2 t'_2 \sigma_2 \cdots t'_r t'_r \sigma_r} \bar{M}'$$

by adjusting the above ID-valuation (7) so that each red ID gets a "twin" ID as follows:

- $\bar{M}'_0$ arises from $\bar{M}_0$ by equipping the additional token on p_0 with a fresh ID I'_0 that is viewed as a *twin of the red ID* I_0;
- for each $j \in [1, r]$, when the (red) I is the source ID of the original (red) transition occurrence t'_j, then the twin ID of I is the source of the added occurrence t'_j; moreover, each (red) destination ID of the original t'_j gets a twin due to the added t'_j.

It is clear that $\bar{M}'$ differs from $\bar{M}_m$ only so that each red ID in $\bar{M}_m(p)$ has an additional twin in $\bar{M}'(p)$ (for each $p \in P$). Since all successors of the red ID I_0 in $\bar{M}_m$ are on the don't care places (i.e., outside the set P_{CR} of crucial places), all successors of its twin ID I'_0 are in $\bar{M}'$ on the don't care places as well; this entails that $(M')_{\downarrow P_{\text{CR}}} = (M_{\text{W}})_{\downarrow P_{\text{CR}}}$. □

Remark 6.6. We note that the main idea of the proof of the next proposition (Proposition 6.7) is related to the proof idea of Replacement Lemma in [9], though it is presented in a different technical framework.

Proposition 6.7. Let $N = (P, T, F)$ be an ord-BIMO net, and let M_0, M'_0 be two markings of N that coincide except of one place p_0 for which we have $2|P| - 1 \le M'_0(p_0) = M_0(p_0) - 1$. If (N, M_0) is non-live then (N, M'_0) is non-live as well.

Proof:
Let N, M_0, M'_0, p_0 be as in the statement, and let us assume that (N, M_0) is non-live; we aim to show that (N, M'_0) is non-live as well. (Now $M'_0(p_0)$ arises from $M_0(p_0)$ by removing a token, while in Proposition 6.5 one token was added.)

We fix $M_0 \xrightarrow{\sigma} M_{\text{W}}$, $P_{\text{CR}} \subseteq P$, and $T_{\text{D}} \subseteq T$ guaranteed by Lemma 4.2 (in particular we recall that $|(M_{\text{W}})_{\downarrow P_{\text{CR}}}| < |P_{\text{CR}}|$). Similarly as in the proof of Proposition 6.5, we aim to show an execution $M'_0 \xrightarrow{\sigma'} M'$ where $(M')_{\downarrow P_{\text{CR}}} = (M_{\text{W}})_{\downarrow P_{\text{CR}}}$, by which the proof will be finished. We assume that $M_0 \xrightarrow{\sigma} M_{\text{W}}$ is of the form

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_m} M_m = M_{\text{W}},$$

and for this execution we fix an ID-valuation

$$\bar{M}_0 \xrightarrow{t_1} \bar{M}_1 \xrightarrow{t_2} \bar{M}_2 \cdots \xrightarrow{t_m} \bar{M}_m. \tag{9}$$

We view an ID I in $\bar{M}_0(p)$ as *black* if there is a place $p' \in P_{\text{CR}}$ and some $\text{I}' \in \bar{M}_m(p')$ such that $\text{I} \rightsquigarrow^* \text{I}'$; hence there are at most $|P_{\text{CR}}|-1$ black IDs in the set $\bigcup_{p\in P} \bar{M}_0(p)$. Now $M_0(p_0) \geq 2|P|$, and we thus have at least $|P|+1$ non-black IDs in $\bar{M}_0(p_0)$ (since $2|P| - (|P_{\text{CR}}| - 1) \geq |P|+1$). We view all non-black IDs in $\bar{M}_0(p_0)$ as *red*, and all their successors (in (9)) are viewed as red as well; this also determines which *transition occurrences* t_i in (9) are *red*. We note that each red transition occurrence changes the distribution of red IDs while not affecting non-red IDs; on the other hand, each non-red transition occurrence does not affect red IDs (though it might need red IDs on its observation places).

We write $\sigma = t_1 t_2 \cdots t_m$ as $\sigma_0 t'_1 \sigma_1 t'_2 \sigma_2 \cdots t'_r \sigma_r$ where t'_i, $i \in [1, r]$, are precisely the red transition occurrences in (9), and we also present (9) as

$$\bar{M}^{\text{R}}_0 \xrightarrow{\sigma_0 t'_1} \bar{M}^{\text{R}}_1 \xrightarrow{\sigma_1 t'_2} \bar{M}^{\text{R}}_2 \cdots \xrightarrow{\sigma_{r-1} t'_r} \bar{M}^{\text{R}}_r \xrightarrow{\sigma_r} \bar{M}^{\text{R}}_{r+1} \tag{10}$$

where $\bar{M}^{\text{R}}_0 = \bar{M}_0$ and $\bar{M}^{\text{R}}_{r+1} = \bar{M}_m$. (The superscript R in $\bar{M}^{\text{R}}_i$ is just a symbol that might be viewed as referring to "oRiginal" id-markings, which differ from the constructed id-markings $\bar{M}'_i$ in (12).) We observe that $\bar{M}^{\text{R}}_r$ and $\bar{M}^{\text{R}}_{r+1}$ have the same distribution of red IDs, since the segment $\bar{M}^{\text{R}}_r \xrightarrow{\sigma_r} \bar{M}^{\text{R}}_{r+1}$ does not contain any red transition occurrence; in particular, in both $\bar{M}^{\text{R}}_r$ and $\bar{M}^{\text{R}}_{r+1}$ there are no red IDs on the "crucial" places, i.e. on the places from P_{CR}.

