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Unfoldings and Coverings of Weighted Graphs

Bruno Courcelle*
LaBRI, CNRS (UMR 5800), University of Bordeaux, France
courcell@labri.fr

Abstract. *Coverings* of undirected graphs are used in distributed computing, and *unfoldings* of directed graphs in semantics of programs. We study these two notions from a graph theoretical point of view so as to highlight their similarities, as they are both defined in terms of surjective graph homomorphisms. In particular, *universal coverings* and *complete unfoldings* are infinite trees that are *regular* if the initial graphs are finite. *Regularity* means that a tree has finitely many subtrees up to isomorphism. Two important theorems have been established by Leighton and Norris for coverings of finite graphs. We prove similar results for unfoldings of finite directed graphs. Moreover, we generalize coverings and similarly, unfoldings to graphs and digraphs equipped with finite or infinite *weights* attached to edges of the covered or unfolded graphs. This generalization yields a canonical "factorization" of the universal covering of any finite graph, that (provably) does not exist without using weights. Introducing ω as an infinite weight provides us with finite descriptions of regular trees having nodes of *countably infinite degree*. Regular trees (trees having finitely many subtrees up to isomorphism) play an important role in the extension of Formal Language Theory to infinite structures described in finitary ways. Our weighted graphs offer effective descriptions of the above mentioned regular trees and yield decidability results. We also generalize to weighted graphs and their coverings a classical factorization theorem of their *characteristic polynomials*.

Keywords: graph unfolding, graph covering, universal covering, regular tree, weighted graph, characteristic polynomial, graph factorization

*Address for correspondence: Bordeaux University, LaBRI, F-33400 Talence, France.

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1. Introduction

We first review informally some basic notions and results. The notion of *covering of an undirected graph* has been introduced by Reidemeister [22] as a discrete analogue of coverings of surfaces. It has proved to be useful in the theory of *distributed computing* where a network is considered as an undirected graph N whose edges represent communication channels. The questions are whether certain problems such as the *election problem* (consisting in distinguishing a unique node of the network) can be solved by a distributed algorithm (of a certain type). This is possible if the graph N is minimal for the *covering relation*, equivalently if the *universal coverings* of N defined from any two different nodes are not isomorphic rooted trees. The universal covering of an undirected graph is an infinite tree. It has a characterization in the sense of Category Theory and can be constructed as the infinite tree of the walks in the graph originated from a node and that do not take the same edge twice in a row (in opposite directions). Starting from any two nodes yields isomorphic trees (without roots). Detailed definitions will be given in Section 4. The universal covering of a finite graph is a *regular tree*, *i.e.*, a tree that has finitely many subtrees *up to isomorphism* (*i.e.*, finitely many isomorphism classes of subtrees). The application of coverings to distributed computing was initiated by Angluin in [2].

Unfoldings of directed graphs are used in the study of abstract programs called *transition systems* in order to represent their semantics [4, 12, 15]. In particular, the *complete unfolding*[1] of a directed graph equipped with a distinguished vertex (representing the "begin" instruction) is a rooted tree that is infinite if the graph has directed cycles. The complete unfolding of the graph representing a transition system S encodes all computations of the program abstracted into S. If the graph is finite, its complete unfolding is a regular tree. Precise definitions will be given in Section 3.

We are interested in unfoldings and coverings from a *graph theoretical point of view*. Both notions are defined in terms of *surjective graph homomorphisms* that are bijective on the neighbourhoods of vertices related by the considered homomorphisms. The notion of neighbourhood is thus a parameter that gives rise to different but related notions: unfoldings, coverings and even others [15]. For unfoldings of directed graphs, the neighbourhood of a vertex x is the set of edges outgoing from x. For coverings of undirected graphs, it is the set of edges incident to x. We study unfoldings and coverings by means of *graph homomorphisms*, *quotient graphs*, *infinite trees* and, in particular, *regular* ones. One of our objectives is to highlight the similarities between the two notions, regarding the definitions and also some results without using any cumbersome categorical framework.

In the theory of coverings, a theorem by Norris [21] states that two regular rooted trees T_x and T_y, that define the universal covering T of a finite undirected graph with p vertices by starting the walks from x and y are isomorphic if their truncations at depth $p-1$ are isomorphic. Another important theorem by Leighton [18] states that, if two finite undirected graphs have isomorphic universal coverings, then they have a *common finite* covering. Its proof is quite difficult. We prove a special case that subsumes the known case of regular graphs [3].

Weighted graphs.

Moreover, we extend the definitions of unfoldings and coverings in the following ways. A directed graphs is *weighted* if each edge has a *weight*, a positive integer or the infinite cardinal ω. An edge

[1] It is simply called *unfolding* in [4, 12, 15].

of weight 3 (resp. ω) *unfolds into* 3 directed edges (resp. countably many) with the same origin. We define complete unfoldings accordingly, and we obtain *regular trees* from finite graphs. These trees have nodes of infinite degree[2] in the case where some edges have weight ω, which generalizes the usual definitions. We call *complete unfolding* what is usually called *the* unfolding (this tree is unique up to isomorphism), and we define as *unfolding* of a weighted directed graph H a weighted directed graph that lies inbetween H and its complete unfolding. "Inbetween" is formally defined in terms of surjective homomorphisms that are locally bijective as explained above. Each regular rooted tree T is the complete unfolding of a finite unique *canonical* weighted directed graph, that can be used as a finite description of T. We extend to weighted directed graphs the theorems by Leighton and Norris described above[3].

We also extend the notion of covering to *weighted undirected graphs*. In this case, weights in $\mathbb{N}_+ \cup \{\omega\}$ are attached to *half-edges*: an edge that is not a loop has two half-edges and thus two weights. A loop is a half-edge (without any matching opposite half-edge) and has a single weight. Each such graph H has a unique *universal covering* (unicity is up to isomorphism) that is an infinite tree T without root denoted by $\boldsymbol{UC}(H)$. It is formally defined from the unfolding of a directed graph, where p parallel directed edges from a vertex x to y replace a weight p attached to an half-edge incident with x whose matching half-edge is incident with y. It is not from walks as easily as in the case of unweighted graphs.

We call *strongly regular* a tree T of the form $\boldsymbol{UC}(H)$ for some finite weighted graph H. This means that T yields finitely many regular rooted trees T_x, up to isomorphism, by taking its different nodes x as roots. This is a new notion. Each strongly regular tree is the universal covering of a *canonical* (it is unique up to isomorphism) finite weighted graph of minimal size, and thus, has a finitary description. It can be seen as a kind of minimal factorization. The infinite rooted binary tree is regular, but it is not strongly regular after forgetting its root.

Our new definitions and main results

1) We define and study coverings and unfoldings in close connection by considering them as two instances of the same notion of a locally bijective homomorphism, based on different types of neighbourhood. In both cases we introduce weights on edges. Infinite weights yield trees of infinite degree having finite descriptions.

2) Our first main result states that two finite graphs have isomorphic universal coverings if and only if they are coverings of a unique minimal weighted graph. Using weighted graphs is here necessary.

3) Our second main theorem extends that by Norris to universal coverings and to complete unfoldings of finite, weighted, graphs and directed graphs.

4) Our third main theorem extends that by Leighton to complete unfoldings of weighted directed graphs. We give an easy proof of it for coverings of graphs in a special case that subsumes the previously known cases and yields new cases.

5) Finite weighted undirected graphs are defined by matrices in a natural way. Our fourth main theorem extends to them a factorization of the characteristic polynomials of their coverings that is

[2]Here, we extend the notion of regular tree that arises from the theory of recursive program schemes [10].

[3]In the forthcoming article [13], we will establish the *first-order definability* of regular trees, among all trees, and also of the universal coverings of finite weighted graphs, as described below. These proofs use our extensions of Norris's Theorem.

known in the case of finite graphs without weights. Hence, our approach fits nicely in Algebraic Graph Theory.

6) We identify as *strongly regular* the universal coverings of the finite weighted graphs. They form a proper subclass of regular trees that we study more in [13].

Summary of the article: Basic definitions are in Section 2. Unfoldings of weighted directed graphs are defined and studied in Section 3. Coverings of weighted undirected graphs are defined and studied in Section 4. We study universal coverings of weighted graphs in Section 5 and we discuss Leighton's Theorem for graphs in Section 6.

2. Basic definitions

This section reviews notation and some easy lemmas. Definitions for graphs and trees are standard, but we make precise some possibly ambiguous terminological points.

2.1. Sets, multisets and weighted sets.

All sets, graphs and trees are finite or countably infinite (of cardinality ω).

The cardinality of a set X is denoted by $|X| \in \mathbb{N}\cup\{\omega\}$. This latter set is equipped with an addition + that is the standard one on $\mathbb{N}$ together with the rule $\omega + x = x + \omega = \omega$ for all x in $\mathbb{N} \cup \{\omega\}$.

We denote by $[p]$ the set $\{1, \ldots, p\}$ and by $\mathbb{N}_+$ the set of positive integers.

A *weighted set* is a pair (X, λ) where X is a set and λ is a mapping $X \to \mathbb{N}_+ \cup \{\omega\}$. We call $\lambda(x)$ the *weight of* x, and, for $Y \subseteq X$, we define[4] $\lambda(Y) := \Sigma\{\lambda(x) \mid x \in Y\}$. A weighted set can be seen as a *multiset*, where $\lambda(x)$ is the number of occurrences of x. From a set X, we get the weighted set denoted by $(X, \mathbf{1})$ where all weights are 1. We define $Set(X, \lambda) := \{(x, i) \mid x \in X, i \in \mathbb{N}_+, 1 \leq i \leq \lambda(x)\}$ so that $\lambda(X) = |Set(X, \lambda)|$.

We denote by $\uplus$ the union of multisets, equivalently of weighted sets: $(X, \lambda) \uplus (Y, \lambda') := (X \cup Y, \lambda'')$ where $\lambda''(x)$ is $\lambda(x) + \lambda'(x)$ if $x \in X \cap Y$ and $\lambda(x)$ or $\lambda'(x)$ otherwise.

Let (X, λ) and (Y, λ') be weighted sets. A surjective mapping $\kappa : X \to Y$ is a *weighted surjection* or a *surjection of multisets*: $(X, \lambda) \to (Y, \lambda')$ if, for every $y \in Y$, we have $\lambda'(y) = \lambda(\kappa^{-1}(y))$, hence is the sum of weights of the x's such that $\kappa(x) = y$. If X is a set, hence, if λ has value 1 for all $x \in X$, a weighted surjection $\kappa : X \to Y$ satisfies $\lambda'(y) = \left|\kappa^{-1}(y)\right|$ for every $y \in Y$. Figure 1 illustrates this notion, see Example 2.2(1).

Lemma 2.1: Let (X, λ) and (Y, λ') be weighted sets.

1) A mapping $\kappa : X \to Y$ is a weighted surjection if and only if there exists a bijection $\kappa' : Set(X, \lambda) \to Set(Y, \lambda')$ such that[5] $\kappa'(x, i) = (y, j)$ implies $\kappa(x) = y$.

2) If there are weighted surjections $\kappa : (X, \lambda) \to (Y, \lambda')$ and $\alpha : Z = (Z, \mathbf{1}) \to (Y, \lambda')$, there exists a weighted surjection $\beta : Z = (Z, \mathbf{1}) \to (X, \lambda)$ such that $\alpha' = \kappa' \circ \beta'$, where α', κ', β' are

[4] For typographical reasons, we use the notation $\Sigma\{\lambda(x) \mid x \in Y\}$ rather than $\sum_{x\in Y} \lambda(x)$ and we will do the same below in Sections 3.1 and 4.2.

[5] To simplify notation, we write $\kappa'(x, i)$ instead of $\kappa'((x, i))$ and we will do the same in many similar cases.

related to α, κ, β as in 1). For each triple p, q, r such that $\kappa(p) = \alpha(r) = q$, we can define β such that $\beta(r) = p$.

3) We have $\lambda(X) = \lambda'(Y)$ if and only if there exists a set $S \subseteq X \times Y$ and a weight function μ on S such that $\mu(S) = \lambda(X) = \lambda'(Y)$ and for every $x \in X$, $\lambda(x) = \mu(S \cap \{(x, y) \mid y \in Y\})$ and similarly, for every $y \in Y$, $\lambda'(y) = \mu(S \cap \{(x, y) \mid x \in X\})$.

Proof:
Let (X, λ) and (Y, λ') be weighted sets.

1) Assume that we have $\kappa : X \to Y$ and a bijection $\kappa' : Set(X, \lambda) \to Set(Y, \lambda')$ as in the statement. Then κ is surjective. For each $y \in Y$, the mapping κ' induces a bijection $Set(\kappa^{-1}(y), \lambda) \to Set(\{y\}, \lambda')$, hence $\lambda'(y) = \lambda(\kappa^{-1}(y))$. Hence, κ is a weighted surjection.

Conversely, let $\kappa : X \to Y$ be a weighted surjection. For each y in Y, since $\lambda'(y) = \lambda(\kappa^{-1}(y))$, we can define a bijection: $Set(\kappa^{-1}(y), \lambda) \to Set(\{y\}, \lambda')$. The union of all these bijections defines κ' as desired.

