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# Exact Wirelength of Embedding 3-Ary $n$-Cubes into Certain Cylinders and Trees

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**Abstract.** Graph embeddings play a significant role in the design and analysis of parallel algorithms. It is a mapping of the topological structure of a guest graph $G$ into a host graph $H$, which is represented as a one-to-one mapping from the vertex set of the guest graph to the vertex set of the host graph. In multiprocessing systems, the interconnection networks enhance the efficient communication between the components in the system. Obtaining minimum wirelength in embedding problems is significant in the designing of networks and simulating one architecture by another. In this paper, we determine the wirelength of embedding 3-ary $n$-cubes into cylinders and certain trees.

**Keywords:** embedding, edge isoperimetric problem, congestion, wirelength, 3-ary $n$-cube

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# 1. Introduction

A multiprocessor is a computer network designed for parallel processing. It has numerous nodes that communicate by passing messages through a network. The pattern of connecting the nodes in a multicomputer is described as an interconnection network. By embedding a guest graph into a host graph, an already formulated algorithm for the guest graph can be modified and used in the embedded host architecture [1]. Embedding and its implications are extensively studied in [1–5]. Embedding has vast applications in the complex connection networks such as network compression [6], visualization [7], clustering [8], link prediction [9] and node classification [10]. The efficiency of a graph embedding is determined by the optimal wirelength of the layout. The wirelength of a graph embedding originate from VLSI designs, data structures, networks that deal with parallel computing systems, biological models, structural engineering and so on [11]. The implementation of 100 billion transistors in a Chip Multi-processor (CMP) has become a reality as microprocessor technology advances into the nanoscale stage [12]. The chip architecture must consider how to efficiently use a high number of transistors. The complexity of chip design is also rising, making it increasingly challenging on improving the overall performance of the system by enhancing the performance of a single processing core. Due to the key benefits of network-on-chip (NoC) such as high integration, low power consumption, cheap cost and compact volume, it has become a widely used approach to designing very large-scale integration (VLSI) systems [13, 14]. Various NoC is analysed for effective communication in CMP [15–18]. The topology structure must meet a few unique requirements for NoC, due to the area restriction on processors, interconnection network and overall wirelength of NoC has emerged as the most pressing problem of its effective communication. It is a secondary factor for NoC to take into account when calculating the cost of their interconnection networks. The cost of wiring for connectivity increases, with network complexity. Consequently, it is preferable to replace NoC with a conventional network for the complex networks serving as a counterpart, where the embedding problem becomes a key feature in analysing NoC performance. The $k$-ary $n$-cube is a parallel architecture used in implementation and message latency [19–21]. This architecture is the hypercube when $k = 2$ and the torus when $k = 3$. Hypercubes have been used in Ipsc/2 and Ipsc/860 and tori in J-Machine, Cray T3D and T3E [22]. The topological properties of $k$-ary $n$-cubes have been explored in [23, 24]. Due to the advantageous topological properties of 3-ary $n$-cube, $Q_n^3$ such as symmetricity, pancyclicity, short message latency and easy implementation it has been utilised to build multicomputers such as the Cray XT5, Blue Gene/L supercomputers [25] and CamCube [26] systems. Embedding problem on 3-ary $n$-cubes is extensively studied on paths, cycles with faulty nodes and links [27, 28]. Further 3-ary $n$-cubes have been embedded into paths, grids [29] and 3D Torus [30]. Fan et al. [31] had studied the fault tolerance of 3-ary $n$-cubes and embedding of the same into torus NoC. In this paper, the optimal wirelength is computed for embedding 3-ary $n$-cubes into certain cylinders and certain trees such as caterpillars, firecracker graphs and banana trees, which enables the efficient communication of 3-ary $n$-cubes onto the above-mentioned network-on-chip.

# 2. Preliminaries

This section consists of the preliminary work required for our subsequent work.

**Definition 2.1.** [32] The edge isoperimetric problem is to find a subset of vertices in a given graph that induces the maximum number of edges among all subsets with the same number of vertices. In otherwords, for a given $r$, $1 \leq r \leq |V_G|$, the problem is to find $I_G(r) = \max_{A \subseteq V, |A|=r} |I_G(A)|$, where $I_G(A) = \{(u, v) \in E : u, v \in A\}$.

**Definition 2.2.** [33] Embedding of graph $G$ into graph $H$ is a one-to-one mapping $f : V(G) \rightarrow V(H)$ such that $f$ induces a one-to-one mapping $P_f : E(G) \rightarrow \{P_f(u, v) : P_f(u, v)$ is a path in $H$ between $f(u)$ and $f(v)$, for every edge $(u, v)$ in $G\}$.