We recall that we aim to construct a suitable execution $M'_0 \xrightarrow{\sigma'} M'$; we will have

$$\sigma' = \sigma_0 (t'_1)^{n_1} \sigma_1 (t'_2)^{n_2} \sigma_2 \cdots (t'_r)^{n_r} \sigma_r \tag{11}$$

for certain multiplicities $n_1, n_2, \ldots, n_r$ (also allowing $n_i = 0$). In the proof of Proposition 6.5 we had $n_i = 2$ for all $i \in [1, r]$ but here (when $M'_0(p_0) = M_0(p_0) - 1$) the situation is more complicated. The idea is that we aim to modify (10) so that we get

$$\bar{M}'_0 \xrightarrow{\sigma_0 (t'_1)^{n_1}} \bar{M}'_1 \xrightarrow{\sigma_1 (t'_2)^{n_2}} \bar{M}'_2 \cdots \xrightarrow{\sigma_{r-1} (t'_r)^{n_r}} \bar{M}'_r \xrightarrow{\sigma_r} \bar{M}'_{r+1} \tag{12}$$

where $\bar{M}'_0$ arises from $\bar{M}_0$ (i.e., from $\bar{M}^{\text{R}}_0$) by removing one red ID from p_0 (hence at least $|P|$ red IDs remain in $\bar{M}'_0(p_0)$), and where the red transition occurrences are used so that (12) is an ID-valuation of a desired execution $M'_0 \xrightarrow{\sigma'} M'$. In (12) we will also view as red precisely those IDs that are successors of the red IDs in $\bar{M}'_0(p_0)$. We will construct (12) so that the red transition occurrences (changing just the distribution of red IDs) will be precisely those in the segments $(t'_i)^{n_i}$ (for all $i \in [1, r]$); each non-red transition occurrence, in a segment σ_i ($i \in [0, r]$), will cause the same ID-change as its corresponding non-red transition occurrence in (10). More concretely, we aim to choose the multiplicities n_i so that σ' (defined in (11)) is performable from M'_0 and allows us to construct a respective ID-valuation (12) so that the following conditions hold for all $i \in [0, r+1]$:

1. the id-marking $\bar{M}'_i$ in (12) coincides with $\bar{M}^{\text{R}}_i$ in (10) when the red IDs are ignored;

2. for each $p \in P$: there is at least one red ID in $\bar{M}'_i(p)$ in (12) if, and only if, in (10) there is a red ID in some of the sets $\bar{M}^{\mathrm{R}}_0(p), \bar{M}^{\mathrm{R}}_1(p), \ldots, \bar{M}^{\mathrm{R}}_i(p)$ and there is a red ID in some of the sets $\bar{M}^{\mathrm{R}}_i(p), \bar{M}^{\mathrm{R}}_{i+1}(p), \ldots, \bar{M}^{\mathrm{R}}_{r+1}(p)$.

Hence in $\bar{M}'_i(p)$ in (12) we aim to remember, by a presence of a red ID, if in the corresponding situation in (10) there has been a red ID on p in the past or at present; but there should be no red ID in $\bar{M}'_i(p)$ if in the corresponding situation in (10) there is no red ID on p at present and in the future.

Conditions 1 and 2 clearly hold in the case $i = 0$, by our definition of $\bar{M}'_0$ (arising from $\bar{M}^{\mathrm{R}}_0$). Moreover, if conditions 1 and 2 hold in the case $i = r{+}1$ (and σ' is performable from M'_0), then we are done, since in this case $(\bar{M}'_{r+1})_{\downarrow P_{\mathrm{CR}}} = (\bar{M}^{\mathrm{R}}_{r+1})_{\downarrow P_{\mathrm{CR}}}$ (due to the fact that there are no red IDs in $\bigcup_{p \in P_{\mathrm{CR}}} \bar{M}^{\mathrm{R}}_{r+1}(p)$), which entails $(M')_{\downarrow P_{\mathrm{CR}}} = (M_{\mathrm{W}})_{\downarrow P_{\mathrm{CR}}}$.

Informally speaking, at most $|P|$ red IDs will suffice to serve as the required "memory"; we will create them from the successors of at most $|P|$ red IDs in $\bar{M}'_0(p_0)$ (recall that $\bar{M}'_0(p_0)$ contains at least $|P|$ red IDs). But we should also take care of the required "cleaning" (to get $(M')_{\downarrow P_{\mathrm{CR}}} = (M_{\mathrm{W}})_{\downarrow P_{\mathrm{CR}}}$); e.g., if $p_0 \in P_{\mathrm{CR}}$, then we have to remove all red IDs from p_0.

To define the multiplicities n_i in (12) rigorously, we first introduce a few technical notions. Referring to (10), for $i \in [0, r{+}1]$ we put

$$\mathrm{R}_i = \{p \in P \mid \bar{M}^{\mathrm{R}}_i(p) \text{ contains a red ID}\}.$$

As we observed after defining (10), we have $\mathrm{R}_r = \mathrm{R}_{r+1}$, and $\mathrm{R}_r \cap P_{\mathrm{CR}} = \emptyset$. We define "first-red" sets and "last-red" sets for all $i \in [0, r{+}1]$:

- $\mathrm{FR}_i = \{p \in P \mid p \in \mathrm{R}_i \text{ and } p \notin \mathrm{R}_0 \cup \mathrm{R}_1 \cdots \cup \mathrm{R}_{i-1}\}$;
- $\mathrm{LR}_i = \{p \in P \mid p \in \mathrm{R}_i \text{ and } p \notin \mathrm{R}_{i+1} \cup \mathrm{R}_{i+2} \cdots \cup \mathrm{R}_{r+1}\}$.

We note that $\mathrm{FR}_0 = \{p_0\}$, the sets FR_i ($i \in [0, r{+}1]$) are pairwise disjoint (hence $|\bigcup_{i \in [0,r+1]} \mathrm{FR}_i| \leq |P|$), and for $i \in [1, r]$ each place in FR_i is a destination place of t'_i (which holds trivially in the case $\mathrm{FR}_i = \emptyset$). For $i \in [0, r{-}1]$, if the set LR_i is nonempty then it is a singleton consisting of the source place of t'_{i+1}. For $i \in [1, r]$, we say that t'_i in (10) is

- a *first-red-destination transition-occurrence*, a *first-rdto* for short, if $\mathrm{FR}_i \neq \emptyset$;
- a *last-red-source transition-occurrence*, a *last-rsto* for short, if $\mathrm{LR}_{i-1} \neq \emptyset$.