2) Let κ and κ' be as in 1). We have a bijection $\alpha' : Z = Set(Z, \mathbf{1}) \to Set(Y, \lambda')$. We define $\beta' : Z = Set(Z, \mathbf{1}) \to Set(X, \lambda)$ by $\beta' := \kappa'^{-1} \circ \alpha'$, from which we get the desired weighted surjection $\beta : (Z, \mathbf{1}) \to (X, \lambda)$ such that $\alpha' = \kappa' \circ \beta'$. The condition on p, q, r is straightforward to satisfy.

3) Assume we have $\lambda(X) = \lambda'(Y)$. Consider any bijection $\mu' : Set(X, \lambda) \to Set(Y, \lambda')$. Then, we define $\mu(x, y)$ as the cardinality of the set $\{((x, i), (y, j)) \mid \mu'(x, i) = (y, j)\}$ if it is not empty. We let $S \subseteq X \times Y$ be the set of all pairs (x, y) such that $\mu'(x, i) = (y, j)$ for some i, j. We obtain the desired weight function on S. The converse is clear. $\square$

In Assertion 3), we call S a *witness of the equality* of weights $\lambda(X) = \lambda'(Y)$. If X and Y are disjoint, we can consider it as a bipartite graph whose edges are between X and Y, and are weighted by μ. The weight $\lambda(x)$ of vertex x is the sum of the weights of its incident edges. See Example 2.2(3).

Examples 2.2: *Weighted relations between weighted sets.*

(1) Let X consist of a, b, c, d of respective weights $2, 3, 4$ and ω and Y consist of u and v of respective weights 5 and ω. The mapping κ: $a \longmapsto u, b \longmapsto u, c \longmapsto v, d \longmapsto v$ is a weighted surjection, illustrated in Figure 1. One possible bijection κ' satisfying Assertion (1) of Lemma 2.1 is: $(a, i) \longmapsto (u, i)$ for $i = 1, 2$, $(b, i) \longmapsto (u, i + 2)$ for $i = 1, 2, 3$, $(c, i) \longmapsto (v, i)$ for $i = 1, \ldots, 4$, $(d, i) \longmapsto (v, i + 4)$ for $i \geq 1$.

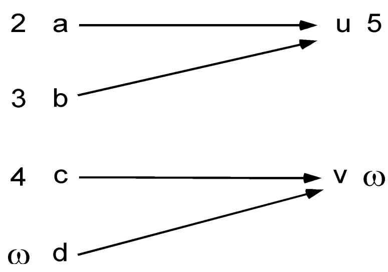

Figure 1. A weighted surjection, see Example 2.2(1).

(2) We examplify Assertion (2). Let X, Y, κ, κ' be as above and $Z := \mathbb{N}_+$. Let $\alpha : Z \to Y$ that maps $i \mapsto u$ for $i = 1, \dots, 5$ and $i \mapsto v$ for $i > 5$. We obtain β' that maps $i \mapsto (a, i)$ for $i = 1, 2$, $i \mapsto (b, i-2)$ for $i = 3, 4, 5$, $i \mapsto (c, i-5)$ for $i = 6, \dots, 9$, and $i \mapsto (d, i-9)$ for $i > 9$. We deduce the weighted surjection $\beta : Z \to Y$. This construction works if we are given $p := c, q := v$ and $r := 7$ (cf. the last point of Assertion (2)). If $p := d$, we can modify accordingly the definition of β'.

(3) To illustrate Assertion (3), we use X consisting of a, b, c, d of respective weights $\omega, 4, 2$ and ω and Y consisting of u, v, w, x, y of respective weights $\omega, 4, 3, 5, 1$. We can take S to consist of (a, u) and (d, u) of weight ω, $(a, v), (c, v), (c, w)$ and (d, y) of weight 1, (b, v) and (b, w) of weight 2 and (d, x) of weight 5. See Figure 2. This is clearly not the unique way to define S.

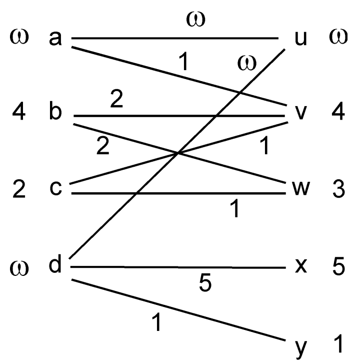

Figure 2. The weighted set S of Example 2.2(3).

If, with the same weighted set X, we take Y consisting of $y_1, \dots, y_n, \dots$ all of weight ω, then we can take S to consist of (b, y_1) of weight 4, (c, y_1) of weight 2 and (a, y_i) and (d, y_i) of weight ω for all i. $\square$

2.2. Graphs

By a *graph* we mean an undirected graph, and we call *digraph* a directed graph, for shortness sake.

A graph is defined as a triple $G = (V, E, Inc)$ where V is the set of vertices, E is the set of edges, and Inc is the *incidence relation*. The notation $e : x - y$ indicates that edge e links vertices x and y, called its *ends*, equivalently, that (e, x) and (e, y) belong to the set $Inc \subseteq E \times V$. A triple (V, E, Inc) defines a graph if and only if V and E are disjoint, $Inc \subseteq E \times V$, and for each $e \in E$, there are one or two vertices $x \in V$ such that $(e, x) \in Inc$.

A pair in Inc is called a *half-edge*. We write $e : x - x$ if e is a *loop at* x, i.e., *incident with* x. It is equivalent to a single half-edge. We denote by $E(x)$ the set of edges incident with x, and by $N(x)$ the set $\{y \in V \mid x - y\}$. We have $x \in N(x)$ if there is a *loop* at x. A graph is *simple* if no two edges have the same set of ends. Hence, it has no two parallel edges. It may have loops, where at most one loop is incident with any vertex.

A *walk* starting at a vertex x is a possibly infinite sequence $x_0, e_1, x_1, \dots, e_n, x_n, \dots$ such that $x = x_0, x_1, \dots, x_n, \dots$ are vertices and each e_i is an edge whose ends are x_{i-1} and x_i. It is a *path* if the vertices $x_0, \ldots, x_n, \ldots$ are pairwise distinct. In both cases, we say that each x_i *is accessible from* x_0. Its *length* is the number of edges. A path $x_0, \ldots, x_n$ defines a *cycle* if $n \geq 2$ and there an edge between x_0 and x_n. Its length is $n+1$.

A *directed graph* (a *digraph*) is defined similarly as a triple $G = (V, E, Inc)$. Its edges are called *arcs*. An arc e is directed from its *tail* x to its *head* y, and we denote this by $e : x \to y$. Its two *half-arcs* are (x, e) and (e, y), which encodes the direction of e. Hence $Inc \subseteq (V \times E) \cup (E \times V)$. A triple (V, E, Inc) defines a digraph if and only if V and E are disjoint, $Inc \subseteq (V \times E) \cup (E \times V)$, and for each $e \in E$, there are vertices $x, y \in V$ such that (x, e) and (e, y) belong to Inc.

A *loop* e at x has two half-arcs (x, e) and (e, x). A digraph is *simple* if, for any x, y, it has no two arcs from x to y. In that case, G can be defined as a pair (V, E) where $E \subseteq V \times V$. To simplify notation, we will also define such G as a pair (V, E) where an arc in E is defined the pair of a tail and a head.

We denote by $E^+(x)$ the set of arcs outgoing from x, and by $N^+(x)$ the set of heads of the arcs in $E^+(x)$. We have $x \in N^+(x)$ if there is a loop at x.

A *directed walk* starting at a vertex x is a possibly infinite sequence $x_0, e_1,\ x_1, \ldots, e_n, x_n, \ldots$ as above such that $x = x_0$ and $e_i : x_{i-1} \to x_i$ for each i. Without ambiguity unless it is reduced to the single vertex x_0, it can be specified as the sequence of arcs $e_1, \ldots, e_n, \ldots$. Its *length* is its number of arcs. It is a *directed path* if the vertices $x_0, \ldots, x_n, \ldots$ are pairwise distinct. We say that each x_i *is accessible from* x_0. A digraph is *strongly connected* if any two vertices are accessible from each other. A directed path $x_0, \ldots, x_n$ defines a *directed cycle* if $n \geq 1$ and there is an arc $x_n \to x_0$.

A *rooted digraph* G has a distinguished vertex called the *root*, denoted by rt_G, from which all vertices are accessible by a directed path. We denote by G/x the induced subgraph of G whose vertices are those accessible from x by a directed path. (The study of rooted trees uses this notion with same notation). We define x as its root.

We denote by $Und(G)$ the graph underlying a digraph G: each arc $e : x \to y$ of G is made into an edge $e : x - y$ of $Und(G)$. Hence, it need not be simple if G is.

We write $V_G, E_G, E_G(x), E^+_G(x), N^+_G(x), Inc_G$ etc. to specify, if necessary, the relevant graph or digraph G.

For graphs and digraphs, *inclusion* is denoted by $\subseteq$, *i.e.* $G = (V, E, Inc) \subseteq H = (V', E', Inc')$ if and only if $V \subseteq V', E \subseteq E'$ and $Inc \subseteq Inc'$. *Induced inclusion* denoted by $\subseteq_i$ holds if, furthermore, E is the set of edges or arcs of E' whose ends, tails and heads are in V. We write then $G = H[V]$.

A *homomorphism* $\eta : G \to H$ of graphs or of digraphs maps V_G to V_H, E_G to E_H, Inc_G to Inc_H and preserves incidences in the obvious way. It maps loops to loops but can map a nonloop edge or arc to a loop. If G and H are rooted, it maps the root of G to that of H. Isomorphism is denoted by $\simeq$ and the isomorphism class of G by $[G]_\simeq$.

If $\eta : G \to H$ is a homomorphism of graphs or of digraphs, we make G into a labelled graph or digraph G_η by equipping each vertex, edge or arc x by the *label* $\eta(x)$. Formally, $G_\eta = (V, E, Inc, \eta)$. Hence, this labelled graph encodes G and η. We will use this notion when H is finite. Other graph labellings will be defined at the relevant places.

We extend the notion of a homomorphism by allowing "forgetful" operations. A homomorphism $Und(G) \to H$ where G is directed and H is not is also considered as a homomorphism $G \to H$. Similar conventions concern labelled graphs.

Definition 2.3: *Quotient graphs and digraphs*

(a) An equivalence relation $\sim$ on a graph $G = (V, E, Inc)$ is an equivalence relation on $V \cup E$ such that each equivalence class is either a set of vertices or a set of edges, and, if e and e' are equivalent edges[6], then each end of e is equivalent to an end of e'.

(b) The *quotient graph* $G/\sim$ is then defined as $(V/\sim, E/\sim, Inc_{G/\sim})$ such that $([e]_\sim, [v]_\sim) \in Inc_{G/\sim}$ if and only if $(e', v') \in Inc$ for some $e' \sim e$ and $v' \sim v$.

(c) The definition is similar for a digraph G: we require that if e and f are equivalent arcs, then the tail (resp. the head) of e is equivalent to that of f. The quotient digraph is defined as for graphs.

(d) In both cases, we have a surjective homomorphism $\eta_\sim : G \to G/\sim$ that maps a vertex, an edge or an arc to its equivalence class. An edge $e : x - y$ is mapped to a loop in $G/\sim$ if $x \sim y$. The same holds for arcs.

Remark 2.4: An equivalence relation $\sim$ on the vertex set V of $G = (V, E, Inc)$ can be extended to edges or arcs as follows: two edges are equivalent if and only if each end of one is equivalent to some end of the other; two arcs are equivalent if and only if their tails are equivalent and so are their heads.

A notion of quotient graph of a digraph follows then by Definition 2.3. $\square$

2.3. Trees

A *tree* is a nonempty simple connected graph without loops or cycles. We call *nodes* its vertices. This convention is useful in the frequent case where we discuss simultaneously a graph and a tree constructed from it.

The set of nodes of a tree T is denoted by N_T. A *subtree* of a tree T is a connected subgraph, hence, it is a tree. A tree has *(locally) finite degree* if each node has finite degree. It has *bounded degree* if the degrees of its nodes are bounded by a same integer.

A *rooted tree* is a tree T equipped with a distinguished node r called its *root*. We denote it sometimes by T_r to specify simultaneously the root and the underlying undirected tree T. In a way depending on r, we direct its edges so that every node is accessible from r by a directed path. If $x \to y$ in T_r, then y is called a *son* of x, and x is the (unique) *father* of y. The *depth* of a node is its distance to the root (the root has depth 0). The *height* of a rooted tree is the least upper-bound of the depths of its nodes. A *star* is a rooted tree of height 1.

Let R be a rooted tree; its root is rt_R. By *forgetting* its root and making its arcs undirected, we get a tree $T := Unr(R)$. Hence, $R = T_{rt_R}$. If x is a node of R, then the digraph R/x is a rooted tree with root x, called the *subtree of R issued from* x. It is induced on the set of nodes accessible from x by a directed path. If $i \in \mathbb{N}$, the *truncation at depth* i *of* R, denoted by $R \restriction i$, is the induced subgraph of R whose nodes are at *distance* at most i from the root, that is, are accessible from it by a (unique) directed path of length at most i. It is a rooted tree with the same root as R and $R \restriction 0$ is the tree reduced to the root rt_R.

A *homomorphism of rooted trees*: $R \to R'$ is a homomorphism of directed graphs that maps rt_R to $rt_{R'}$. A homomorphism from a rooted tree R to a tree T is defined as a homomorphism of trees: $Unr(R) \to T$.