**Definition 2.3.** [33] For an edge $e \in E(H)$, let $c_f(e)$ denote the number of edges $(u, v)$ of $G$ such that $e$ is in the path $P_f(u, v)$ between vertices $f(u)$ and $f(v)$ in $H$. The wirelength of an embedding $f$ of $G$ into $H$ is given by $WL_f(G, H) = \sum_{e \in E(H)} c_f(e)$. The wirelength of embedding $G$ into $H$ is defined as $WL(G, H) = min\{WL_f(G, H) : f$ is an embedding from $G$ to $H\}$.

**Remark 2.4.** For any set $S$ of edges of $H$, $c_f(S) = \sum_{e \in S} c_f(e)$.

**Remark 2.5.** $\sum_{v \in V(G_i)} deg_G(v)$ denotes the sum of degree of all vertices in $G_i$, where $deg_G(v)$ is the number of edges incident on a vertex $v$.

**Lemma 2.6.** ( [34], Congestion Lemma) Let $f$ be an embedding of an arbitrary graph $G$ into $H$. Let $S$ be an edge cut of $H$ such that the removal of edges of $S$ separates $H$ into two components $H_1$ and $H_2$ and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also $S$ satisfies the following conditions:

1. For every edge $(a, b) \in G_i, i = 1, 2, P_f(a, b)$ has no edges in $S$.

2. For every edge $(a, b)$ in G with $a \in G_1$ and $b \in G_2$, $P_f(a, b)$ has exactly one edge in $S$.

3. $G_1$ and $G_2$ are maximum subgraphs.

Then, $c_f(S) = \sum_{v \in V(G_1)} deg_G(v) - 2|E(G_1)| = \sum_{v \in V(G_2)} deg_G(v) - 2|E(G_2)|$ and $c_f(S)$ is minimum.

**Remark 2.7.** In Lemma 2.6, if $G$ is a regular graph then $G_1$ is a maximum subgraph of $G$ implies that $G_2$ is also a maximum subgraph of $G$.

**Lemma 2.8.** ( [35], $k$-Partition Lemma) Let $f : G \rightarrow H$ be an embedding. Let $[kE(H)]$ denote a multiset of edges of $H$ with each edge in $H$ repeated exactly $k$ times. Let $S_1, S_2, ..., S_r$ be a partition of $[kE(H)]$ such that each $S_i$ is an edge cut of $H$ satisfying the Congestion Lemma. Then

$$WL_f(G, H) = \frac{1}{k} \sum_{i=1}^{r} c_f(S_i).$$

# 3. 3-Ary $n$-cube, $Q_n^3$

**Definition 3.1.** [36] The 3-ary $n$-cube, $Q_n^3$ $(n \geq 1)$ is defined to be a graph on $3^n$ vertices, each of the form $x = (x_{n-1}, x_{n-2}, ..., x_0)$, where $0 \leq x_i \leq 2$ for $0 \leq i \leq n-1$. Two vertices are joined by an edge if and only if there exists $j$, $0 \leq j \leq n-1$, such that $x_j = y_j \pm 1$ $(mod\ 3)$ and $x_i = y_i$, for every $i \in \{0, 1, ..., j-1, j+1, ..., n-1\}$.

It is also recursively defined as the cartesian product of $n$ cycles of order 3,

$$Q_n^3 = C_3 \otimes C_3 \otimes ... \otimes C_3 (n \text{ times}).$$

Thus,

$$Q_n^3 = \begin{cases} C_3, & \text{if } n = 1. \\ C_3 \otimes Q_{n-1}^3, & \text{otherwise.} \end{cases}$$

Each $Q_n^3$ contains three copies of $Q_{n-1}^3$ as subgraphs. Recursively each $Q_{n-1}^3$ has three copies of $Q_{n-2}^3$ as subgraphs. Thus we can partition $Q_n^3$ into 3 disjoint isomorphic copies $Q_{n-1}^3(0)$, $Q_{n-1}^3(1)$, $Q_{n-1}^3(2)$, where $Q_{n-1}^3(k)$, $\forall\ 0 \leq k \leq 2$ denotes the subgraph induced by the vertices $\{(x = x_{n-1}, x_{n-2}, ..., x_i, .., x_0) \in V(Q_n^3) | x_i = i\}$, for any $i = 0, 1, 2$. Each $Q_{n-1}^3(k)$ is a convex set of $Q_n^3$. $Q_n^3$ has $k^{n-1}$ edges, having a perfect matching between $Q_{n-1}^3(k)$ and $Q_{n-1}^3(k+1)$, $\forall\ 0 \leq k \leq 2$. $Q_{n-1}^3(k)$ and $Q_{n-1}^3(k+1)$ are adjacent subcubes, and the edges between them are called 'bridges'. The $n$ dimensional $Q_n^3$ is $2n$-regular [36]. See Figure 1.