Since the sets FR_i are pairwise disjoint and $\mathrm{FR}_0 = \{p_0\}$, there are at most $|P|{-}1$ first-rdtos. We define a *causality relation* $\lhd$ *on the set of first-rdtos*:

$$t'_i \lhd t'_j \text{ if the source place of } t'_j \text{ belongs to } \mathrm{FR}_i \text{ (and thus is a destination place of } t'_i).$$

Hence each first-rdto t'_i either has p_0 as the source place, in which case there is no j such that $t'_j \lhd t'_i$, or there is precisely one j ($j < i$) such that $t'_j \lhd t'_i$. Viewing the relation $\lhd$ as a directed graph, it has the form of a forest (a set of directed disjoint trees). By $\lhd^*$ we denote the reflexive and transitive closure of $\lhd$, and for each first-rdto t'_i we put

$$\mathrm{POST}_{\lhd^*}(t'_i) = \{j \mid t'_i \lhd^* t'_j\}.$$

We define the numbers n_i, $i \in [1, r]$, in (12) as follows, depending on t'_i in (10):

- if t'_i is a first-rdto but not a last-rsto, then $n_i = |\text{POST}_{\lhd^*}(t'_i)|$;
- if t'_i is a last-rsto, with the source place p_s, then $n_i = |\{\text{I} \mid \text{I} \text{ is a red ID in } \bar{M}'_{i-1}(p_s)\}|$;
- if t'_i is neither a first-rdto nor a last-rsto, then $n_i = 0$.

We note that we have not excluded that some t'_i in (10) is both a first-rdto (some of its destination places gets a red ID for the first time) and a last-rsto (its source place gets rid of red IDs for the rest of the execution). We also note that $n_i \geq 1$ for each first-rdto t'_i.

To show the validity of the above conditions 1 and 2, it is useful to add the following condition, for the cases $i \in [1, r]$:

3. If t'_i is a first-rdto (in (10)) and p_s is its source place, then

$$\text{the set } \bar{M}'_{i-1}(p_s) \text{ (in (12)) contains at least } 1 + |\text{POST}_{\lhd^*}(t'_i)| \text{ red IDs.}$$

This condition should thus hold also in the case when t'_i is both a first-rdto and a last-rsto.

Now we show that the chosen n_i indeed fulfill our goals, i.e., σ' (in (11)) is performable from M'_0 and we construct a respective ID-valuation (12) so that conditions $1-3$ are satisfied for all $i \in [0, r+1]$.

We first assume that condition 3 holds for all $i \in [1, r]$, and under this assumption we show that 1 and 2 are then satisfied for all $i = 0, 1, \ldots, r+1$; we use an induction on i. We have already noted that 1 and 2 are satisfied for $i = 0$ (where condition 3 does not apply). In the induction step we fix $j \in [0, r]$ and assume that $\sigma_0(t'_1)^{n_1}\sigma_1(t'_2)^{n_2} \cdots \sigma_{j-1}(t'_j)^{n_j}$ is performable from M'_0 and conditions 1 and 2 are satisfied for $i = j$, in the so far constructed ID-valuation $\bar{M}'_0 \xrightarrow{\sigma_0(t'_1)^{n_1}} \bar{M}'_1 \cdots \xrightarrow{\sigma_{j-1}(t'_j)^{n_j}} \bar{M}'_j$. To extend the validity to the case $i = j + 1$, we first define the segment $\bar{M}'_j \xrightarrow{\sigma_j} \bar{M}'_{\text{A}}$ in (12) by mimicking the segment $\bar{M}^{\text{R}}_j \xrightarrow{\sigma_j} \bar{M}^{\text{R}}_{\text{A}}$ in (10); the substrict A is just auxiliary. Both segments thus perform the same (non-red) ID-changes, which is possible due to conditions 1 and 2 for the case $i = j$; in particular we note that for each red ID used in observation places in the segment $\bar{M}^{\text{R}}_j \xrightarrow{\sigma_j} \bar{M}^{\text{R}}_{\text{A}}$ there is a respective red ID that can be used in $\bar{M}'_j \xrightarrow{\sigma_j} \bar{M}'_{\text{A}}$. In the case $j = r$ we have $\bar{M}'_{\text{A}} = \bar{M}'_{r+1}$, and 1 and 2 obviously hold for $i = j+1$ as well. If $j < r$, then conditions 1 and 2 for $i = j$ and condition 3 for $i = j+1$ (which is so far just assumed) guarantee that we can add a segment $\bar{M}'_{\text{A}} \xrightarrow{(t'_{j+1})^{n_{j+1}}} \bar{M}'_{j+1}$ in which all transition occurrences are red. Moreover, both conditions 1 and 2 are satisfied for $i = j+1$:

- Condition 1 follows from the fact that the segment $\bar{M}'_j \xrightarrow{\sigma_j} \bar{M}'_{\text{A}}$ mimics the segment $\bar{M}^{\text{R}}_j \xrightarrow{\sigma_j} \bar{M}^{\text{R}}_{\text{A}}$ (making precisely the same ID-changes for non-red IDs), while the segments $\bar{M}^{\text{R}}_{\text{A}} \xrightarrow{t'_{j+1}} \bar{M}^{\text{R}}_{j+1}$ and $\bar{M}'_{\text{A}} \xrightarrow{(t'_{j+1})^{n_{j+1}}} \bar{M}'_{j+1}$ affect only red IDs.

- If t'_{j+1} is a first-rdto but not a last-rsto (in (10)), then $n_{j+1} \geq 1$ and performing $(t'_{j+1})^{n_{j+1}}$ in (12) leaves at least one red ID in the source place of t'_{j+1}; hence condition 2 is surely kept.
- If t'_{j+1} is a first-rdto and a last-rsto, then $n_{j+1} \geq 1$ and performing $(t'_{j+1})^{n_{j+1}}$ in (12) leaves no red ID in the source place of t'_{j+1}; hence condition 2 is kept as well.
- If t'_{j+1} is not a first-rdto (while it might be or not be a last-rsto), keeping condition 2 is also obvious.