[6]An edge $e : x - y$ and a loop $f : z - z$ are equivalent if $x \sim y \sim z$.

Lemma 2.5: An isomorphism of rooted trees $\eta : R \to R'$ induces, for each $u \in N_R$, an isomorphism: $R/u \to R'/\eta(u)$ and, in particular, a bijection $N^+_R(u) \to N^+_{R'}(\eta(u))$ such that $R/v \simeq R'/\eta(v)$ if $v \in N^+_R(u)$.

3. Unfoldings of directed graphs

Certain abstract programs can be formalized as *transition systems* that are finite directed graphs with information attached to vertices and arcs. A vertex of the graph is a *state* of the corresponding transition system. An *initial state* r is specified. The tree of directed walks starting at r collects all possible computations of the corresponding transition system. It is called its *unfolding* [4, 12, 15].

We will consider unfoldings from a graph theoretical point of view, without offering any new application to semantics. We will generalize them and define unfoldings of digraphs whose arcs have *weights*. In particular, an arc of weight ω with head y unfolds into countably many arcs whose heads yield y by the unfolding homomorphism. We will obtain a notion of regular tree that generalizes the classical one in that the nodes can have infinite outdegrees. These trees are the unfoldings of finite, weighted and rooted digraphs.

In this section, all trees are rooted and thus directed in a canonical way. In [4, 12, 15] the *unfolding* of a rooted digraph G is what we will call its *complete unfolding*. We will call *unfolding* of such a digraph G a rooted digraph that lies inbetween, via surjective homomorphisms, the digraph G and its complete unfolding, denoted by $\mathit{Unf}(G)$. This terminology is thus similar to that concerning *coverings* and *universal coverings*.

The main contributions of this section are the use of possibly infinite weights, the decidablility of isomorphism in Theorem 3.14, and two theorems similar to those by Norris and Leighton for universal coverings of finite undirected graphs, see Theorems 3.20 and 3.22.

Equality of trees and digraphs will be understood in the strict sense: same nodes or vertices, and same arcs. Equality via an isomorphism is specified explicitely in statements and proofs, and denoted by $\simeq$.

3.1. Weighted directed graphs and their unfoldings

We will equip digraphs with weights in $\mathbb{N}_+ \cup \{\omega\}$. We recall from Section 2.2 that a digraph can be defined as a pair (V, E) where each arc e is an ordered pair of vertices.

Definition 3.1: *Weighted digraphs.*

A *weighted digraph* is a triple $G = (V, E, \lambda)$ such that (V, E) is a digraph whose set of arcs E is *weighted*, that is, equipped with a *weight function* $\lambda : E \to \mathbb{N}_+ \cup \{\omega\}$. We denote by $\overline{E^+}(u)$ the weighted set $(E^+(u), \lambda)$ and by $\overline{N^+}(u)$ the weighted set $(N^+(u), \lambda')$ such that $\lambda'(v) = \Sigma\{\lambda(e) \mid e : u \to v\}$.

A digraph[7] is a weighted digraph whose arcs have all the weight 1.

[7] *Digraph* will mean "without weights" and possibly with parallel arcs.

A weighted digraph is *simple* or *rooted* if the underlying digraph is. If x is a vertex of a weighted digraph G, then G/x (cf. Section 2.2) is a rooted and weighted digraph with root x. If G is strongly connected, the digraphs G/x have all the same vertices and arcs as G.

In the special case where G is simple digraph, then $\overline{E^+}(u) = (E^+(u), \mathbf{1})$, $\overline{N^+}(u) = (N^+(u), \mathbf{1})$ and the head mapping is a bijection $E^+(u) \to N^+(u)$. $\square$

We can handle parallel arcs by means of weights. That is, an arc (x,y) of weight $\lambda(x,y) > 1$ encodes $\lambda(x,y)$ parallel arcs from x to y.

Definition 3.2: *Unfolding*

Let H and G be rooted and weighted digraphs.

(a) A surjective homomorphism $\eta : G \to H$ is an *unfolding* of H if it induces a weighted surjection $\overline{E_G} \to \overline{E_H}$. In particular, if $u \in V_G$ and $\eta(u) = x$, then η induces a weighted surjection $\overline{E_G^+}(u) \to \overline{E_H^+}(x)$. If G and H are simple digraphs, then η induces a bijection $E_G^+(u) \to E_H^+(x)$ and a bijection $N_G^+(u) \to N_H^+(x)$.

We will also say that G is an *unfolding* of H or that H *unfolds into* G. From the accessibility condition in the definition of a rooted digraph, unfoldings only concern connected graphs. They are called *op-fibrations* by Boldi and Vigna [9].

(b) An unfolding $G \to H$ is *complete* if G is a rooted tree without weights (equivalently, all weights are 1). We will also say that G is *a complete unfolding* of H or that H *unfolds completely into* G.

Examples 3.3: (1) A loop of weight 1 (resp. 2) unfolds completely into an infinite directed path (resp. into the infinite binary rooted tree).

(2) An arc $x \to y$ of weight ω such that x is taken as root unfolds (not completely) into any finite star, where at least one arc has weight ω. It unfolds completely into a star S_ω, *i.e.,* any tree whose root has ω sons that are leaves[8]. If in addition, there is a loop $y \to y$ of weight 1, this rooted and weighted digraph unfolds completely into the union of ω infinite directed paths with the same origin, that are otherwise disjoint.

Proposition 3.4: (1) If $\eta : G \to H$ and $\kappa : H \to K$ are unfoldings, then $\kappa \circ \eta$ is an unfolding $G \to K$.

(2) If $\eta : G \to H$ is an unfolding, $u \in V_G$ and $x = \eta(u)$, then η is an unfolding[9] $G/u \to H/x$.

Proof:

(1) The composition $\kappa \circ \eta$ induces a weighted surjection $E_G \to E_K$ as κ and η do the same $E_H \to E_K$ and $E_G \to E_H$ respectively. This observation proves the assertion.

(2) Clear from Definition 3.2. $\square$

The following theorem implies that every rooted and weighted digraph H has, up to isomorphism, a unique complete unfolding.

[8]Any two such trees are isomorphic. By thinking of trees up to isomorphism, which is adequate since any two complete unfoldings of a rooted digraph are isomorphic, we can also write *the* star S_ω.

[9]This is a short expression for "the restriction of η to G/u is an unfolding $G/u \to H/x$". Similar shortenings will be used at other places.

Theorem 3.5: Let H be a rooted and weighted digraph.
1) H has a complete unfolding.
2) If $\beta : T \to H$ is a complete unfolding, then:

> (U) For every unfolding $\kappa : G \to H$, there is a complete unfolding $\eta : T \to G$ such that $\beta = \kappa \circ \eta$.

3) Any two complete unfoldings of H are isomorphic.
4) If $\beta : T \to H$ is an unfolding such that Condition (U) holds, then T is a rooted tree, hence a complete unfolding of H.

Properties 2) and 4) show that the complete unfoldings of H are characterized by a universal property in the sense of Category Theory. One can speak of *the* complete unfolding of *H*, well-defined up to isomorphism. The following notion helps to approximate, level by level, a complete unfolding. The *height* of a rooted tree is the least upper-bound of the distances of its nodes to the root.

Definition 3.6: *Depth-limited unfoldings.*

Let A be a rooted tree of *height* at most i (cf. Section 2.3) and H be a rooted and weighted digraph. An i*-unfolding* $\eta : A \to H$ is a homomorphism (it is not necessarily surjective) satisfying the following condition:

> For every node u of A at distance at most $i-1$ from the root, if $\eta(u) = x$ and e is an arc of H with tail x, then $\left|\eta^{-1}(e) \cap E_A^+(u)\right| = \lambda_H(e)$. $\square$

The complete unfolding of a rooted unweighted digraph H can be constructed as the tree of finite walks starting from the root. As weights in digraphs represent parallel arcs, this construction must be adapted. This is the purpose of the following definition, that replaces parallel arcs with sets of parallel ones.

Definition 3.7: *The expansion of a weighted digraph.*

Let $H = (V, E, \lambda)$ be a weighted digraph. Its *expansion* is the digraph $Exp(H) = (V, Set(E, \lambda))$ having the arc $(e, i) : x \to y$ if $e : x \to y$ in H and $(e, i) \in Set(E, \lambda)$. (The mapping Set is defined in Section 2.1) The digraph $Exp(H)$ is infinite if some arc has weight ω, and/or, of course, if V is infinite. If H has a root, then $Exp(H)$ has the same root.

We now prove Theorem 3.5.

Proof:
Let H be a rooted and weighted digraph.

1) The rooted digraph $Exp(H)$ is an unfolding of H, and we denote by ε the corresponding homomorphism $Exp(H) \to H$. By Proposition 3.4, we need only construct a complete unfolding T of $Exp(H)$. We define it as the tree of directed walks in $Exp(H)$ that start from rt_H, the common root of $Exp(H)$ and H. The father of a node $(e_1, \ldots, e_p)$ is $(e_1, \ldots, e_{p-1})$.

Let $\alpha : T \to Exp(H)$ map $(e_1, \ldots, e_p)$ to the head of e_p; if $p = 0$, then $(e_1, \ldots, e_p)$ is the empty walk, mapped to rt_H; the arc from $(e_1, \ldots, e_{p-1})$ to $(e_1, \ldots, e_p)$ is mapped to e_p. We say that this arc of T is *of type* e_p.

Then α is a complete unfolding $T \to Exp(H)$ and $\beta := \varepsilon \circ \alpha$ yields a complete unfolding $T \to H$. We will denote T by $Unf(H)$. Note that $Unf(H)$ is a concrete tree made of walks in $Exp(H)$.

If H is a rooted tree, then $Exp(H) \simeq H$ and α and β are isomorphisms as one checks easily.

2) We let $\beta : Unf(H) \to H$ be the particular complete unfolding constructed in 1) and $\kappa : G \to H$ be any unfolding.

By induction on i, we construct for each i, an i-unfolding $\eta_i : Unf(H) \upharpoonright i \to G$ such that $\kappa \circ \eta_i$ is the restriction of β to $Unf(H) \upharpoonright i$ (the restriction of $Unf(H)$ to nodes at distance at most i from the root) in such a way that η_{i+1} extends η_i. The union of the mappings η_i will be a complete unfolding $\eta : Unf(H) \to G$ such that $\beta = \kappa \circ \eta$.

We construct η_{i+1} from η_i as follows. Let $u = (e_1, \ldots, e_i) \in N_{T\upharpoonright i}$ be mapped to $w \in V_G$ by η_i. There is a weighted surjection $\mu_u : N^+_{Unf(H)}(u) \to \overline{N^+_G}(w)$ such that $\kappa \circ \mu_u$ is the restriction of β to $N^+_{Unf(H)}(u)$. Its existence follows from Lemma 1.1(2), as $N^+_{Unf(H)}(u)$ is a set, equivalently, the weighted set $\overline{N^+_{Unf(H)}}(u) = (N^+_{Unf(H)}(u), \mathbf{1})$. Then, we let η_{i+1} be the union of η_i and all such mappings μ_u for all nodes u of $Unf(H)$ at depth i.

To prove 3) and to complete the proof of 2), we let $\kappa : G \to H$ be a complete unfolding, hence, G is a tree. Then, the complete unfolding $\eta : Unf(H) \to G$ is an isomorphism. Hence, any two complete unfoldings of H are isomorphic and 2) holds for any complete unfolding β of H.

4) Let $\beta : T \to H$ be an unfolding such that Condition (U) holds. Let G be a complete unfolding of H. There is an unfolding $\gamma : T \to G$. Since G is a tree, T is also a tree, hence a complete unfolding of H. □

We will reserve the notation $Unf(H)$ to the complete unfolding defined as a tree of walks in H. It is defined in [4, 12, 15], but not characterized by a universality property.

3.2. Complete unfoldings and regular trees

The notion of an infinite regular tree is important in applications to semantics, in particular because the complete unfolding of a finite transition system is regular [4, 10, 11], and more generally for the monadic second-order logic of infinite structures, see [12, 15]. We will consider regular trees that are complete unfoldings of finite digraphs.

A graph, a digraph or a tree can have labels attached to its vertices, nodes, edges or arcs.

Definition 3.8: *Regular trees.*

A rooted, possibly labelled, tree T is *regular*[10] if it has finitely many subtrees T/x (inheriting the possible labels of T), *up to isomorphism*, which we will denote by *u.t.i.*, that is, if the set of isomorphism classes $\{[T/x]_\simeq \mid x \in N_T\}$ is finite. In the latter case, its cardinality is the *regularity index* of T, denoted by $Ind(T)$. If T is regular, each subtree T/x is regular of no larger index because $(T/x)/y = T/y$ for $y \leq_T x$ (which means that x is on the directed path from the root to y). □

[10]Slightly different notions of regular trees are studied in [10, 12, 15]. However, they have in common the finiteness of the set of subtrees T/x up to isomorphism.

Every finite tree is regular. A rooted tree of height 1 (a *star*) is regular of index 2. We will prove that the complete unfolding of a finite, rooted and weighted digraph H, that may have infinite weights, is regular of index at most $|V_H|$ and has a canonical "factorization" in terms of a finite weighted digraph analoguous to the minimal automaton of a regular language.