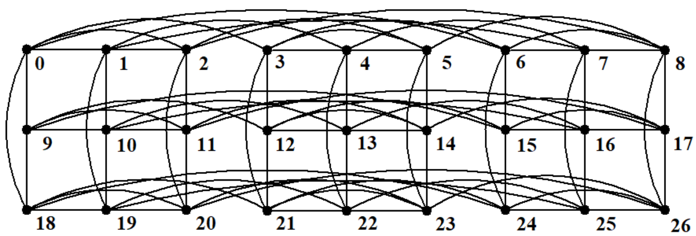


Figure 1. 3-ary 3-cube, $Q_3^3$.

**Definition 3.2.** [37] The Lexicographic order on a set of $n$-tuples with integer entries is defined as follows: We say that $(x_1, ..., x_n)$ is greater than $(y_1, ..., y_n)$ if there exist an index $i$, $1 \leq i \leq n$, such that $x_j = y_j$ for $1 \leq j < i$ and $x_i > y_i$.

Sergei et al. [37] has studied the edge isoperimetric problem for the torus $C_3 \times C_3$ which was solved in [38, 39] by introducing a new characteristic, called $\delta-$*sequence* which is defined as follows: For a graph $G = (V, E)$ with $1 \leq k \leq |V|$, we define

$$\delta(k) = I(k) - I(k-1), \text{ with } \delta(1) = 0,$$

where $I(k)$ is the maximum number of edges induced by any $k$ vertices.

Further $\delta_G = (\delta(1), \delta(2)...., \delta(|V|))$ is called the $\delta-$sequence of $G$. The $\delta-$*sequence* of $C_3 \times C_3$ is (0,1,2,1,2,3,2,3,4). This gives an optimal order for the maximum subgraph for $C_3 \times C_3$ by lexicographic ordering.

**Theorem 3.3.** [40] If the cartesian product of $G \times G$ is optimal with vertices of lexicographic ordering then it is optimal for $G^n$ for any $n \geq 3$.

The following corollary of Theorem 3.3 solves the edge isoperimetric problem in $Q_n^3$, $n \geq 2$.

**Corollary 3.4.** The Lexicographic ordering of vertices of $Q_n^3$, $n \geq 2$, is an optimal ordering for inducing maximum subgraphs in $Q_n^3$.

**Remark 3.5.** Let $lex_k = \{0, 1, 2, ..., k-1\}$, $1 \leq k \leq 3^n$ denote the first $k$ vertices in $Q_n^3$, $n \geq 2$ with lexicographic ordering.

**Theorem 3.6.** If $G$ is a 3-ary $n$-cube, $Q_n^3$, $n \geq 2$, then $I_G(k) = k_1 3^{k_1} + (k_2+1)3^{k_2} + (k_3+2)3^{k_3} + ... + (k_r + (r-1))3^{k_r}$, $k_i = 0, 1, 2, ..., n,\ 1 \leq i \leq r$; where $I_G(k)$ is the number of edges induced in any maximum subgraph on $k$ vertices and $k = 3^{k_1} + 3^{k_2} + 3^{k_3} + ... + 3^{k_r}$, $k_1 \geq k_2 \geq k_3 \geq .... \geq k_r$.

**Proof:**
Consider $Q_n^3(k)$ where $k = 3^{k_1} + 3^{k_2} + 3^{k_3} + ... + 3^{k_r}$. $Q_n^3(k)$ contains $Q_{k_1}^3$, $Q_{k_2}^3$,...,$Q_{k_r}^3$ where $Q_{k_i}^3, \forall\, i > 1$ is adjacent with $Q_{k_{i-1}}^3, ..., Q_{k_2}^3, Q_{k_1}^3$. There are $3^{k_i}$ edges between $Q_{k_i}^3$ and each of $Q_{k_j}^3$, $\forall\, j = 1, 2, ..., i-1$. Thus there exist $(i-1)3^{k_i}$ edges between $Q_{k_i}^3$ and $Q_{k_j}^3$, $\forall\, j = 1, 2, ..., i-1$. Further $Q_{k_i}^3$, $\forall\, i = 1, 2, 3, ..., r$ also has $k_i 3^{k_i}$ edges in it. This implies that $Q_{k_i}^3$ contributes $(k_i 3^{k_i} + (i-1)3^{k_i}) = (k_i + (i-1))3^{k_i}$ edges to $I_{Q_n^3}(k)$. Hence the Lemma. □

## 4. Embedding of $Q_n^3$ into cylinder $C_3 \times P_{3^{n-1}}$

**Definition 4.1.** [5] Let $P_\alpha$ and $C_\alpha$ denote a path and cycle on $\alpha$ vertices respectively. The 2-dimensional grid is defined as $P_{\alpha_1} \times P_{\alpha_2}$, where $\alpha_i \geq 2$ is an integer for each $i = 1, 2$. The cylinder $C_{\alpha_1} \times P_{\alpha_2}$, where $\alpha_1, \alpha_2 \geq 3$ is a $P_{\alpha_1} \times P_{\alpha_2}$ grid with a wraparound edge in each column.