It remains to deal with condition 3, which should hold for all $i \in [1, r]$ for which t'_i are first-rdtos; we recall that there are at most $|P| - 1$ first-rdtos. Now we use an induction based on the relation $\triangleleft$ (defined on the set of first-rdtos); we recall that $\triangleleft$ can be naturally viewed as a set of disjoint directed trees. We fix the source place p of some first-rdto t'_i and define

$$\mathrm{S}_p = \{i \in [1, r] \mid t'_i \text{ is a first-rdto in (10) for which } p \text{ is the source place}\}.$$

Let $\mathrm{S}_p = \{i_1, i_2, \dots, i_k\}$ where $i_1 < i_2 \cdots < i_k$. We note that if there exists a last-rsto t'_j whose source place is p, then $i_k \leq j$. We have either $p = p_0$, or there is a unique i_0 ($i_0 < i_1$) such that $t'_{i_0} \triangleleft t'_i$ for all $i \in \mathrm{S}_p$ (in which case p is a destination place of t'_{i_0}, belonging to FR_{i_0}). We deal with these two cases below (implicitly using the fact that the definition of $\triangleleft$ entails that the sets $\textsc{Post}_{\triangleleft^*}(t'_{i_1})$, $\textsc{Post}_{\triangleleft^*}(t'_{i_2})$, ..., $\textsc{Post}_{\triangleleft^*}(t'_{i_k})$ are pairwise disjoint):

- $p = p_0$ (basis of our induction)

  Since $\sum_{i \in \mathrm{S}_{p_0}} |\textsc{Post}_{\triangleleft^*}(t'_i)| \leq |P| - 1$ and there are at least $|P|$ red IDs in $\bar{M}'_0(p_0)$, condition 3 clearly holds for all $i \in \mathrm{S}_{p_0}$.

- $t'_{i_0} \triangleleft t'_i$ for all $i \in \mathrm{S}_p$ (induction step, assuming that condition 3 holds for i_0)

  We note that $\bar{M}'_{i_0}(p)$ contains n_{i_0} red IDs, hence at least $|\textsc{Post}_{\triangleleft^*}(t'_{i_0})|$ red IDs. (Here we use the inductive assumption for i_0, which guaranteed that in the segment $(t'_{i_0})^{n_{i_0}}$ of (12) we could indeed make all transition occurrences red.) Since

  $$n_{i_0} \geq |\textsc{Post}_{\triangleleft^*}(t'_{i_0})| = 1 + \sum_{i, t'_{i_0} \triangleleft t'_i} |\textsc{Post}_{\triangleleft^*}(t'_i)| \geq 1 + \sum_{i \in \mathrm{S}_p} |\textsc{Post}_{\triangleleft^*}(t'_i)|,$$

  condition 3 holds for all $i \in \mathrm{S}_p = \{i_1, i_2, \dots, i_k\}$.

□

6.2. An ord-IMO net is structurally live iff there is a live $\{0, 1\}$-marking

In this section we prove the following fact for ord-IMO nets (thus proving another part of Theorem 2.5 that refers to Table 1); Section 6.3 shows that this does not hold for ord-BIO nets.

Lemma 6.8. (For structural liveness of ord-IMO nets the $\{0, 1\}$ markings are decisive)
An ord-IMO net $N = (P, T, F)$ is structurally live iff there is a marking $M_0 : P \to \{0, 1\}$ such that (N, M_0) is live.

Proof:
The "if" direction is trivial, so we now deal with the "only-if" direction. For the sake of contradiction we assume a fixed ord-IMO net $N = (P, T, F)$ for which a fixed marking M_0 is live but all $M'_0 : P \to \{0, 1\}$ are non-live.

By Proposition 5.1 (and the fact that all $M \in [M_0\rangle$ are live if M_0 is live) we can assume that M_0 is optimal. Since M_0 is thus self-coverable and the ord-IMO net N is conservative, there must be some full $\sigma = t_1 t_2 \cdots t_m$ (in which each transition from T occurs at least once) such that

$$M_0 \xrightarrow{\sigma} M_0, \text{ i.e. } M_0 \xrightarrow{t_1 t_2 \cdots t_m} M_0. \tag{13}$$

This easily shows that all strongly connected components of $Relax(N)$ are pairwise isolated (there is no edge from one scc to another scc). Hence for each M and all $M' \in [M\rangle$ we have $|M'_{\downarrow P_C}| = |M_{\downarrow P_C}|$ for all components C of $Relax(N)$. (The number of tokens in each component is stable.)

Since M_0 is optimal, by Lemma 5.3 for each component C that is rich at M_0 we have $M_0(p) \geq 1$ for all $p \in P_C$, and for each component C that is poor at M_0 (hence $|(M_0)_{\downarrow P_C}| < |P_C|$) we have $M_0(p) \in \{0, 1\}$ for all $p \in P_C$. Let M'_0 be the "carrier-marking" related to M_0, i.e., for each $p \in P$ we have

$$M'_0(p) = 0 \text{ if } M_0(p) = 0, \text{ and } M'_0(p) = 1 \text{ if } M_0(p) \geq 1.$$

We say that M'_0 arises from M_0 by removing "superfluous tokens" (from rich components), and note that (N, M'_0) has the same rich components as (N, M_0). We also recall that M'_0 is non-live by our assumption. Lemma 4.2 and its proof thus yield

$$M'_0 \xrightarrow{\sigma'} M_{\text{W}}, P_{\text{CR}}, T_{\text{D}}$$

where M_{W} is a DL-marking for which the transitions from T_{D} are dead and the transitions from $T_L = T \smallsetminus T_{\text{D}}$ are live. Moreover, due to the proof of Lemma 4.2 we can choose M_{W} so that P_{CR} is the union of the sets P_C for the components C of $Relax(N_{\downarrow(P,T_L)})$ that are poor in $(N_{\downarrow(P,T_L)}, M_{\text{W}})$.

We now aim to show that there is $\bar{M}_0 \in [M_0\rangle$ such that $\bar{M}_0 \geq M'_0$ and $(\bar{M}_0)_{\downarrow P_{\text{CR}}} = (M'_0)_{\downarrow P_{\text{CR}}}$; then the execution $M_0 \xrightarrow{*} \bar{M}_0 \xrightarrow{\sigma'} M'_{\text{W}}$ contradicts the assumption that M_0 is live (since $(M'_{\text{W}})_{\downarrow P_{\text{CR}}} = (M_{\text{W}})_{\downarrow P_{\text{CR}}}$ and transitions from T_{D} are thus dead at M'_{W}). The proof will thus be finished.