Let G be a weighted digraph. Let $\approx$ be the equivalence relation[11] on V_G such that $x \approx y$ if and only if $Unf(G/x) \simeq Unf(G/y)$. According to the definitions of Section 1.2, the quotient $H := G/\approx$ is the simple digraph defined as follows: $V_H := \{[x]_\approx \mid x \in V_G\}$ and $E_H := \{([x]_\approx, [y]_\approx) \mid G$ has an arc $x \to y\}$. If G is rooted, we take $rt_H := [rt_G]_\approx$. We have a surjective homomorphism $\eta : G \to H$. We now define weights on the arcs of H. If e is an arc $x \to y$ of G, we define $\lambda'_G(e) := \Sigma\{\lambda_G(f) \mid f : x \to z$ is an arc of G for some $z \approx y\}$.

Lemma 3.9: Let x, x' be vertices of G such that $x \approx x'$.

(1) If there is an arc $x \to y$ for some y, then there is one $x' \to y'$ such that $y' \approx y$.

(2) If e is an arc $x \to y$, e' is an arc $x' \to y'$ such that $y' \approx y$, then $\lambda'_G(e) = \lambda'_G(e')$.

Proof:

(1) Assume $x \approx x'$. Let $\alpha : Unf(G/x) \to G/x$ by the unfolding homomorphism, mapping the root $\overline{x}$ of $Unf(G/x)$ to x, and similarly $\alpha' : Unf(G/x') \to G/x'$ mapping the root $\overline{x'}$ of $Unf(G/x')$ to x'. Let μ be an isomorphism $Unf(G/x) \simeq Unf(G/x')$. It maps $\overline{x}$ to $\overline{x'}$.

Let e be an arc $x \to y$ of G. There is u in $Unf(G/x)$ such that $\alpha(u) = y$ and $\overline{x} \to u$ in $Unf(G/x)$. Let $y' := \alpha'(\mu(u))$. We have $\overline{x'} \to \mu(u)$, hence an arc $x' \to y'$ in G.

We have $Unf(G/y) \simeq Unf(G/x)/u \simeq Unf(G/x')/\mu(u) \simeq Unf(G/y')$. Hence, $y' \approx y$.

(2) If e is an arc $x \to y$ of G and with the same notation as in (1), we observe that, since α is an unfolding, $\lambda'_G(e)$ is the number of sons u of the root of the tree $Unf(G/x)$ such that $Unf(G/x)/u \simeq Unf(G/y)$. Then, if e' is an arc $x' \to y'$, we have similarly that $\lambda'_G(e')$ is the number of sons u' of the root of the tree $Unf(G/x')$ such that $Unf(G/x')/u' \simeq Unf(G/y')$. Since $y' \approx y$, we have $\lambda'_G(e) = \lambda'_G(e')$. □

Definition 3.10: *The canonical quotient of a rooted and weighted digraph.*

Let G be a rooted and weighted digraph and $H := G/\approx$ as above. The mapping η such that $\eta(x) := [x]_\approx$ if $x \in V_G$ and $\eta(e) := ([x]_\approx, [y]_\approx)$ if e is an arc $x \to y$ of G is a homomorphism $G \to H$ that is surjective by Lemma 3.9(1).

We define a weight function on H by $\lambda_H([x]_\approx, [y]_\approx) := \lambda'_G(e)$ for any arc $e : x \to z$ of G such that $z \approx y$. It is well-defined by Lemma 3.9(2).

Furthermore, if G is vertex-labelled, then $x \approx y$ implies that x and y have same label. The quotient digraph $H := G/\approx$ is vertex-labelled and the homomorphism $\eta : G \to H$ preserves labels.

We define the *size* $|G|$ of a digraph G as $|V_G| + |E_G|$. □

Proposition 3.11: (1) The homomorphism $\eta : G \to G/\approx$ is an unfolding.

(2) If G is finite, then $G/\approx$ is, up to isomorphism, the unique rooted and weighted digraph of minimal size of which G is an unfolding.

[11] If G is rooted so that $Unf(G)$ is defined, an equivalent expression of $x \approx y$ is $Unf(G)/u \simeq Unf(G)/v$ where u, v are nodes of $Unf(G)$ such that $\alpha(u) = x$, $\alpha(v) = y$ and $\alpha : Unf(G) \to G$ is the complete unfolding.

Proof:
(1) As observed in Definition 3.10, the homomorphism $\eta : G \to G/\approx$ is surjective. It is an unfolding by the proof of Lemma 3.9(2).

(2) If G is finite, then $|G| \geq |H|$. If $\alpha : G \to K$ is an unfolding, then there is an unfolding $\beta : K \to H$ that we define as follows:

$\beta(u) := [x]_\approx$ where $u \in V_K$ and $\alpha(x) = u$;

$\beta(e) := ([x]_\approx, [y]_\approx)$ where $e : u \to v$ is an edge of K, $\alpha(x) = u$, $\alpha(y) = v$ and $x \to y$ is an edge of G.

It is easy to see that β is an unfolding. Hence, $|K| \geq |H|$. If $|K| = |H|$, it is an isomorphism. $\square$

Example 3.12: Figure 3 shows to the left a weighted digraph G with vertex set $\{s, u, v, w, x, y, z\}$. The weights that are not shown are equal to 1. We have $s \approx w \approx y$ and $u \approx v \approx x \approx z$. The quotient digraph $G/\approx$ is shown to the right. $\square$

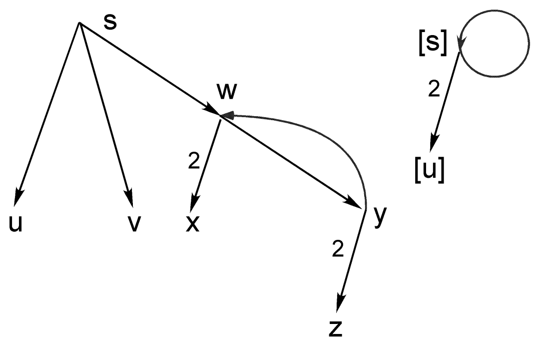

Figure 3. A digraph G and its quotient $G/\approx$, cf. Example 3.12.

We now consider the case where R is a regular tree. Theorem 3.14 will prove that $\approx$ and $G/\approx$ are computable if G is finite.

Theorem 3.13: (1) A rooted tree T is regular of index at most p if it is the complete unfolding of a finite, rooted and weighted digraph having p vertices.

(2) Conversely, a regular tree T is the complete unfolding of a unique rooted and weighted simple digraph having $Ind(T)$ vertices.

(3) If $\eta : T \to H$ is a complete unfolding of a rooted and weighted digraph having p vertices ($p \in \mathbb{N}_+$), then the labelled rooted tree T_η (where each node u is labelled by $\eta(u)$) is regular of index at most p.

Proof:
(1) Let $\eta : T = Unf(H) \to H$ be the unfolding homomorphism where H is a rooted and weighted digraph having p vertices. If $u, v \in N_T$ and $\eta(u) = \eta(v) = x$, then $T/u \simeq T/v$ because these two trees are complete unfoldings of H/x by Proposition 3.4(2). It follows that T is regular and its index is at most the number of vertices of H.

(2) Conversely, let T be a regular tree of index p. Let $\approx$ be the equivalence relation on N_T such that $u \approx v$ if and only if $T/u \simeq T/v$. We have $T/u \simeq \mathit{Unf}(T/u)$. The quotient construction of Definition 3.10 shows that T is the complete unfolding of the finite, rooted and weighted digraph $T/\approx$, that has p vertices.

(3) Easy extension of (1). $\square$

Finite, rooted and weighted digraphs can be used as finite descriptions of regular trees. Although an arc of weight $p \in \mathbb{N}_+$ can be replaced (cf. Definition 3.7) by p parallel arcs and a loop of weight $q \in \mathbb{N}_+$ by q loops, the use of weights gives more concise descriptions. Furthermore, the weight ω makes it possible to describe trees of infinite degree in finitary ways, by means of finite arc-labelled digraphs[12]. The following result shows that this description is effective.

Theorem 3.14: Given a finite weighted digraph H and two vertices $x, y \in V_H$, one can decide whether $\mathit{Unf}(H/x) \simeq \mathit{Unf}(H/y)$.

We need a few technical definitions and lemmas.

Definition 3.15: *Equivalent weighted sets.*

Let R be an equivalence relation on a set V and $X, Y \subseteq V$.

(a) Let $\overline{X} = (X, \lambda)$ and $\overline{Y} = (Y, \lambda')$ be weighted sets.

We write $\overline{X} \sim \overline{Y} \pmod R$ if:

(C) For every equivalence class C of R, we have $\lambda(C \cap X) = \lambda'(C \cap Y)$.

Equivalently, for every $x \in X$, there is $y \in Y$ such that $\lambda([x]_R \cap X) = \lambda'([y]_R \cap Y)$ and $(x, y) \in R$, and similarly, for every $y \in Y$, there is $x \in X$ such that (x, y) satisfies the same property. This is an equivalence relation. Condition (C) implies that $\lambda(X) = \lambda'(Y)$.

(b) A *witness* of the equivalence $\overline{X} \sim \overline{Y} \pmod R$ is a set $S \subseteq X \times Y$ with weight function μ, that is the (disjoint) union of witnesses of the weight equalities $\lambda(C \cap X) = \lambda'(C \cap Y)$ for all equivalence classes C of R, (cf. Lemma 2.1(3). $\square$

We say that an equivalence relation R *refines* an equivalence relation R' on the same set if each class of R' is a union of classes of R. This is written $R \subseteq R'$, by considering equivalence relations as sets of pairs.

In the following two lemmas, $V, R, \overline{X}$ and $\overline{Y}$ are as in the previous definition.

Lemma 3.16: If $\overline{X} \sim \overline{Y} \pmod R$ and $R \subseteq R'$, then $\overline{X} \sim \overline{Y} \pmod{R'}$.

Proof:

Each class C' of R' is the union of (disjoint) classes $C_1, C_2, \dots$ of R. Hence, $\lambda(C' \cap X) = \lambda(C_1 \cap X) + \lambda(C_2 \cap X) + \dots$ and similarly for Y. The result follows. $\square$

[12]In Section 4, weights on half-edges of graphs will be even more important, as they will allow us to describe, as universal coverings of finite weighted graphs, trees of finite degree that are *not* universal coverings of any finite graph. Furthermore, weights ω will yield trees with nodes of infinite degree.

Lemma 3.17: Assume that $\overline{X} \sim \overline{Y} \pmod R$. Let U and W be sets, and κ and η be weighted surjections[13], respectively $U \to \overline{X}$ and $W \to \overline{Y}$. There is a bijection $\ell : U \to W$ such that $(\kappa(u), \eta(\ell(u)) \in R$ for all u in U. Furthermore, for any $u_0 \in U$ and $w_0 \in W$ such that $(\kappa(u_0), \eta(w_0)) \in R$, one can find ℓ as above such that $\ell(u_0) = w_0$.

Proof:
We have a bijection $\gamma : Set(X, \lambda) \to Set(Y, \lambda')$ such that[14] $\gamma(x, i) = (y, j)$ implies $(x, y) \in R$, and bijections $\kappa' : U \to Set(X, \lambda)$ and $\eta' : W \to Set(Y, \lambda')$. We define $\ell := \eta'^{-1} \circ \gamma \circ \kappa'$.

For proving the last assertion, we choose γ such that $\gamma(\kappa(u_0), i) = (\eta(w_0), j)$ for some i, j. $\square$

The bijection ℓ is uniquely defined if $\lambda([x]_R \cap X) = \lambda'([y]_R \cap Y) = 1$ for all $x \in X$ and $y \in Y$, but not otherwise. We now prove Theorem 3.14.

Proof:
Let H be a finite weighted digraph[15] and $\approx$ be the equivalence relation on V_H such that $x \approx y$ if and only if $Unf(H/x) \simeq Unf(H/y)$. We recall from Section 2 that $N_H^+(x)$ is the set of heads of the arcs with tail x and that $\overline{N_H^+}(x) := (N_H^+(x), \eta)$ where $\eta(y)$ is the sum of the weights $\lambda_H(e)$ of the arcs $e : x \to y$ (cf. Definition 3.2, H may have parallel arcs).

Claim 1: The equivalence relation $\approx$ satisfies the following property, that we state for an arbitrary equivalence relation R on V_H:

(E): If xRy then $\overline{N_H^+}(x) \sim \overline{N_H^+}(y) \pmod R$.

Proof: This follows from Lemma 2.4 and the definitions. $\square$

Claim 2: If R is an equivalence relation on V_H that satisfies Property (E), then $R \subseteq \approx$.

Proof: We consider $(x, y) \in R$, and we let $\kappa : T \to H/x$ and $\kappa' : T' \to H/y$ be the unfolding homomorphisms where $T := Unf(H/x)$ and $T' := Unf(H/y)$.

For each i, we construct by induction an isomorphism $\eta_i : T \restriction i \to T' \restriction i$ such that $(\kappa(u), \kappa'(\eta_i(u))) \in R$ for every node u of $T \restriction i$, and η_{i+1} extends η_i. The common extension of these isomorphisms will be an isomorphism $T \to T'$, proving that $x \approx y$.

We let η_0 map rt_T to $rt_{T'}$. We have $(x, y) = (\kappa(rt_T), \kappa'(\eta_0(rt_T))) \in R$, as was to be verified.

We now define η_{i+1} extending η_i.