**Lexicographic ordered embedding:** The lexicographic ordered embedding $lex : Q_n^3 \rightarrow C_3 \times P_{3^{n-1}}$ with labels 0 to $3^n - 1$ is an assignment of labels to the vertices of $Q_n^3$ in lexicographic order and the vertices of $C_3 \times P_{3^{n-1}}$ as follows: Vertices in $r^{th}$ column are labeled as $3(r-1)+0,\ 3(r-1)+1,\ 3(r-1)+2$ from top to bottom, where $r = 1, 2, .., 3^{n-1}$.

**Embedding Algorithm A:**
*Input:* The 3-ary $n$-cube, $Q_n^3$ and the cylinder $C_3 \times P_{3^{n-1}}$ on $3^n$ vertices.
*Algorithm:* Lexicographic ordered embedding of $Q_n^3$ into $C_3 \times P_{3^{n-1}}$.
*Output:* The embedding $lex$ of 3-ary $n$-cube, $Q_n^3$ into cylinder $C_3 \times P_{3^{n-1}}$ on $3^n$ vertices is with minimum wirelength.

**Notation.** $C^i_{lex} = \{0, 1, 2, ..., 3i - 1\}$, for $i = 1, 2, ..., 3^{n-1} - 1$ denotes the first $i$ column vertices of $C_3 \times P_{3^{n-1}}$ with vertices labeled as in Embedding Algorithm A. From Remark 3.5, it is clear that $C^i_{lex} = lex_{3i}$. The following lemma is a consequence of Corollary 3.4.

**Lemma 4.2.** $C^i_{lex}$ induces maximum subgraph in $Q^3_n$ for $i = 1, 2, .., 3^{n-1} - 1$.

**Notation.** $R^j_{lex} = \{j, 3+j, ..., 3(3^{n-1})+j\}$, for $j = 0, 1, 2$ denotes the $j^{th}$ row vertices of $C_3 \times P_{3^{n-1}}$ with the lexicographic ordered embedding of $Q^3_n$ into $C_3 \times P_{3^{n-1}}$.

**Lemma 4.3.** $R^j_{lex}$ induces maximum subgraph in $Q^3_n$ for $j = 0, 1, 2$.

**Proof:**
From Lemma 4.2, we know that the lexicographic ordering columnwise induces a maximum subgraph. Hence to prove this lemma we have to show that the vertices in each row is isomorphic to subgraph induced by lexicographic ordering $0, 1, 2, ..., 3^{n-1} - 1$. For $j = 0, 1, 2$, define $\varphi^j : R^j_{lex} \to lex_{3^{n-1}}$ by $\varphi^j(3k + l) = 3l + k + j$. If the $n$-tuple representation of integer $3k + l$ is $(\gamma_1, \gamma_2, ..., \gamma_n)$, then the $n$-tuple representation of integer $3l+k+j$ is $(\gamma_2, \gamma_3, ..., \gamma_n, \gamma_1+j)$. Thus if the $n$-tuple representation in two numbers $x$ and $y$ differ in exactly one bit, then it also holds good for $f(x)$ and $f(y)$. This implies that $(x, y)$ is an edge in $Q^3_n$ if and only if $(f(x), f(y))$ is an edge in $Q^3_n$. Thus $R^j_{lex}$ and $lex_{3^{n-1}}$ are isomorphic, which implies that $R^j_{lex}$ induces a maximum subgraph in $Q^3_n$. □

**Theorem 4.4.** The wirelength $WL(Q^3_n, C_3 \times P_{3^{n-1}})$ is minimum for lexicographic ordered embedding $lex$ of $Q^3_n$ into $C_3 \times P_{3^{n-1}}, n \geq 2$.