We first note that for each component C' of $Relax(N_{\downarrow(P,T_L)})$ we have that $P_{C'}$ is a subset of P_C for some component C of $Relax(N)$; in other words, each component of $Relax(N)$ is partitioned into some components of $Relax(N_{\downarrow(P,T_L)})$. Since for each component C of $Relax(N)$ that is rich at (the above carrier-marking) M'_0 we have $|M_{\downarrow P_C}| = |(M'_0)_{\downarrow P_C}| = |P_C|$ for all $M \in [M'_0\rangle$, the partition of C into the components of $Relax(N_{\downarrow(P,T_L)})$ must contain at least one component C' that is rich in $(N_{\downarrow(P,T_L)}, M_{\text{W}})$. Hence for each component C of $Relax(N)$ that is rich at M'_0 there is a place $p_{\text{NC}} \in P_C \smallsetminus P_{\text{CR}}$ (hence p_{NC} is a don't care place in $(N_{\downarrow(P,T_L)}, M_{\text{W}})$).

We recall that the rich components in (N, M'_0) coincide with the rich components in (N, M_0). Our goal to show $M_0 \xrightarrow{*} \bar{M}_0$ where $(\bar{M}_0)_{\downarrow P_{\text{CR}}} = (M'_0)_{\downarrow P_{\text{CR}}}$ will be realized by moving all superfluous tokens in each component C that is rich in (N, M_0) onto a chosen place $p_{\mathrm{NC}} \in P_C \setminus P_{\mathrm{CR}}$. It is clearly sufficient to show how to modify the execution (13) so that M_0 is transformed just so that one superfluous token from some $p \in P_{\mathrm{CR}}$ is moved outside P_{CR}, onto a particular p_{NC}. Such a process can be repeated until we get the desired $\bar{M}_0$, which finishes the proof.

Hence we fix some $p \in P_{\mathrm{CR}}$ where $M_0(p) \geq 2$ (and thus $M_0(p) > M_0'(p)$); by our choice of M_0 we have $p \in P_C$ for a component C that is rich in (N, M_0). We fix some $p_{\mathrm{NC}} \in P_C \setminus P_{\mathrm{CR}}$ (whose existence has been discussed). There must be a path from p to p_{NC} (in $Relax(N)$); let $t_1' t_2' \cdots t_r'$ be the sequence of transitions on this path. We consider the execution

$$M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 \cdots \xrightarrow{\sigma_r} M_r \tag{14}$$

where σ_i ($i \in [1, r]$) arises from σ in (13) by omitting all transitions contained in rich components, except of one occurrence of t_i' (hence t_1' occurs once in σ_1, t_2' occurs once in σ_2, etc.). Since all places in rich components are marked in M_0, it is easy to check that (14) is a valid execution, and that M_r coincides with M_0 except that $M_r(p) = M_0(p) - 1$ (hence $M_r(p) \geq 1$) and $M_r(p_{\mathrm{NC}}) = M_0(p_{\mathrm{NC}}) + 1$. Hence the carrier-markings of M_r and M_0 are the same, namely M_0', but the amount of superfluous tokens in M_r is less than in M_0. □

6.3. A structurally live ord-BIO net in which all $\{0, 1\}$-markings are non-live

Here we show that Lemma 6.8 cannot be extended to ord-BIO nets, by providing a concrete example. Roughly speaking, if we wanted to mimic the proof of Lemma 6.8 in the case of BIO nets, a prob-

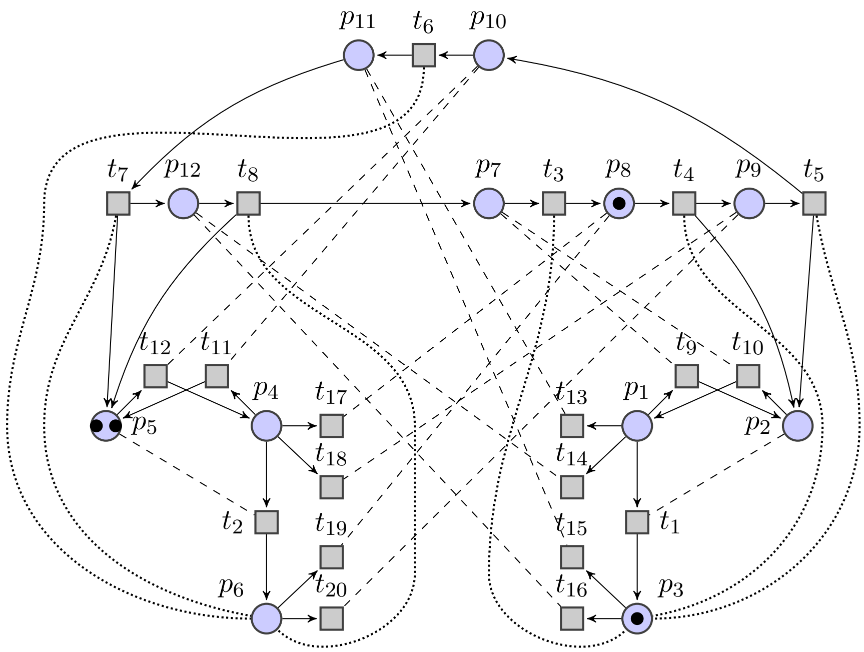

Figure 8. A structurally live ord-BIO net N in which all $M : P \longrightarrow \{0, 1\}$ are non-live.

lem would arise at moving superfluous tokens to don't care places: such moving can create further superfluous tokens due to branching transitions.

Our example net N is depicted in Figure 8; for lucidity, instead of drawing two observation edges (p,t) and (t,p) we draw just one, dotted or dashed, edge between p and t without end-arrows. The respective properties of N are formulated by the next two propositions.

Proposition 6.9. The net $N = (P, T, F)$ in Figure 8 is structurally live.