Consider v in $T \restriction (i + 1)$ at depth $i + 1$ and its father u. Then $w := \eta_i(u)$ is a node of $T' \restriction i$. Furthermore κ induces a weighted surjection $N_T^+(u) \to \overline{N_H^+}(\kappa(u))$, and similarly, κ' induces a weighted surjection $N_T^+(w) \to \overline{N_H^+}(\kappa'(w))$. By the inductive property of η_i, we have $(\kappa(u), \kappa'(w)) \in R$. Hence, by Property (E), we have $\overline{N_H^+}(\kappa(u)) \sim \overline{N_H^+}(\kappa'(u)) \pmod R$. By Lemma 3.17, there is a bijection $\ell_u : N_T^+(u) \to N_T^+(w)$ such that $(\kappa(s), \kappa'(\ell_u(s)) \in R$ for each s in $N_T^+(u)$ (s is in $T \restriction (i + 1)$). We define $\eta_{i+1}(s) := \ell_u(s)$ for every son s of u in T .

[13] A weighted surjection of a set X onto a weighted set is well-defined by considering that each element of X has weight 1.
[14] We recall that we write $\gamma(x, i)$ for $\gamma((x, i))$.
[15] It is not necessarily rooted.

We do that for all nodes v at depth $i+1$ in T. We obtain the desired extension with the inductive property $(t, \eta_{i+1}(t)) \in R$ for every node t of $T \upharpoonright (i+1)$. $\square$

There are finitely many equivalence relations R on V_H. For each of them, one can check if it satisfies Property (E) and contains the pair (x, y). Then $x \approx y$ if and only if one of them has these two properties. $\square$

The following algorithm is similar to the minimization of finite deterministic automata. It will help to prove Theorem 3.20.

Algorithm 3.18: *Deciding the isomorphism of complete unfoldings.*

Input: A finite weighted digraph[16] H.

Output: The equivalence relation $\approx$ on V_H such that $x \approx y$ if and only if $\mathit{Unf}(H/x) \simeq \mathit{Unf}(H/y)$.

Method: We define a decreasing[17] sequence of equivalence relations $R_i, i \geq 0$ on V_H as follows:

$$R_0 = V_H \times V_H;$$
$$R_{i+1} = R_i \cap \{(x,y) \mid \overline{N_H^+}(x) \sim \overline{N_H^+}(y) \pmod{R_i}\}.$$

We have $R_{i+1} = R_i$ for some $i := i_{\max}$, and we output R_i as the desired result.

Proposition 3.19: Algorithm 3.18 is correct and terminates with $i_{\max} \leq |V_H| - 1$.

Proof:

Let R be the intersection of the relations R_i. It is clear that if $R_{i+1} = R_i$, then $R_{i+2} = R_{i+1}$ etc... so that, $R_i = R$. This guarantees termination.

Each step such that $R_{i+1} \neq R_i$ splits at least one equivalence class of R_i. Such a splitting cannot be done more than $|V_H| - 1$ times.

We now prove the correctness, *i.e.*, that $\approx = R$.

We prove that $\approx \subseteq R_i$ for all i. This is clear for $i = 0$. Assume now $\approx \subseteq R_i$. If $x \approx y$, then $\overline{N_H^+}(x) \sim \overline{N_H^+}(y) \pmod{\approx}$, hence $\overline{N_H^+}(x) \sim \overline{N_H^+}(y) \pmod{R_i}$ by Lemma 3.16, and so, $(x, y) \in R_{i+1}$. Hence, $\approx \subseteq R$.

The relation R satisfies Property (E), hence $R \subseteq \approx$ by Claim 2 in the proof of Theorem 3.14. $\square$

The following result is similar to a theorem by Norris [21] about universal coverings that we presented in the introduction and that we will generalize in Section 4 to weighted graphs. See [13], it implies that, for every regular tree, there is a first-order sentence using the generalized quantifier "there exists ω elements x that satisfy..." of which it is the unique model that is a rooted tree.

Theorem 3.20: Let H be a finite weighted digraph with p vertices. If $x, y \in V_H$, then:

$$\mathit{Unf}(H/x) \upharpoonright (p-1) \simeq \mathit{Unf}(H/y) \upharpoonright (p-1) \text{ implies } \mathit{Unf}(H/x) \simeq \mathit{Unf}(H/y).$$

[16] It need not be connected. In order to decide whether $\mathit{Unf}(G/x) \simeq \mathit{Unf}(G'/y)$ where $G \neq G'$, we can use this algorithm by taking for H the union of G and a disjoint copy of G'.

[17] It is decreasing for set inclusion. Hence, the equivalence R_{i+1} refines R_i.

Proof:
We use the relations R_i of Algorithm 3.18. We know by Proposition 3.19 that $\approx = R_{p-1}$.

Claim: If $Unf(H/x) \upharpoonright (p-1) \simeq Unf(H/y) \upharpoonright (p-1)$, then $(x,y) \in R_{p-1}$.

Proof: By using induction, we prove that for every i:
$Unf(H/x) \upharpoonright i \simeq Unf(H/y) \upharpoonright i$ implies $(x,y) \in R_i$.
If $i = 0$, this fact holds because $(x,y) \in R_0$ for all x, y.

We prove the case $i+1$ by assuming that we have an isomorphism $\alpha : Unf(H/x) \upharpoonright (i+1) \to Unf(H/y) \upharpoonright (i+1)$. Hence $Unf(H/x) \upharpoonright i \simeq Unf(H/y) \upharpoonright i$ and $(x,y) \in R_i$ by the induction hypothesis.

We now check that $\overline{N_H^+}(x) \sim \overline{N_H^+}(y) \pmod{R_i}$ in order to obtain that $(x,y) \in R_{i+1}$.

Let $\eta : Unf(H/x) \to H/x$ and $\eta' : Unf(H/y) \to H/y$ be complete unfoldings. We have $\eta(rt_{Unf(H/x)}) = x$ and $\eta'(rt_{Unf(H/y)}) = y$.

For each son u of $rt_{Unf(H/x)}$, α defines an isomorphism:
$Unf(H/x)/u \upharpoonright i \to Unf(H/y)/\alpha(u) \upharpoonright i$,
where $\alpha(u)$ is a son of $rt_{Unf(H/y)}$. But $Unf(H/x)/u = Unf(H/\eta(u))$ and $Unf(H/y)/\alpha(u) = Unf(H/\eta'(\alpha(u)))$. Hence $(\eta(u), \eta'(\alpha(u))) \in R_i$ by induction.

Then $N_H^+(x)$ is the set of such $\eta(u)$ and $N_H^+(y)$ is that of such $\eta'(\alpha(u))$. By counting occurrences, we obtain $\overline{N_H^+}(x) \sim \overline{N_H^+}(y) \pmod{R_i}$. Hence, $(x,y) \in R_{i+1}$. □

If $Unf(H/x) \upharpoonright (p-1) \simeq Unf(H/y) \upharpoonright (p-1)$, we have $(x,y) \in R_{p-1}$ by the claim, hence $x \approx y$ as was to be proved since $R_{p-1} = \approx$ by Proposition 3.18. □

Remark 3.21: By the proof of Proposition 3.19, $Unf(H/x) \upharpoonright (p-1) \simeq Unf(H/y) \upharpoonright (p-1)$ implies $Unf(H/x) \upharpoonright i \simeq Unf(H/y) \upharpoonright i$ for all i. We might think that this implies $Unf(H/x) \simeq Unf(H/y)$. This argument is correct only if $Unf(H/x)$ and $Unf(H/y)$ have finite degree, by using König's Lemma, as in the proof of Lemma 2.7 of [17].

However, this implication is false for trees with nodes of infinite degree. Let T be the union of the finite paths $0 \to (1,i) \to (2,i) \to \cdots \to (i,i)$ for all $i \in \mathbb{N}_+$, and T' be T together with the infinite path $0 \to 1 \to 2 \to \cdots \to i \to \ldots$. They are not isomorphic, but $T \upharpoonright i \simeq T' \upharpoonright i$ for each i. Theorem 3.14 is used for proving Theorem 3.20. To prove its Claim 2, we cannot use König's Lemma because the trees $Unf(H/x)$ and $Unf(H/y)$ need not have finite degree. Instead, we construct a sequence of isomorphisms:

$$\eta_i : Unf(H/x) \upharpoonright i \to Unf(H/y) \upharpoonright i \text{ such that } \eta_{i+1} \text{ extends } \eta_i.$$

Their common extension yields an isomorphism: $Unf(H/x) \to Unf(H/y)$. □

The following theorem is similar to that of Leighton about coverings ([18], see below Theorem 4.10), and much easier to prove.

Theorem 3.22: Given two finite, rooted and weighted digraphs G and H, the following properties are equivalent:

1) G and H are unfoldings of a finite rooted and weighted digraph,

2) G and H have isomorphic complete unfoldings,
3) G and H have a common finite unfolding.
They are decidable.

Proof:
Without loss of generality, we assume that G and H are disjoint.

1) $\Longrightarrow$2) If G and H are unfoldings of a finite rooted and weighted digraph M, then the complete unfolding of M is a complete one of both G and H by Theorem 3.5(2).

2) $\Longrightarrow$3) Let $\gamma : T \to G$ and $\eta : T \to H$ be complete unfoldings of G and H.

If $u \in N_T$, then $T/u \simeq \mathit{Unf}(G/\gamma(u)) \simeq \mathit{Unf}(H/\eta(u))$ by Proposition 3.4(2).

We define $\approx$ as the equivalence relation on $V_G \cup V_H$ such that $x \approx y$ if and only if $\mathit{Unf}((G \cup H)/x) \simeq \mathit{Unf}((G \cup H)/y))$, where $\mathit{Unf}((G \cup H)/x) = \mathit{Unf}(G/x)$ if $x \in V_G$ and similarly for H as G and H are disjoint.

For helping to understand the technical details, we first present the proof for the special case where there are no two distinct nodes u, v in T with same father, and such that $T/u \simeq T/v$. This fact implies that all arcs in G and H have weight 1. In such a case:

(*) if $u \in N_T$, the relation $T/v \simeq \mathit{Unf}(G/y)$ defines by Lemma 2.4 a bijection between the sons v of u in T and the vertices y in $N^+_G(\gamma(u))$. A similar fact holds for H with the vertices y in $N^+_H(\eta(u))$.

We define a digraph L as follows. Its set of vertices is $V_L := \{(x, y) \mid x \in V_G, y \in V_H$ and $\mathit{Unf}(G/x) \simeq \mathit{Unf}(H/y)\}$. For each $(x, y) \in V_L$, the relation $\approx$ defines, by Fact (*) above, a bijection between $N^+_G(x)$ and $N^+_H(y)$.We define in L an arc $(x, y) \to (x', y')$ (of weight 1) if $x' \in N^+_G(x)$ and $y' \in N^+_H(y)$ (and of course $x' \approx y'$).

We now define $K := L/(rt_G, rt_H)$. It is a finite and rooted digraph. The projection π_1 such that $\pi_1(x, y) := x$ is an unfolding $K \to G$. The other projection is an unfolding $K \to H$.

We now consider the general case. The construction is similar, but the definition of the arcs $(x, y) \to (x', y')$ of L is more complicated because the relation $\approx$ is not necessarily a bijection between $N^+_G(x)$ and $N^+_H(y)$.

We define V_L as above. For each $(x, y) \in V_L$, we have $\overline{N^+_G}(x) \sim \overline{N^+_H}(y)(\mathrm{mod} \approx)$ by Lemma 2.4. We choose a witness $(S_{x,y}, \mu_{x,y})$ of $\overline{N^+_G}(x) \sim \overline{N^+_H}(y)(\mathrm{mod} \approx)$, cf. Definition 3.15(b). We define in L an arc $(x, y) \to (x', y')$ of weight $\mu_{x,y}(x', y')$ for each (x', y') in $S_{x,y}$. We now define $K := L/(rt_G, rt_H)$. It is rooted and weighted with at most $|V_G|.|V_H|$ vertices.

Claim: K is an unfolding of G, and, similarly, of H.

Proof of claim: Let π map a vertex (x, y) of K to the vertex x of G, and an arc $(x, y) \to (x', y')$ to the arc $x \to x'$ of G. We make a few observations.

(1) If $(x, y) \in V_L$ and $x - x'$ is an arc of G, there is an arc $(x, y) \to (x', y')$ in L. If $(x, y) \in V_K$ then (x', y') and the arc $(x, y) \to (x', y')$ are in K that is a subgraph of L.

(2) If x is a vertex in G, there is a directed path from rt_G to x and, by (1), a directed path in L from the root (rt_G, rt_H) to $(x, y) \in V_L$ for some $y \in V_H$. All vertices and arcs of this path are in K.

It follows that π is a surjective homomorphism: $K \to G$. We now check Definition 3.2. We verify the following condition.

(**) For every $(x, y) \in V_K$ and $x' \in N_G^+(x)$, we have:

$\lambda_G(x, x') = \Sigma\{\lambda_K((x, y), (x', y')) \mid (x', y') \in V_K\}$.

By the definition of K, $\lambda_K((x, y), (x', y')) = \mu_{x,y}(x', y')$, and the pairs (x', y') are in $S_{x,y}$. The weighted set $(S_{x,y}, \mu_{x,y})$ is chosen so that $\lambda_G(x, x') = \Sigma\{\mu_{x,y}(x', y') \mid (x', y') \in S_{x,y}\}$. This proves (**), the claim and point 3).