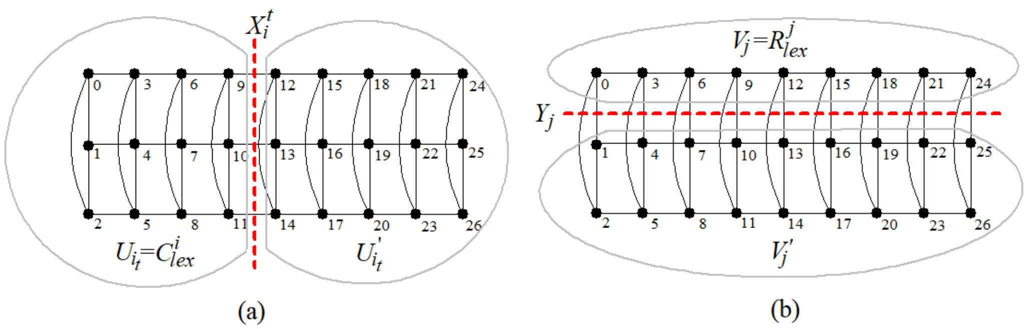


Figure 2. (a) Vertical edge cuts $X^t_i$, $1 \leq i \leq 8$, $1 \leq t \leq 2$ of Cylinder $C_3 \times P_9$ with lexicographic ordering. (b)Horizontal edge cuts $Y_j$, $1 \leq j \leq 2$ of Cylinder $C_3 \times P_9$ with lexicographic ordering.

**Proof:**
Consider the lexicographic embedding $lex : Q^3_n \to C_3 \times P_{3^{n-1}}$ given in the Embedding Algorithm A. $X^t_i$, $i = 1, 2, .., 3^{n-1} - 1$ and $t = 1, 2$, shown in Figure 2(a) is the vertical edge cut of the cylinder $C_3 \times P_{3^{n-1}}$. Removal of $X^t_i$ disconnects $C_3 \times P_{3^{n-1}}$ into two components $U_{i_t}$ and $U'_{i_t}$, where $V(U_{i_t}) = C^i_{lex}$. $Y_j$, $0 \leq j \leq 2$ as shown in Figure 2(b) are the horizontal edge cuts of the cylinder

$C_3 \times P_{3^{n-1}}$. Thus $Y_j$ disconnects $C_3 \times P_{3^{n-1}}$ into two components $V_j$ and $V_j^{'}$, where $V(V_j) = R_{lex}^j$. See Figure 2(b). Let $S_{i_t}$ and $S_{i_t}^{'}$ be the preimages of $U_{i_t}$ and $U_{i_t}^{'}$ in $Q_n^3$ under lexicographic ordering respectively. The edge partition $X_i^t$ satisfies the first two conditions of the congestion lemma. To satisfy condition (iii) of the congestion lemma, it is enough to prove that the edges induced by the preimages $S_{i_t}$ and $S_{i_t}^{'}$ are maximum subgraphs. That is, congestion $c_f(X_i^t)$ is minimum, where $S_{i_t}$ is the subgraph induced by the vertices of $C_{lex}^i$. By Lemma 4.2, $S_{i_t}$ is a maximum subgraph in $Q_n^3$. Hence by the Congestion Lemma $c_f(X_i^t)$ is minimum for $i = 1, 2, .., 3^{n-1} - 1$. Similarly, let $T_j$ and $T_{j'}$ be the preimages of $V_j$ and $V_{j'}$ in $Q_n^3$ under lexicographic ordering respectively. By Lemma 4.3, $T_j$ is a maximum subgraph induced by the vertices of $R_{lex}^j$. Hence by the Congestion Lemma $c_f(Y_j)$ is minimum for $j = 0, 1, 2$. Partition Lemma consequently implies that $WL(Q_n^3, C_3 \times P_{3^{n-1}})$ is minimum. □

**Theorem 4.5.** The minimum wirelength of embedding $Q_n^3$ into $C_3 \times P_{3^{n-1}}$ is given by

$$WL(Q_n^3, C_3 \times P_{3^{n-1}}) = 3^{n-1}\Big(2\big(3^{n-1} - 1\big) + 3\Big).$$

**Proof:**
By Congestion Lemma and 2-Partition Lemma,

$$\begin{aligned} WL(Q_n^3, C_3 \times P_{3^{n-1}}) &= \frac{1}{2}\left(\sum_{t=1}^{2} \sum_{i=1}^{(3^{n-1})-1} c_{lex}(X_i^t) + \sum_{j=0}^{2} c_{lex}(Y_j)\right) \\ &= \frac{1}{2}\left(4\Big(3^{n-1}\Big)\Big(3^{n-1} - 1\Big) + 6\Big(3^{n-1}\Big)\right) \\ &= 3^{n-1}\Big(2\big(3^{n-1} - 1\big) + 3\Big). \end{aligned}$$

□

## 5. Embedding of $Q_n^3$ into certain trees

A tree is an acyclic connected graph. Trees are the most basic graph-theoretic models utilised in various domains, including automatic classification, information theory, data structure and analysis, artificial intelligence, algorithm design, operation research, combinatorial optimization, electrical network theory and network design [11]. We have embedded 3-ary $n$-cubes into certain trees such as caterpillar, firecracker graph and banana tree which are well known in the literature by satisfying the property of some graph variants [41–43]. The research on caterpillars and their embeddings [44, 45] reveal that embedding problems are not simple. For instance, in [46, 47] the authors demonstrated the *NP*-completeness of determining the least dilation of embedding a caterpillar into chain. These predominant use of trees in networks motivated us to study the embedding of 3-ary $n$-cubes into certain trees mentioned above. In a tree traversal, labeling the vertices first time one visits is called preorder traversal.