Proof:
Let M_0 be the marking of N depicted in Figure 8, i.e., $M_0(p_3) = M_0(p_8) = 1$, $M_0(p_5) = 2$, and $M_0(p) = 0$ for all $p \in P \smallsetminus \{p_3, p_5, p_8\}$. The set of places $\{p_7, p_8, p_9, p_{10}, p_{11}, p_{12}\}$ can be viewed as a "control part" in which precisely one token is moving; i.e., for all $M \in [M_0\rangle$ we have

1. $\sum_{i\in[7,12]} M(p_i) = 1$.

The first three of "control transitions" $t_3, t_4, t_5, t_6, t_7, t_8$ require an observation token in p_3, while the last three require an observation token in p_6; we also note that each of t_4 and t_5 adds a fresh token to p_2, while each of t_7 and t_8 adds a fresh token to p_5. We now add the following conditions that are satisfied for all $M \in [M_0\rangle$:

2. if $\sum_{i\in[7,9]} M(p_i) = 1$, then $M(p_5) \geq 2$;
3. if $\sum_{i\in[10,12]} M(p_i) = 1$, then $M(p_2) \geq 2$;
4. if $M(p_7) = 1$, then $M(p_1) + M(p_2) \geq 2$ or $M(p_3) \geq 1$;
5. if $M(p_{10}) = 1$, then $M(p_4) + M(p_5) \geq 2$ or $M(p_6) \geq 1$;
6. if $M(p_8) + M(p_9) = 1$, then $M(p_3) \geq 1$;
7. if $M(p_{11}) + M(p_{12}) = 1$, then $M(p_6) \geq 1$;
8. if $M(p_9) = 1$, then $M(p_2) \geq 1$;
9. if $M(p_{12}) = 1$, then $M(p_5) \geq 1$.

We easily check that the conditions $1 - 9$ are satisfied for M_0 (i.e., for $M = M_0$). Now let M_1 satisfy $1 - 9$, and let $M_1 \xrightarrow{t} M_2$ (for some $t \in T$). It is a routine to verify that M_2 satisfies $1 - 9$ as well: E.g., if $t = t_3$, then $M_1(p_7) = 1$, $M_1(p_3) = M_2(p_3) \geq 1$, and $M_2(p_8) = 1$; M_2 thus obviously satisfies the conditions $1 - 9$ (in particular 6).

In fact, we have shown that if M satisfies $1 - 9$, then each $M' \in [M\rangle$ satisfies $1 - 9$ as well. Now we show that if M satisfies $1 - 9$, then no transition is dead at M; this entails that each M satisfying $1 - 9$ (which includes M_0) is live.

We fix some M satisfying $1 - 9$. By condition 1, there is exactly one control transition $t \in \{t_3, t_4, t_5, t_6, t_7, t_8\}$ whose source place is marked (by one token) at M; we perform a case analysis:

- If $t = t_3$ (hence $M(p_7) = 1$), then by condition 4 we have either $M(p_3) \geq 1$, or we can mark p_3 by transition t_1 that might need to be preceded by t_9 or t_{10}; then t_3 can be executed.

- If $t = t_6$ (hence $M(p_{10}) = 1$), then by condition 5 we have either $M(p_6) \geq 1$, or we can mark p_6 by transition t_2 that might need to be preceded by t_{11} or t_{12}; then t_6 can be executed.
- If $t \in \{t_4, t_5\}$, then t can be executed by condition 6.
- If $t \in \{t_7, t_8\}$, then t can be executed by condition 7.

From M we can thus perform all control transitions so that both p_3 and p_6 are marked afterwards. Then we can be further executing just the control transitions, which is increasing the number of tokens on p_2 and p_5. This makes clear that all transitions (including the "token-consuming transitions" $t_{13}, t_{14}, \ldots, t_{20}$) can become enabled when we start from M. □

Proposition 6.10. For the net $N = (P, T, F)$ in Figure 8, each marking $M : P \to \{0, 1\}$ is non-live.

Proof:
We first show that any marking M satisfying one of the following two conditions

a) $M(p_1) + M(p_2) \leq 1$ and $M(p_3) = 0$,

b) $M(p_4) + M(p_5) \leq 1$ and $M(p_6) = 0$

is non-live. In the case a), t_1, t_3, t_4, t_5 are dead at M, since each of t_3, t_4, t_5 needs to its enabling that t_1 is performed earlier, while t_1 needs that t_4 or t_5 is performed earlier. The case b) is analogous, here t_2, t_6, t_7, t_8 are dead at M.

Now we fix a marking $M_0 : P \longrightarrow \{0, 1\}$ and show that M_0 is non-live, by analysing the following cases C1 and C2.

C1 $M_0(p_8) + M_0(p_9) > 0$ or $M_0(p_{11}) + M_0(p_{12}) > 0$.

If $M_0(p_{11}) + M_0(p_{12}) > 0$, then we have $M_0 \xrightarrow{\sigma} M$ where σ contains no other transitions than $t_{13}, t_{14}, t_{15}, t_{16}$, and $M(p_1) = M(p_3) = 0$, while $M(p_2) = M_0(p_2) \leq 1$. Hence M is non-live by the above case a), which entails that M_0 is non-live as well. If $M_0(p_8) + M_0(p_9) > 0$, then we get the case b) analogously.

C2 $M_0(p_8) + M_0(p_9) = 0$ and $M_0(p_{11}) + M_0(p_{12}) = 0$.

We assume that $M_0 \xrightarrow{\sigma t} M$ is a shortest execution where $t \in \{t_3, t_6\}$; if it does not exist, then M_0 is non-live. For concreteness, we now assume that $t = t_6$, and consider the following two cases separately:

- $M_0(p_7) = 0$.
  We observe that $M(p_2) \leq M_0(p_2) \leq 1$ (since t_3 and t_6 do not occur in σ, and thus t_4, t_5, t_7, t_8, and t_9 cannot occur in σ either). Since $M(p_{11}) = 1$, we have $M \xrightarrow{\sigma'} M'$ where $\sigma' = (t_{13})^{M(p_1)}(t_{15})^{M(p_3)}$ and $M'(p_1) = M'(p_3) = 0$. Since $M'(p_2) \leq 1$, the above case a) shows that M' is non-live, which entails that M_0 is non-live as well.