3) $\Longrightarrow$1) Assume that $\gamma : T \to G$ and $\eta : T \to H$ are complete unfoldings.

Let $\approx$ be the equivalence relation on N_T such that $u \approx v$ if and only if $T/u \simeq T/v$. We define M as the weighted graph $T/\approx$, cf. Definition 3.11 and the proof of Theorem 3.12. There are unfoldings: $\gamma' : G \to M$ and $\eta' : H \to M$. We omit details.

The decidability follows from Theorem 3.19. □

Remarks 3.23: In the proof of 2) $\Longrightarrow$3), T is a complete unfolding of K by Theorem 3.5. Note however that in this proof, K is not defined in a unique way, in particular because the weighted relations $(S_{x,y}, \mu_{x,y})$ are not uniquely defined. It is however in the special case we first considered.

4. Coverings

In this section and the next two ones, we will consider undirected graphs, simply called *graphs*, and their coverings. We recall from Section 2.1 that a graph G is defined as a triple (V, E, Inc) where the elements of Inc (a subset of $E \times V$) are its *half-edges*. This description allows graphs with parallel edges and loops. An edge e is a loop at a vertex x if and only if $(e, x) \in Inc$ and there is no pair (e, y) in Inc such that $y \neq x$. We denote by $Inc(x)$ the set of half-edges (e, x) for some $e \in E$. Its cardinality is the *degree* of x, where a loop at x counts for one.

We will use *trees* (undirected and without root) and *rooted trees*, in particular the regular trees considered in the previous section. Trees and graphs may be labelled.

The main contributions of this section are the definition of weighted graphs, that can be seen as graph interpretations of degree matrices. We extend coverings to weighted graphs. If two finite graphs have a common (finite) covering, they cover a common (finite) weighted graph (Theorem 4.10). Regarding characteristic polynomials, we obtain an extension of a known factorization result (Section 4.3). We postpone to Section 5 the study of universal coverings of weighted graphs.

As in Section 3, equality of trees and graphs is understood in the strict sense: same nodes or vertices, and same edges or arcs. Equality up to isomorphism is specified explicitely and denoted by $\simeq$.

4.1. Coverings of graphs: definitions and known results

We mainly review known definitions and facts from [2, 3, 7, 8, 16, 17, 18, 21]. Our main reference for all assertions is [16] by Fiala and Kratochvíl.

We define the *adjacency matrix* A_G of a finite graph G such that $V_G = [p] := \{1, \ldots, p\}$ for some p as follows: $A_G[x, y] = A_G[y, x]$ is the number of edges between x and y and $A_G[x, x]$ is the number of loops at x.

Definition 4.1: *Covering.*

(a) Let G, H be graphs. A *covering* $\gamma : G \to H$ is a surjective homomorphism such that, if $\gamma(x) = y$, then γ defines a bijection $E_G(x) \to E_H(y)$. We will also say that G is a *covering of* H.

(b) Let G, H be finite, $V_G = [p]$ and $V_H = [q]$. A surjective mapping $\gamma : V_G \to V_H$ can be represented by a $p \times q$-matrix B_γ such that $B_\gamma[i, j] := \texttt{if } \gamma(i) = j \texttt{ then } 1 \texttt{ else } 0$. Each row of this matrix has a unique 1 and each column has at least one 1. Then, γ defines a covering if and only if $A_G B_\gamma = B_\gamma A_H$. □

An edge covers a loop incident to a single vertex. More generally, a k*-regular* graph, *i.e.* such that all vertices have degree k, covers k loops incident to a single vertex.

Proposition 4.2: Let $\gamma : G \to H$ be a covering.

(1) If $\delta : H \to K$ is a covering, then $\delta \circ \gamma : G \to K$ is a covering.

(2) If G and H are finite and H is connected, then, either γ is an isomorphism or $|V_G| > |V_H|$ and then, $|V_G| \,/\, |V_H| = |Inc_G| \,/\, |Inc_H|$ and this number is a positive integer.

(3) If H is a tree and G is connected, then G is a tree and γ is an isomorphism.

Proof:
Assertion (2) is due to Reidemeister (see [7, 8, 22]). Here is a proof sketch (cf. Section 2.1 in [16]).

Let T be a spanning tree of H. It does not include the loops that are irrelevant to connectedness. Then, $\gamma^{-1}(E_T)$ is a set of edges of G. By the definition of a covering, it is the union of k pairwise disjoint trees, all isomorphic to T by γ. This union includes all vertices, hence $|V_G| \,/\, |V_H| = k$.

We now prove that $|Inc_G| \,/\, |Inc_H| = k$.

Consider the edges e of G such that $\gamma(e)$ is a loop at x in H. Such an edge may link two vertices in different connected component of G. We have the pair $(\gamma(e), x)$ in Inc_H and a single pair of the form (e, u) such that $\gamma(u) = x$ in each connected component of G. Hence, there are k such edges e.

Consider now the edges e of G such that $\gamma(e) : x - y$ is not a loop in H. We have $(\gamma(e), x)$ and $(\gamma(e), y)$ in Inc_H. Each e yields exactly two pairs (e, u) and (e, v) in Inc_G such that $\gamma(u) = x$ and $\gamma(v) = y$. There are exactly $2k$ such pairs (e, u) and (e', v) in Inc_G. Hence, $|Inc_G| \,/\, |Inc_H| = k$.

(3) This is known from [16, 22] if H is finite. Assume now that H is infinite. Consider as in (2) the edges of $\gamma^{-1}(E_H)$ of G. They form a union of spanning trees T of G. There are no other edges in G. As G is connected, it is a tree. □

Definition 4.3: *Degree matrix*

(a) For every finite graph G, there is a unique partition $(B_1, \ldots, B_p)$ of V_G having a minimum number of classes, such that for every $i, j \in [p]$, every vertex x in B_i has the same number of neighbours, say $r_{i,j}$, in B_j. It is called the *degree (refinement) partition*. It can be computed in polynomial time [6].

(b) Let $\alpha : V_G \to [p]$ maps a vertex x to the integer i such that $x \in B_i$. We call α a *good indexing* of V_G. The numbers $r_{i,j}$ can be organized into a $p \times p$ matrix $M_{G,\alpha}$ such that and $M_{G,\alpha}[i,j] = r_{i,j}$. It is called the *degree (refinement) matrix* of G. This matrix may not be symmetric. □

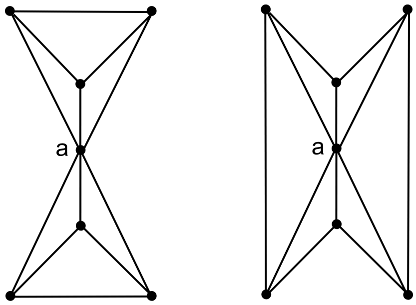

Figure 4. None of these graphs covers any smaller graph. See Example 4.4.

Example 4.4: The two graphs of Figure 4 have degree partition (B_1, B_2) where $B_1 = \{a\}$ and B_2 consists of the six other vertices. The corresponding matrix is $M := \begin{bmatrix} 0 & 6 \\ 1 & 2 \end{bmatrix}$. As both have 7 vertices, a prime number, they cannot cover any graph apart themselves by Proposition 4.2(2) (an observation made by Boldi and Vigna in [9]). They cover a common weighted graph H whose weight matrix is M, see Example 4.20(4). □

Lemma 4.5: If G and H are finite, if α is a good indexing of V_H and $\gamma : G \to H$ is a covering, then $\alpha \circ \gamma$ is a good indexing of V_G and $M_{G,\alpha\circ\gamma} = M_{H,\alpha}$.

Proof:
Because of γ, the graphs G and H have same degree matrices (for some appropriate numbering of the components of the degree partition, cf. [16], Section 4.1). Then $\alpha \circ \gamma$ is a good indexing of V_G and the equality $M_{G,\alpha\circ\gamma} = M_{H,\alpha}$ follows from the definitions. □

Definition 4.6: *Universal coverings*

(a) A covering of a graph H that is a tree is called a *universal covering* of H (hence H is connected, cf. Remark 4.19).

(b) Every connected graph H has a universal covering constructed as follows. For a vertex x of H, we define $UC(H,x)$ as the rooted tree of all finite walks in H that start at x and do not use a same edge (including a loop) twice in a row. The tree $Unr(UC(H,x))$ is obtained by forgetting the root of $UC(H,x)$ and its orientation. It is a covering of H, hence a universal one. We have $Unr(UC(H,x)) \simeq Unr(UC(H,y))$ for any two vertices x and y ([16], Section 4.2). Examples are given below. □

Examples 4.7: (1) An edge is the universal covering of a single loop. A path with 4 vertices is that of an edge with a loop at one of its ends.

(2) If H consists of two parallel edges, then $Unr(UC(H,x))$ is a *biinfinite path, i.e.*, the union of two infinite paths originating from a same node. (A biinfinite path is somehow isomorphic to $\mathbb{Z}$). Equivalently, it is the unique tree *u.t.i.* (up to isomorphism) whose nodes have all degree 2. It is also the universal covering of two loops at a same vertex or of any cycle.

(3) The universal covering of a connected k-regular graph is the infinite tree whose nodes have all degree k. This is clear from the construction recalled in Definition 4.6(b). $\square$

We recall that if u is a node of a tree T, then T_u is the rooted tree obtained by taking u as the root.

Proposition 4.8: Let H, H' be graphs.

(1) If $\gamma : T \to H$ is a universal covering and $u \in N_T$, then $T_u \simeq UC(H, \gamma(u))$.

(2) If there an isomorphism of H to H' maps x to y, then $UC(H,x)$ is isomorphic to $UC(H', y)$.

Proof:

(1) We will prove below a generalization of this fact for weighted graphs.

(2) This is clear from the descriptions of $UC(H,x)$ and $UC(H',y)$ in terms of walks. $\square$

By Assertion (1) and Definition 4.6(b), all universal coverings of H are isomorphic. One can speak of *the* universal covering of H, denoted[18] by $\boldsymbol{UC}(H)$.

Remark 4.9: The converse to Assertion (2) does not hold when $H = H'$. Take for a counter-example the union of the two graphs of Figure 4 with an edge between the two vertices marked a, that we will call x and y. Then $UC(H,x) \simeq UC(H,y)$ but there exists no automorphism of H that maps x to y. $\square$

The relevance to distributed computing can be stated as follows: if x and y are two nodes of a network represented by a graph H and $UC(H,x) \simeq UC(H,y)$, then, no computation in H (following certain rules, see [2]) can distinguish x from y. It follows that an election algorithm that would select x would also select y, hence would not be correct.

Theorem 4.10: Let G, H be finite and connected graphs. The following properties are equivalent.

(i) G and H have a common finite covering,

(ii) G and H have isomorphic universal coverings,

(iii) $M_{G,\alpha} = M_{H,\beta}$ for some good indexings α and β of V_G and V_H.

The implication (iii)$\Longrightarrow$(i) has a difficult proof by Leighton in [18]. We will prove in Theorem 6.1 below is a special case of it from which follows that of regular graphs, known from Angluin and Gardiner [3].

If G and H have the same number of vertices, them (iii) implies that they are *fractionally isomorphic* by Theorem 6.5.1 of the book [23]. We will not develop this aspect in the present article.

[18]The use of boldface letters is intended to recall that $\boldsymbol{UC}(H)$ is only defined up to isomorphism. Most proofs about universal coverings will be done from the concrete trees $Unr(UC(H,x))$.

We will interpret a degree matrix $M_{G,\alpha}$ and a good indexing α of a graph G as a covering $\alpha : G \to M$ where M is a finite *weighted graph*. Furthermore, we will allow infinite weights and obtain universal coverings that are trees of infinite degree, as in Section 3 for unfoldings.

Definition 4.11: *Equivalences on graphs that yield coverings.*

We recall from Section 1 that an equivalence relation $\sim$ on a graph $G = (V, E, Inc)$ is an equivalence relation on $V \cup E$ such that each equivalence class is a set of vertices or of edges, and, if e and e' are equivalent edges, then each end of e is equivalent to an end of e'. The *quotient graph* is then defined as $G/\sim := (V/\sim, E/\sim, Inc_{G/\sim})$ such that $([e]_\sim, [v]_\sim) \in Inc_{G/\sim}$ if and only if $(e', v') \in Inc_G$ for some $e' \sim e$ and $v' \sim v$.

We say that such an equivalence $\sim$ is *strong* if, whenever x and x' are equivalent vertices, it defines a bijection between $E(x)$ and $E(x')$. $\square$

Proposition 4.12: (1) If $\sim$ is a strong equivalence on a graph G, then the surjection $\alpha : V \cup E \to (V/\sim) \cup (E/\sim)$ that maps x to its equivalence class $[x]_\sim$ is a covering $G \to G/\sim$.

(2) Every connected graph H is isomorphic to $T/\sim$ where T is its universal covering and $\sim$ is a strong equivalence relation on T.

Proof:
(1) The proof is straightforward.

(2) We let $\gamma : T \to H$ be a universal covering where $T = (N, E, Inc)$. We define $x \sim y$ for $x, y \in N \cup E$ if and only if $\gamma(x) = \gamma(y)$. Then $T/\sim$ is isomorphic to H. $\square$

Quotients of trees will be studied in Sections 5.2 and 6.