### 5.1. Wirelength of embedding $Q_n^3$ in caterpillar

**Definition 5.1.** [5] A *caterpillar* is a tree which will be a path if all its leaves are deleted. The path which is retained is called the backbone of the caterpillar.

**Embedding Algorithm B:**
*Input:* The 3-ary $n$-cube, $Q_n^3$ and 2-regular caterpillar denoted by 2-CAT on $3^n$ vertices.
*Algorithm:* Label the vertices of 3-ary $n$-cube, $Q_n^3$ and caterpillar using lexicographic ordering and preorder traversal respectively.
*Output:* The embedding $lex$ of 3-ary $n$-cube, $Q_n^3$ into caterpillar on $3^n$ vertices is with minimum wirelength.

**Lemma 5.2.** The edge cuts $S_i$, $1 \leq i \leq 3^{n-1} - 1$ and $T_j$, $1 \leq j \leq 2(3^{n-1})$ as shown in Figure 3 induce maximum subgraphs in $Q_n^3$.

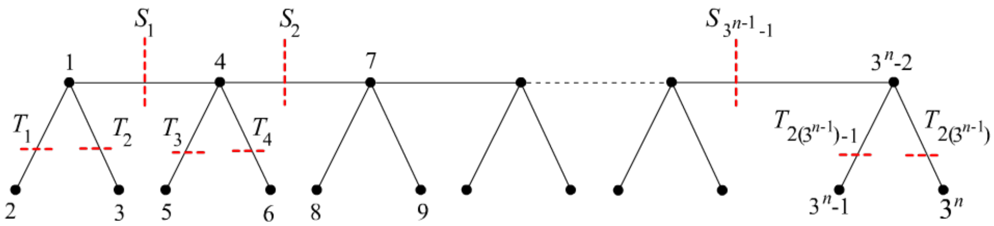


Figure 3. Edge cuts of Caterpillar.

**Proof:**
By Theorem 3.3, the lexicographic ordering of vertices of $Q_n^3$ gives the optimal order for inducing the maximum subgraph. The edge cut $S_i$ removes the edges in the backbone of the caterpillar, such that each $S_i$ disconnects it into two components of lexicographic ordering which induce a maximum subgraph in $Q_n^3$. The edge cut $T_j$ disconnects the caterpillar with exactly one vertex as one of the components. Hence $S_i, \forall\ i = 1, 2, ..., 3^{n-1} - 1$ and $T_j, \forall\ j = 1, 2, ..., 2(3^{n-1})$ induce maximum subgraph in $Q_n^3$. □

**Lemma 5.3.** The Embedding Algorithm B gives minimum wirelength of embedding $Q_n^3$ into 2-regular caterpillar.

**Proof.** By Lemma 5.2 the edge cuts $S_i$ and $T_j$ satisfy conditions of the Congestion Lemma. Therefore $c_f(S_i)$ and $c_f(T_j)$ are minimum. Then the partition lemma implies that wirelength is minimum.

**Theorem 5.4.** The minimum wirelength of embedding $Q_n^3$ into caterpillar is given by

$$WL(Q_n^3, \text{2-CAT}) = 2(3^{n-1})(3^{n-1} - 1) + (4n)(3^{n-1}).$$

**Proof:**
By Congestion Lemma and Partition Lemma,

$$\begin{aligned} WL(Q_n^3, \text{2-CAT}) &= \sum_{i=1}^{3^{n-1}-1} c_f(S_i) + \sum_{j=1}^{2(3^{n-1})} c_f(T_j) \\ &= \sum_{i=1}^{3^{n-1}-1} ((2n)(3i) - 2|E(3i)|) + (2n)(2(3^{n-1})) \\ &= 2(3^{n-1})(3^{n-1}-1) + (4n)(3^{n-1}). \end{aligned}$$

□

## 5.2. Wirelength of embedding $Q_n^3$ in Firecracker graph

**Definition 5.5.** [48] A *firecracker graph* $F_{n,k}$ is a graph obtained by the concatenation of $n$, $k$-stars by linking one leaf from each.

In what follows, we consider concatenation of $3^{n-1}$ number of 3-stars.