- $M_0(p_7) = 1$.
  Here $M(p_{11}) = M(p_7) = 1$ (since neither t_3 nor t_8 occurs in σ). We have $M \xrightarrow{\sigma'} M'$ where $\sigma' = (t_{10})^{M(p_2)}(t_{13})^{M(p_1)+M(p_2)}(t_{15})^{M(p_3)}$ and $M'(p_1) = M'(p_2) = M'(p_3) = 0$. Again, we apply the above case a) to M', and deduce that M_0 is non-live.

The case $t = t_3$ (instead of $t = t_6$) is analogous; here the above case b) applies. □

7. Extension to non-ordinary nets

In this section we finish proving Theorem 2.5 that is captured by Table 1 (in Section 2.3). This will be accomplished by proving the next lemma.

Lemma 7.1. (Remaining results to fill Table 1)
Let $N = (P, T, F)$ be a BIMO net in which the maximum edge-weight is w. We have:

1. N is structurally live iff there is $M : P \to \{0, \dots, w \cdot |P|\}$ such that (N, M) is live.
2. (N, M) is live iff (N, M') is live whenever $M'(p) = M(p)$ for all $p \in P$ such that $\min\{M(p), M'(p)\} < 2 \cdot w \cdot |P|$.
3. If N is an IMO net, then it is structurally live iff there is $M : P \to \{0, \dots, w\}$ such that (N, M) is live.

We recall that the maximum edge-weight in IO nets is 2, which entails that the bounds for IMO nets in Table 1 entail the bounds for IO nets. We prove Lemma 7.1 by a simple construction (illustrated in Figure 9) that gives an ord-X net to a given X net, for X$\in$ {IO, IMO, BIO, BIMO}, and by recalling the results for ordinary nets.

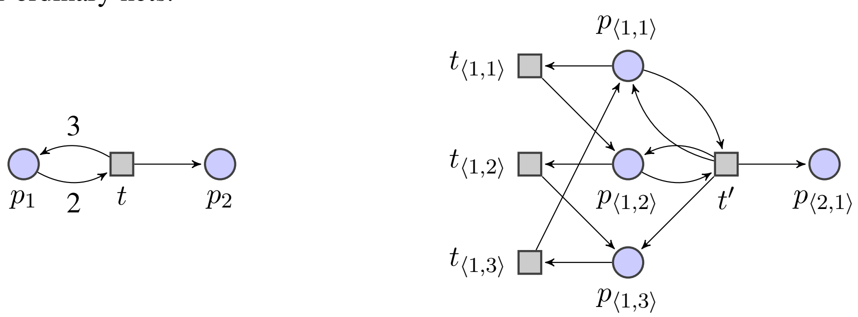

Figure 9. Transforming a BIMO transition $t : p_1 \xrightarrow{\wr p_1 \int} \wr p_1, p_1, p_2 \int$ to an ord-BIMO transition $t' : p_{\langle 1,1\rangle} \xrightarrow{\wr p_{\langle 1,2\rangle} \int} \wr p_{\langle 1,1\rangle}, p_{\langle 1,3\rangle}, p_{\langle 2,1\rangle} \int$.

An ord-BIMO net N' related to a given BIMO net N. Given a BIMO net $N = (P, T, F)$, we define the function $W_{\text{MAX}} : P \longrightarrow \mathbb{N}$ such that

$$W_{\text{MAX}}(p) = \max(\{F(p,t) \mid t \in T\} \cup \{F(t,p) \mid t \in T\}).$$

(For instance, in Figure 9 we have $W_{\text{MAX}}(p_1) = 3$.)

From $N = (P, T, F)$, where $P = \{p_1, p_2, \ldots, p_m\}$, we create an ord-BIMO net $N' = (P', T', F')$ as follows (cf. Figure 9):

- P' arises from P so that each place $p_i \in P$ with $W_{\text{MAX}}(p_i) \geq 1$ is replaced by new places $p_{\langle i,1 \rangle}, p_{\langle i,2 \rangle}, \ldots, p_{\langle i, W_{\text{MAX}}(p_i) \rangle}$.
- $T' = \{t' \mid t \in T\} \cup \bigcup_{i \in [1,m]} \{t_{\langle i,1 \rangle}, t_{\langle i,2 \rangle}, \ldots, t_{\langle i, W_{\text{MAX}}(p_i) \rangle}\}$ where
  - for each $i \in [1, m]$ we have $t_{\langle i,j \rangle} : p_{\langle i,j \rangle} \to p_{\langle i,j+1 \rangle}$ for all $j \in [1, W_{\text{MAX}}(p_i){-}1]$, and $t_{\langle i, W_{\text{MAX}}(p_i) \rangle} : p_{\langle i, W_{\text{MAX}}(p_i) \rangle} \to p_{\langle i,1 \rangle}$;
  - each edge (p_i, t) in N, with weight $F(p_i, t)$, gives rise to ordinary edges $(p_{\langle i,1 \rangle}, t')$, $(p_{\langle i,2 \rangle}, t'), \ldots, (p_{\langle i, F(p_i,t) \rangle}, t')$ in N';
  - each edge (t, p_i) in N, with weight $F(t, p_i)$, gives rise to ordinary edges $(t', p_{\langle i,1 \rangle})$, $(t', p_{\langle i,2 \rangle}), \ldots, (t', p_{\langle i, F(p_i,t) \rangle})$ in N'.

It is easy to verify that the above ord-BIMO net N' related to a BIMO net N is an ord-X net if N is an X net, for X$\in$ {BIO, IMO,IO}.