4.2. Coverings of weighted graphs

We extend to weighted graphs the notion of covering. The two graphs of Example 4.4 cover a same weighted graph but no same graph. The case of finite weighted graphs will be of particular interest, because they provide us with finite descriptions of certain regular trees.

Definitions 4.13: *Weighted graphs and weight matrices.*

(a) A *weighted graph* is a quadruple $G = (V, E, Inc, \lambda)$ such that (V, E, Inc) is a simple graph (it has no two parallel edges and no two loops at a same vertex) and λ is a *weight function*: $Inc \to \mathbb{N}_+ \cup \{\omega\}$. The two halves of an edge may have different weights.

A graph G is made into a weighted graph $W(G)$ as follows: p parallel edges between x and y are fused into a single edge whose two half-edges have weight p; similarly, p loops at x are fused into a single one at x of weight p. A simple graph is a weighted graph whose weights are all 1.

(b) A finite weighted graph G with vertex set equal to (or indexed by) $[p]$ can be represented by the *weight matrix* $M_G : [p] \times [p] \to \mathbb{N} \cup \{\omega\}$ such that $M_G[x, y] := \lambda_G(e, x)$ if $e : x - y$. Then the sum of weights of the half-edges is the sum of coefficients of M_G.

Definition 4.14: *Coverings of weighted graphs*

Let H and G be a weighted graphs. A *covering* $\gamma : G \to H$ is a surjective homomorphism of unweighted graphs such that, if $x \in V_G$, $\gamma(x) = y$ and $e \in E_H(y)$, then:

$\lambda_H(e, y) = \Sigma\{\lambda_G(e', x) \mid e' \in E_G(x), \gamma(e') = e\}$,

or equivalently, γ induces a weighted surjection $(Inc_G(x), \lambda_G) \to (Inc_H(y), \lambda_H)$.

We will say that G is a *covering of* H.

Remarks 4.15: (1) If in Definition 4.14, G be a graph, then $\lambda_H(e, y) = |\{e' \mid e' \in E_G(x), \gamma(e') = e\}|$, and, equivalently, γ induces a weighted surjection $Inc_G(x) \to (Inc_H(y), \lambda_H)$. The degree of x in G is the sum of weights of the half-edges in $Inc_H(\gamma(x))$.

(2) If H is a simple graph, then G is a graph and the condition implies that γ is injective on each set $Inc_G(x)$, whence bijective: we get the notion of covering of Section 4.1.

(3) Each graph G covers the weighted graph $W(G)$. $\square$

Coverings of finite weighted graphs, even having infinite weights, can also be expressed in terms of weight matrices (as for graphs in terms of adjacency matrices, cf. Definition 4.1).

Let G and H be finite weighted graphs and $\alpha : V_G \to V_H$ be surjective, where $V_G = [p]$ and $V_H = [q]$. This mapping is represented by the matrix B_α (as in Definition 4.1) such that:

$B_\alpha[i, j] := \texttt{if } \alpha(i) = j \texttt{ then } 1 \texttt{ else } 0.$

The following proposition is straightforward from the definition. For defining the product of two matrices, we use the rules $\omega + x = \omega$ for every x, $\omega.x = \omega$ if $x > 0$ and $\omega.0 = 0$. We need no substractions.

Proposition 4.16: A homomorphism $\alpha : G \to H$ is a covering if and only if $M_G B_\alpha = B_\alpha M_H$.

Remark 4.17: Here is a method to build a graph G that covers a finite or infinite weighted graph H. It is similar to the construction of the proof of Proposition 4.2(2). Given $H = (V, E, Inc, \lambda)$, we construct $G = (V', E', Inc')$, as follows (it is unweighted).

We choose a set V' and a surjective mapping $\alpha : V' \to V$. For each $x \in V'$ and $(e, \alpha(x)) \in Inc$, we create $\lambda(e, \alpha(x))$ (yet abstract) half-edges incident with x, defined as pairs $((e, i), x)$ for $i = 1, 2, \ldots, \lambda(e, \alpha(x))$. In this way, we have defined Inc'. We let α map $((e, i), x)$ to $(e, \alpha(x))$.

We choose a partial matching M on Inc' satisfying the following property:

A pair in M is of the form $(((e, i), y),((e, j), z))$ such that $y \neq z$ and $e : \alpha(y) - \alpha(z)$ is an edge of H, and this pair defines an edge f in G; we define $\alpha(f) := e$.

If $((e, i), y)$ is not matched in M, then e is a loop in H incident with $\alpha(y)$; $((e, i), y)$ is a loop f of G and define $\alpha(f) := e$.

There are numerical constraints on V' and α, as we will see in Theorem 4.24.

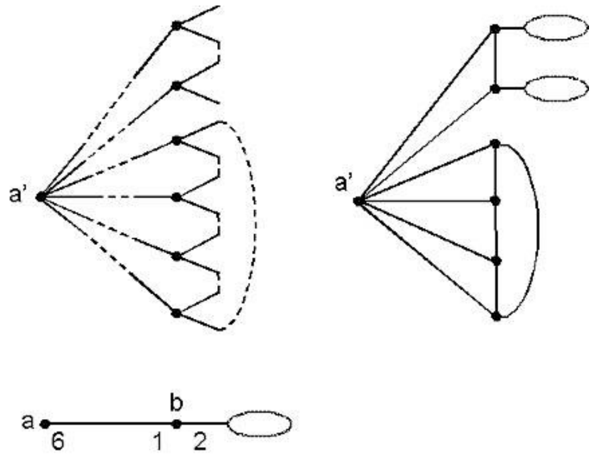

Figure 5. See Remark 4.17

For an example illustrating this construction, Figure 5 shows the weighted graph H of Example 4.4 with vertices a and b. It shows above an intermediate step H in the construction of G, where $\alpha^{-1}(a) = \{a'\}$ and $\alpha^{-1}(b)$ consists of the six other vertices. The half-edges are solid lines. The matching is shown by dotted lines. The two half-edges that are not matched yield loops in the final graph shown to the right. Their drawing recalls that they count for one in the degree of their vertices. The graph G covers the same weighted graph H as the two graphs of Figure 4. $\square$

As for graphs (Proposition 4.2(1)), we have:

Proposition 4.18: Let G, H, K be weighted graphs. If $\gamma : G \to H$ and $\delta : H \to K$ are coverings, then so is $\delta \circ \gamma : G \to K$. The same holds if G is a graph, or if G and H are graphs.

Remark 4.19: If two disjoint weighted graphs are coverings of H, then, their union is a covering of H. If $\gamma : G \to H$ is a covering and G is connected, then H is connected because γ maps every path in G to a walk in H. If H is not connected, then G is the union of (disjoint) coverings of its connected components. It follows that we need only consider connected coverings of connected weighted graphs.

Examples 4.20: 1) The complete bipartite graph $K_{3,4}$ (with 3+4 vertices) covers H consisting of one edge whose half-edges have weights 4 and 3. Although H is a tree, Proposition 4.2(3) does not hold. Proposition 4.2(2) does not either: $|V_G|$ need not be a multiple of $|V_H|$ when G is a covering of a weighted graph H.

2) A graph G consisting of 3 parallel edges covers the graph $W(G)$ consisting of an edge whose two half-edges have weight 3, that itself covers a loop of weight 3.

3) If H has a loop of weight p at a vertex x, then H is covered by the weighted graph built as follows: we remove the loop at x, obtaining thus H'; we take the union of H' and a disjoint copy of it where x' is the copy of x and we add one edge between x and x' whose two half-edges have weight p.

4) The two graphs of Figure 4, Example 4.4 cover both the weighted graph H shown in Figure 5.

5) The graph G consisting of two vertices, x and y, an edge $e : x - y$ and loops f and g at x and y with weights $\lambda(e, x) = 3$, $\lambda(e, y) = 2$, $\lambda(f, x) = 4$ and $\lambda(g, y) = 5$, covers H consisting of a single vertex with a loop of weight 7.

The Kronecker product of a weighted graph H by an edge is a weighted bipartite graph, whose universal covering is that of H. We will use this notion in Section 6.

Definition 4.21: *Kronecker product by an edge.*

Let H be a weighted graph. Its *Kronecker (or categorical) product* by K_2 (a single edge) is the weighted bipartite graph $H \times K_2$ defined as follows. Its vertex set is $V_{H\times K_2} := V_H \times \{1, 2\}$, partitioned into $(V_H \times \{1\}, V_H \times \{2\})$. For each edge e of H between x and $y \neq x$, $H \times K_2$ we have the edge $e_{x,y} : (x, 1) - (y, 2)$ (and also $e_{y,x} : (y, 1) - (x, 2)$). A loop e at x yields a unique non-loop edge $e_{x,x} : (x, 1) - (x, 2)$. The weight $\lambda_G(e_{x,y}, (x, i))$ is $\lambda_H(e, x)$ for $i = 1, 2$. □

Lemma 4.22: Let G and H be weighted graphs.

(1) There is a covering $H \times K_2 \to H$.

(2) From a covering $\alpha : H \to G$ one can define a covering $\alpha' : H \times K_2 \to G \times K_2$.

Proof:
(1) The mapping π: $(x, i) \longmapsto x$, $e_{x,y} \longmapsto e$ is a covering: $H \times K_2 \to H$. If H is connected and bipartite, then $H \times K_2$ has two connected components, that are isomorphic. Each of them is a covering of H.

(2) From $\alpha : H \to G$, we define $\alpha' : H \times K_2 \to G \times K_2$ by $\alpha'(x, i) := (\alpha(x), i)$ and $\alpha'(e_{x,y}) := \alpha(e)_{\alpha(x),\alpha(y)}$. □

In particular, every (finite) weighted graph is covered by a (finite) weighted bipartite graph.

Example 4.23: *Weighted graphs, weight matrices and coverings.*

Every matrix $W : [p] \times [p] \to \mathbb{N} \cup \{\omega\}$ such that $W[x, y] = 0$ implies $W[y, x] = 0$ is the weight matrix of a finite weighted graph with p vertices. The matrix $M := \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ is the weight matrix of H having one edge $e : x - y$, weights $\lambda(e, x) = 3$, and $\lambda(e, y) = 2$ and a loop at x of weight 1. It is covered by the graph G equal to $K_{2,3}$ with an additional edge between the two vertices of degree 3. Then we have:

$$B_\alpha = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad M_G = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_G B_\alpha = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 2 & 0 \\ 2 & 0 \\ 2 & 0 \end{bmatrix} = B_\alpha M_H$$

where $\alpha(1) = \alpha(2) = 1$ and $\alpha(3) = \alpha(4) = \alpha(5) = 2$. □

The following theorem is stated without proof in [18] but is essential in this article (which proves a part of Theorem 4.10 ; see also [16], Section 4.1).

Theorem 4.24: Given a finite weighted graph H with finite weights, one can decide if it is covered by a finite unweighted graph G. If this is the case, one can construct G loop-free.

Proof:
Let $H = ([p], E, Inc, \lambda)$ be a finite weighted graph with finite weights. We first assume that H has no loops.

Assume that $\gamma : G \to H$ is a covering where G is a finite graph.

For each i, let $w_i := \left|\gamma^{-1}(i)\right|$. Let $e_{i,j} : i - j$ be an edge of H, with $i < j$. Let $m_{i,j} = \lambda(e_{i,j}, i)$ and $m_{j,i} = \lambda(e_{i,j}, j)$. We have $\left|\gamma^{-1}(e_{i,j})\right| = m_{i,j}.w_i = m_{j,i}.w_j$.

Consider the system Σ_H of equations of the form $m_{i,j}.x_i = m_{j,i}.x_j$, with one equation for each edge $e_{i,j}$. It is satisfied by the numbers $(w_1, \ldots, w_p)$. This system may have no solution. We give an example after the proof.

Claim 1: If Σ_H has a solution $(w_1, \ldots, w_p)$ in positive integers, then this p-tuple is equal to $(\left|\gamma^{-1}(1)\right|, \ldots, \left|\gamma^{-1}(p)\right|)$ for some finite covering γ of H by a graph G.

Proof : We define G from $(w_1, \ldots, w_p)$. Its vertices are the pairs (i, s) where $i \in [p] = V$ and $s \in [w_i]$. For an edge $e_{i,j}$ of H, we let $m := m_{i,j}.w_i = m_{j,i}.w_j$. We define as follows m edges $f_1, \ldots, f_m$ between the vertices (i, s) and (j, s') where $s \in [w_i]$ and $s' \in [w_j]$.

We partition $[m]$ into pairwise disjoint intervals[19] ;
$[m] = I_1 \cup I_2 \cup \cdots \cup I_{w_i}$, where all intervals I_q have size $m_{i,j}$, and also
$[m] = J_1 \cup J_2 \cup \cdots \cup J_{w_j}$, where all intervals J_q have size $m_{j,i}$.

For $k \in [m]$, we define an edge $f_{i,j,k}$ between (i, s) and (j, s') if and only if $k \in I_s \cap J_{s'}$. We define $\gamma(f_{i,j,k}) := e_{i,j}$. Hence, γ is a surjective homomorphism.