**Embedding Algorithm C:**
*Input:* The 3-ary $n$-cube, $Q_n^3$ and firecracker graph, $F_{3^{n-1},3}$ on $3^n$ vertices.
*Algorithm:* Label the vertices of 3-ary $n$-cube, $Q_n^3$ and firecracker graph, $F_{3^{n-1},3}$ using lexicographic ordering and preorder traversal respectively.
*Output:* The embedding $lex$ of 3-ary $n$-cube, $Q_n^3$ into firecracker graph, $F_{3^{n-1},3}$ on $3^n$ vertices is with minimum wirelength.

**Lemma 5.6.** The edge cut $S_i$, $\forall\ i = 1, 2, ..., 3^{n-1} - 1$ of $F_{3^{n-1},3}$ as shown in Figure 4 induces maximum subgraph in $Q_n^3$.

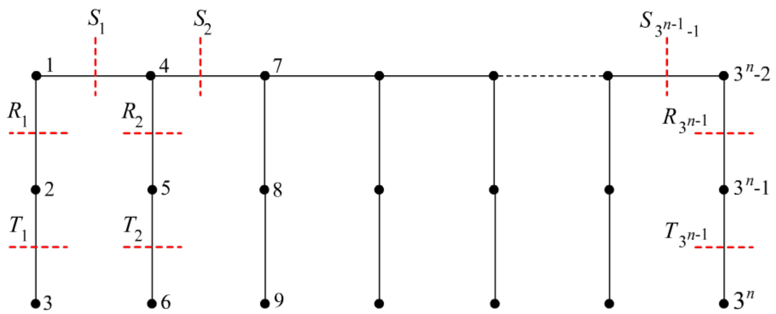


Figure 4. Edge cuts of $F_{3^{n-1},3}$.

**Proof:**
The removal of edges in $S_i$, $1 \leq i \leq 3^{n-1}-1$ disconnects $F_{3^{n-1},3}$ into two components whose inverse images under $lex$ induce lexicographic ordering of the corresponding subgraphs of $Q_n^3$. This implies that the inverse images are maximum subgraphs of $Q_n^3$. □

**Lemma 5.7.** The edge cuts $R_j$ and $T_k, \forall\, j,k = 1,2,...,3^{n-1}$ of $F_{3^{n-1},3}$ as shown in Figure 4 induce maximum subgraph in $Q_n^3$.

**Proof:**
The result is obvious as one of the components due to the cuts is either a singleton set or an edge. □

By Congestion Lemma and Partition Lemma, we arrive at the following result.

**Theorem 5.8.** Minimum wirelength is induced by the embedding algorithm of $Q_n^3$ into $F_{3^{n-1},3}$ on $3^n$ vertices.

**Theorem 5.9.** The minimum wirelength of embedding $Q_n^3$ into $F_{3^{n-1},3}$ is given by

$$WL(Q_n^3, F_{3^{n-1},3}) = 2\big(3^{n-1}\big)\Big(\big(3^{n-1}-1\big) + \big(2n-1\big) + n\Big).$$

**Proof:**
By Congestion Lemma and Partition Lemma,

$$\begin{aligned} WL(Q_n^3, F_{3^{n-1},3}) &= \sum_{i=1}^{(3^{n-1})-1} c_f(S_i) + \sum_{j=1}^{3^{n-1}} c_f(R_j) + \sum_{k=1}^{3^{n-1}} c_f(T_k) \\ &= 2\big(3^{n-1} \times (3^{n-1}-1)\big) + (4n-2)(3^{n-1}) + (2n)(3^{n-1}) \\ &= 2\big(3^{n-1}\big)\Big(\big(3^{n-1}-1\big) + \big(2n-1\big) + n\Big). \end{aligned}$$ □

## 5.3. Wirelength of embedding $Q_n^3$ in banana tree

**Definition 5.10.** [48] A banana tree $B_{n,k}$ is a graph formed by linking one leaf of each of $n$ copies of a $k$-star graph to a single root vertex that is different from all of the stars.

**Embedding Algorithm D:**
*Input:* The 3-ary $n$-cube, $Q_n^3$ and banana tree, $B_{2,\lfloor\frac{3^n}{2}\rfloor}$ on $3^n$ vertices.
*Algorithm:* Label the vertices of 3-ary $n$-cube, $Q_n^3$ and banana tree, $B_{2,\lfloor\frac{3^n}{2}\rfloor}$ using lexicographic ordering and preorder traversal respectively.
*Output:* The embedding $lex$ of 3-ary $n$-cube, $Q_n^3$ into banana tree, $B_{2,\lfloor\frac{3^n}{2}\rfloor}$ on $3^n$ vertices is with minimum wirelength.