We say that a marking M of $N = (P, T, F)$, where $P = \{p_1, p_2, \ldots, p_m\}$, and a marking M' of $N' = (P', T', F')$ are *related*, which we denote by $M \approx M'$, if for each $p_i \in P$ with $W_{\text{MAX}}(p_i) \geq 1$ we have

$$M(p) = \textstyle\sum_{j \in [1, W_{\text{MAX}}(p_i)]} M'(p_{\langle i,j \rangle}).$$

It is straightforward to verify that $M_1 \approx M_2$ entails:

- if $M_1 \xrightarrow{t} M_1'$, then $M_2 \xrightarrow{\sigma t'} M_2'$ for some σ consisting of occurrences of $t_{\langle i,j \rangle}$ ($i \in [1, m]$, $j \in [1, W_{\text{MAX}}(p_i)]$), and we have $M_1' \approx M_2'$;
- if $M_2 \xrightarrow{t_{\langle i,j \rangle}} M_2'$, then $M_1 \approx M_2'$;
- if $M_2 \xrightarrow{t'} M_2'$, then $M_1 \xrightarrow{t} M_1'$ where $M_1' \approx M_2'$;
- M_1 is live in N iff M_2 is live in N'.

The previous results for ordinary nets and the above construction thus entail Lemma 7.1.

Moreover, Lemma 4.2 can be generalized as follows:

Lemma 7.2. (For BIMO nets, if M_0 is non-live, then there is a simple witness $M_{\text{W}} \in [M_0\rangle$)
For a BIMO net $N = (P, T, F)$, with the maximum edge-weight w, a marking M_0 is non-live iff there are

- $M_{\text{W}} \in [M_0\rangle$ (a witness marking),
- $P_{\text{CR}} \subseteq P$ (a set of crucial places), and
- a nonempty set $T_{\text{D}} \subseteq T$ (a set of dead transitions)

such that

1. $0 \leq M_{\mathrm{W}}(p) \leq w$ for each $p \in P_{\mathrm{CR}}$;
2. $N_{\downarrow(P_{\mathrm{CR}}, T \smallsetminus T_{\mathrm{D}})}$ is an IMO net (and is thus conservative);
3. all transitions from T_{D} are dead in $(N_{\downarrow(P_{\mathrm{CR}},T)}, (M_{\mathrm{W}})_{\downarrow P_{\mathrm{CR}}})$.

8. Structural liveness for BIMO nets is in PSPACE

The previous results allow us to give a straightforward proof of the following theorem; we note that we assume a standard presentation of BIMO nets, with edge-weights given in binary.

Theorem 8.1. The structural liveness problem (SLP) for BIMO nets is in PSPACE (and is thus PSPACE-complete).

Proof:
We suggest a nondeterministic algorithm, Algorithm 1; its input consists of a BIMO net N and a marking M_0 of N (with the values $M_0(p)$ given in binary), it works in polynomial space, and it can finish successfully if, and only if, (N, M_0) is non-live. (This establishes the claim, since NPSPACE=PSPACE.)

Inspecting the presented pseudocode, the fact that Algorithm 1 works in polynomial space is obvious, including the check at line 25: since $N_{\downarrow(P_{\mathrm{CR}},T)}$ is an IMO net, and thus a conservative net, determining whether ${}^{\bullet}t$ can be covered from $\mathrm{M}_{\downarrow P_{\mathrm{CR}}}$ is clearly solvable in polynomial space (when we again recall that PSPACE=NPSPACE).

Lemma 7.2 and Theorem 2.5(2) guarantee that Algorithm 1 can return $true$ if, and only if, (N, M_0) is non-live:

- If (N, M_0) is non-live, then there are $M_{\mathrm{W}} \in [M_0\rangle$, P_{CR}, T_{D} as given in Lemma 7.2. In this case Algorithm 1 can simply perform a respective execution $M_0 \xrightarrow{\sigma} M_{\mathrm{W}}$, by repeatedly choosing the cases $c = 1$ and $c = 2$; forgetting the precise marking values above the bound $2 \cdot w \cdot |P|$ (due to the line 7) does not prevent this since there are the lines $18 - 19$ in the case $c = 2$. Finally the case $c = 3$ is chosen, with the respective P_{CR} and some $t \in T_{\mathrm{D}}$.
- If (N, M_0) is live, then all markings stored in M must be live: due to Theorem 2.5(2), and the monotonicity of net executions (if $M_1 \xrightarrow{\sigma} M_2$, then $M_1 + M_3 \xrightarrow{\sigma} M_2 + M_3$), all such markings are reachable from live markings (and thus are live themselves). This fact prevents Algorithm 1 from returning $true$.

□

9. Additional remarks

As mentioned in the introduction, IO nets model the IO protocols, hence a subclass of the general population protocols. The nets modelling the general population protocols, the *pp-nets* for short, are also conservative, hence the liveness problem (LP) is also PSPACE-complete for them.

Algorithm 1: (Nondeterministically) verify non-liveness of a marked BIMO net

Input: a BIMO net $N = (P, T, F)$, where $P = \{p_1, p_2, \ldots, p_m\}$ and w is the maximum edge-weight; and a marking $M_0 : P \to \mathbb{N}$.

Output: at least one computation returns $true$ if, and only if, (N, M_0) is non-live.

```
begin
    M ← M_0 {M is a program variable containing the current marking};
    (B_1, ..., B_m) ← (false, ..., false) {each place p_i has an attached boolean variable B_i};
    repeat
        for i ← 1 to m do
            if M(p_i) > 2 · w · |P| then
                M(p_i) ← 2 · w · |P|; B_i ← true {B_i will not change anymore};
            end
        end
        choose (nondeterministically) c ∈ {1, 2, 3};
        if c = 1 then
            choose a transition t that is enabled at the current marking M stored in M;
            {the computation fails if there is no such t};
            M ← M' where M -t-> M';
        end
        if c = 2 then
            choose i ∈ [1, m];
            if M(p_i) < 2 · w · |P| and B_i = true then
                M(p_i) ← M(p_i) + 1;
            end
        end
        if c = 3 then
            choose P_CR ⊆ P such that N↓(P_CR, T) is an IMO net;
            choose t ∈ T;
            if t is dead in (N↓(P_CR, T), M↓P_CR) then
                return true;
            end
        end
    until false {hence the cycle repeats forever if not finishing with a return or a fail};
end
```

On the other hand, in [14] we elaborate an extension of the lower-bound proof from [6] to show that the structural liveness problem (SLP) is EXPSPACE-hard for the pp-nets.

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