**Lemma 5.11.** The edge cuts $R_j$ and $T_k,\ \forall\, j,k = 1,2$ of $B_{2,\lfloor\frac{3^n}{2}\rfloor}$ as shown in Figure 5 induce maximum subgraphs in $Q_n^3$.

**Proof:**
The removal of edges in $R_j$ and $T_k,\ \forall\, j,k = 1,2$ disconnects $B_{2,\lfloor\frac{3^n}{2}\rfloor}$ into two components whose inverse images under $lex$ induce lexicographic ordering of the corresponding subgraphs of $Q_n^3$. This implies that the inverse images are maximum subgraphs of $Q_n^3$. □

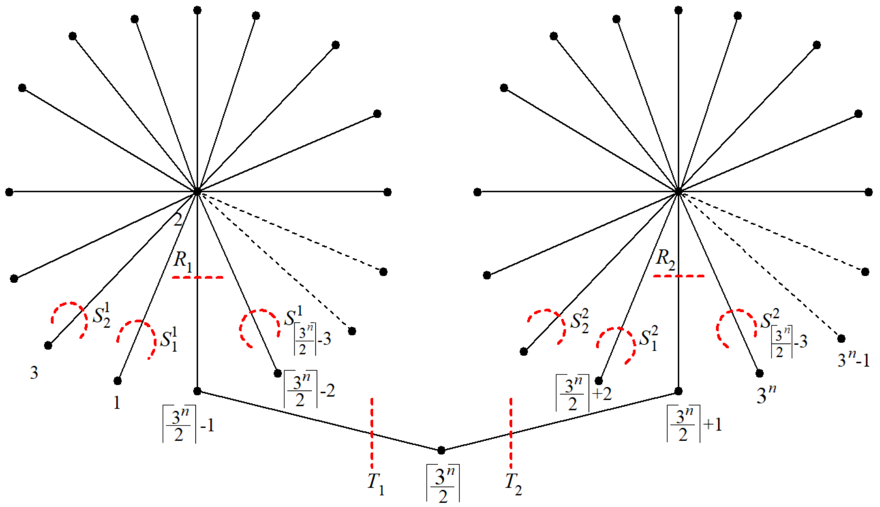


Figure 5. Edge cuts of $B_{2,\lfloor \frac{3^n}{2} \rfloor}$.

**Lemma 5.12.** The edge cuts $S_i^1$ and $S_i^2$, $\forall\ i = 1, 2, ..., \lceil \frac{3^n}{2} \rceil - 3$ of $B_{2,\lfloor \frac{3^n}{2} \rfloor}$ as shown in Figure 5 induce maximum subgraph in $Q_n^3$.

**Proof:**
The result is obvious as one of the components due to the cuts is a singleton set. $\square$

By Congestion Lemma and Partition Lemma, we arrive at the following result.

**Theorem 5.13.** Minimum wirelength is induced by the embedding algorithm of $Q_n^3$ into $B_{2,\lfloor \frac{3^n}{2} \rfloor}$ on $3^n$ vertices.

**Theorem 5.14.** The minimum wirelength of embedding $Q_n^3$ into $B_{2,\lfloor \frac{3^n}{2} \rfloor}$ is given by

$$WL\Big(Q_n^3, B_{2,\lfloor \frac{3^n}{2} \rfloor}\Big) = 4n\Big(\Big\lceil \frac{3^n}{2} \Big\rceil - 3\Big) + 4\Big(\Big\lceil \frac{3^n}{2} \Big\rceil - 2\Big) + 4\Big(\Big\lceil \frac{3^n}{2} \Big\rceil - 1\Big).$$

**Proof:**
By Congestion Lemma and Partition Lemma,

$$\begin{aligned} WL\Big(Q_n^3, B_{2,\lfloor \frac{3^n}{2} \rfloor}\Big) &= \sum_{i=1}^{\lceil \frac{3^n}{2} \rceil -3} c_f(S_i^1) + \sum_{i=1}^{\lceil \frac{3^n}{2} \rceil -3} c_f(S_i^2) + \sum_{j=1}^{2} c_f(R_j) + \sum_{k=1}^{2} c_f(T_k) \\ &= 4n\Big(\Big\lceil \frac{3^n}{2} \Big\rceil - 3\Big) + 4\Big(\Big\lceil \frac{3^n}{2} \Big\rceil - 2\Big) + 4\Big(\Big\lceil \frac{3^n}{2} \Big\rceil - 1\Big). \end{aligned}$$ $\square$

# 6. Conclusion

The optimal wirelength of 3-ary $n$-cube into certain cylinders and trees such as caterpillars, firecracker graphs and banana trees are determined in this paper.

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