For each vertex (i, s), if $e_{i,j}$ is an edge in H, then the edges $f_{i,j,k}$ such that $\gamma(f_{i,j,k}) := e_{i,j}$ are those such that $k \in I_s \cap J_{s'}$ for some s'. There are $m_{i,j}$ such edges. Similarly for each vertex (j, s') such that $e_{i,j}$ is an edge in H (hence where $i < j$) , there are $m_{j,i}$ edges $f_{i,j,k}$ such that $\gamma(f_{i,j,k}) := e_{i,j}$: they are those such that $k \in I_s \cap J_{s'}$ for some s. Hence, G is a finite covering of H. □

Claim 2: A system Σ_H has a solution in positive integers if and only if it has one in rational numbers. This is decidable and a solution in positive integers can be computed if there is one. If H is a tree, then Σ_H has a solution.

Proof : We first decide if Σ_H has a solution in real numbers. We eliminate unknowns one by one.

To eliminate an unknown x, we list the equations where it occurs: say $ax = by, cx = dz, \ldots, ex = fu$. Then, any solution must satisfy $ba^{-1}y = dc^{-1}z = \cdots = fe^{-1}u$. We replace the equations containing x by the new equations $ba^{-1}y = dc^{-1}z = \cdots = fe^{-1}u$. The new system has one less unknown and has a solution if and only if Σ_H has one. From it, we get the value of x. We may obtain two equations concerning the same variables, say $gy = hz$, and $g'y = h'z$, where g, h, g', h' are positive rational numbers. We have no solution if $gh' \neq g'h$: we can stop the construction and report a negative answer. Otherwise, we discard one of these two equations.

[19] We use intervals to be easy and concrete, but any two partitions will work. They yield different nonisomorphic graphs.

If there is a solution, there is one in positive rational numbers. To obtain one in positive integers, it suffices to multiply all its components by the least common multiple of the denominators.

If H is a tree, then, at each step, we can eliminate an unknown that belongs to a single equation, equivalently, that corresponds to a leaf. Hence, this step does not create any new equation. The resulting system still corresponds to a tree. We continue in the same way and we get a solution. □

We now complete the main proof for weighted graphs with loops. Loops do not create constraints: if we add to a weighted graph L a loop of weight q incident with a vertex x, and if a covering γ of L by a graph G has been found, then we need only add q loops to G, incident to each vertex in $\gamma^{-1}(x)$. We do that for all loops of the given graph H and we get a covering as wanted.

If we replace the obtained graph G by $G \times K_2$ of Lemma 4.22, we obtain a loop-free graph that covers G hence also H. □

Example 4.25: Let H be the cycle C_3 with vertices 1,2,3 and weights on its half-edges such that we get the equations $2x_1 = 3x_2, 4x_2 = 3x_3, x_3 = 5x_1$. This system has no solution in positive integers. This means that H is not covered by any finite graph. It is covered by the infinite tree described as follows. Its set of nodes is $N_1 \cup N_2 \cup N_3$ where N_1, N_2, N_3 are infinite and pairwise disjoint; each node in N_1 has 2 neighbours in N_2 and 5 in N_3, each node in N_2 has 3 neighbours in N_1 and 4 in N_3, and each node in N_3 has 1 neighbour in N_1 and 3 in N_3. This tree does not cover any finite graph. □

The following corollary is a key fact in the proof of Theorem 4.10 by Leighton [18]. It is an immediate consequence of the proof of Theorem 4.24. If H is a graph, then the corresponding p-tuple is (1,...,1) by Proposition 4.2(2).

Corollary 4.26: Let H be a weighted graph with finite weights and vertex set $[p]$. If it has finite coverings by graphs, then there is a unique p-tuple $(n_1, \ldots, n_p) \in (\mathbb{N}_+)^p$ such that $\{(kn_1, \ldots, kn_p) \mid k \in \mathbb{N}_+\}$ is the set of p-tuples $(\left|\gamma^{-1}(1)\right|, \ldots, \left|\gamma^{-1}(p)\right|)$ such that $\gamma : G \longrightarrow H$ is a covering where G is a finite graph.

4.3. Characteristic polynomials

It is known that if G is a covering of H where G and H are finite graphs, then the characteristic polynomial of H is a factor of that of G ([16], Theorem 4). We extend this result to finite weighted graphs.

Definitions 4.27: *Characteristic polynomials.*

(a) The *characteristic polynomial* P_M of a $p \times p$ matrix M with coefficients in a ring with multiplicative unit, typically $\mathbb{Z}$,$\mathbb{R}$ or $\mathbb{C}$, is defined as the determinant of the matrix $M - xI_p$ where I_p is the $p \times p$ (diagonal) unity matrix, denoted by $\det(M - xI_p)$. It is a polynomial in x of degree p. The *characteristic polynomial* P_G of a finite graph G is defined as that of its adjacency matrix A_G that is symmetric with coefficients in $\mathbb{N}$. The coefficients of P_G are in $\mathbb{Z}$.

(b) We define the *characteristic polynomial* of a finite weighted graph H with finite weights as $P_H := \det(M_H - xI_p)$ where M_H is its weight matrix, having coefficients in $\mathbb{N}$. For an example, if H is as in Example 4.4, Remark 4.15(3) and Example 4.20(4), then $P_H = -x(2-x) - 6 = x^2 - 2x - 6$.

Theorem 4.28: If G and H are finite weighted graphs with finite weights and G covers H, then P_H is a factor of P_G.

Proof:
Immediate consequence of Proposition 4.16 and the following one. □

The representation of a surjective map $\alpha : [q] \longrightarrow [p]$ by a $q \times p$ matrix B_α is in Definition 4.1(b).

Proposition 4.29: Let M and N be, respectively, $q \times q$ and $p \times p$ matrices over a ring with multiplicative unit. Let $\alpha : [q] \longrightarrow [p]$ be a surjective mapping. If $MB_\alpha = B_\alpha N$, then P_N is a factor of P_M.

Proof:
We transform the matrix $M - xI_q$ by row and column operations into a matrix M'' such that $\det(M - xI_q) = \det(M'')$.

We do that in such a way that M'' has the block structure $\begin{bmatrix} N - xI_p & R \\ 0 & S \end{bmatrix}$. It follows that $\det(M - xI_q) = \det(N - xI_p).\det(S)$, hence $P_M = P_N.\det(S)$.

We can organize M in such a way that $i \in \alpha(i)$ for each $i \in [p]$. This means that i is the smallest element of each set $\alpha(i)$. For each such i, we add to the i-th column of M, all its j-th columns, for $j \in \alpha(i), j > i$.

We obtain a matrix M' with same determinant as $M - xI_q$. Since $MB_\alpha = B_\alpha N$, the first p elements of the j-th line of M' are the same as those of the $\alpha(j)$-th one . By substracting the i-th line from each j-th line, for all $i \in [p], j \in \alpha(i), j > i$, we get a matrix M'' of the desired form, with same determinant as $M - xI_q$ and M''. This concludes the proof. □

Example 4.30: (1) For the matrices $N = M_H$ and $M = M_G$ of Example 4.23, we have $q = 5, p = 2$, and:

$$M - xI_5 = \begin{bmatrix} -x & 1 & 1 & 1 & 1 \\ 1 & -x & 1 & 0 & 0 \\ 1 & 1 & -x & 1 & 1 \\ 1 & 0 & 1 & -x & 0 \\ 1 & 0 & 1 & 0 & -x \end{bmatrix}, M' = \begin{bmatrix} 1-x & 3 & 1 & 1 & 1 \\ 2 & -x & 1 & 0 & 0 \\ 1-x & 3 & -x & 1 & 1 \\ 2 & -x & 1 & -x & 0 \\ 2 & -x & 1 & 0 & -x \end{bmatrix},$$

$$M'' = \begin{bmatrix} 1-x & 3 & 1 & 1 & 1 \\ 2 & -x & 1 & 0 & 0 \\ 0 & 0 & -1-x & 0 & 0 \\ 0 & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & -x \end{bmatrix} = \begin{bmatrix} N - xI_2 & R \\ 0 & S \end{bmatrix}$$

so that $\det(M - xI_5) = \det(N - xI_2).\det(S)$. One can check[20] that:

$\det(N - xI_2) = (x + 2)(x - 3), \det(S) = -x^2(x + 1)$ and
$\det(M - xI_5) = -x^2(x + 1)(x + 2)(x - 3)$.

(2) If G is a weighted graph with p vertices, then $P_{G\times K_2}(x) = (-1)^p P_G(x).P_G(-x)$ where $G \times K_2$ is the Kronecker product (Definition 4.21). This fact can be proved by using the algorithm of the previous proposition.

5. Universal coverings of weighted graphs

We will construct the universal covering of a weighted graph from an unfolding of an associated weighted and rooted digraph. This construction will enlighten the relationships between universal coverings and complete unfoldings. It extends the description given for graphs in Definition 4.6(2), based on walks that do not traverse an edge twice in a row. Because of weights, this construction is no longer convenient.

Furthermore, we will use in a straighforward manner the results of Section 3.2 about complete unfoldings, in particular our adaptation of Norris's Theorem (Theorem 3.20), to obtain a corresponding result about universal coverings of finite weighted graphs. We will also define *strongly regular graphs*, a new notion linked with coverings of finite weighted graphs.

5.1. Universal coverings of weighted graphs

Definition 5.1: *Universal coverings of weighted graphs.*

A covering of weighted graphs $\gamma : G \to H$ is *universal* if G is a tree (without weights), which implies that H is connected. We also say that G is a *universal covering of* H.

We will prove that any two universal coverings of a connected and weighted graph are isomorphic. We first give some examples.

Examples 5.2: 1) An infinite tree whose nodes all have degree p where $1 < p \leq \omega$ is a universal covering of a loop of weight $p > 1$. All nodes of the tree are mapped to the vertex at the loop. It is also a universal covering of an edge whose half-edges both have weight p.

2) A tree such that every node of degree 3 is adjacent to a node of degree 4 and vice-versa is a universal covering of $K_{3,4}$ and also, of an edge whose half-edges have weights 4 and 3.

3) A tree consisting of one node adjacent to ω leaves is a universal covering of an edge whose half-edges have weights 1 and ω.

4) A universal covering γ of the graph H consisting of a path $x - y - z$ with a loop at x, all weights being 1, is the path $z_1 - y_1 - x_1 - x_2 - y_2 - z_2$ with $\gamma(x_1) = \gamma(x_2) = x$, $\gamma(y_1) = \gamma(y_2) = y$ and $\gamma(z_1) = \gamma(z_2) = z$.

[20] By using for instance https://www.dcode.fr/matrix-characteristic-polynomial

5) A biinfinite path (cf. Example 4.9(2)), is a universal covering of the following weighted graphs:

(a) a cycle (in particular two parallel edges) whose half-edges have weight 1, or an edge with both half-edges of weight 2,

(b) the weighted graph H as in 4) except that the weight of the half-edge at z is 2,

(c) one loop of weight 2 or two loops of weight 1 incident to a same vertex,

(d) a path P with ends x and $y \neq x$ such that, either x and y have both a loop of weight 1, or x has a loop of weight 1 and the half-edge (f, y) on[21] P has weight 2, or the half-edges (e, x) and (f, y) on P has both weight 2. $\square$

We will describe a construction of a universal covering for weighted graphs and prove a characterization similar to that of complete unfoldings of Theorem 3.5, that entails unicity, *u.t.i.*, of universal coverings.

Definition 5.3: *The symmetric weighted digraph of a weighted graph and its expansion.*

(a) Let $H = (V, E, Inc, \lambda)$ be a connected and weighted graph, for which we fix a linear order $\leq$ on V. The associated *symmetric weighted digraph* is $Sym(H) := (V, E', \lambda')$ defined as follows. For each edge $e : x - y$ of E, we define the following arcs of E' and their weights:

> if $x < y$ (e is not a loop), we define[22] $e^+ : x \to y$ and $e^- : y \to x$, of respective weights $\lambda(e, x)$ and $\lambda(e, y)$,
>
> if $x = y$ (e is a loop), we define $e^\ell : x \to x$ of weight $\lambda(e, x)$.

(b) We define $ES(H)$ as the expansion of $Sym(H)$ (cf. Definition 3.7). It is the (unweighted) digraph (V, E'') defined as follows, directly from H. For each edge $e : x - y$ of E, we define the following arcs of E'':

> $(e^+, i) : x \to y$ if $x < y$, for $i \in \mathbb{N}_+$, $1 \leq i \leq \lambda(e, x)$,
>
> $(e^-, i) : y \to x$ if $x < y$, for $i \in \mathbb{N}_+$, $1 \leq i \leq \lambda(e, y)$,
>
> $(e^\ell, i) : x \to x$ if $x = y$ (e is a loop) for $i \in \mathbb{N}_+$, $1 \leq i \leq \lambda(e, x)$.

The digraphs $Sym(H)$ and $ES(H)$ are strongly connected as H is connected.

(c) Let $\iota : ES(H) \to H$ be the homomorphism[23] that is the identity on $V_H = V_{ES(H)}$ and is defined as follows on the arcs of $ES(H)$:

> $\iota(e^+, i) := e$, $\iota(e^-, i) := e$ and $\iota(e^\ell, i) := e$.

For each $x \in V_H$, it induces a weighted surjection of the set $E^+_{ES(H)}(x)$ onto $(Inc_H(x), \lambda_H)$. $\square$

Any vertex x of the weighted digraph $Sym(H)$ can be taken as a *root*. We obtain a *rooted* digraph denoted by $Sym(H)_x$, similarly as for T_x, Section 2.3. The accessibility condition of Section 2 is satisfied because $Sym(H)$ is strongly connected. We define $ES(H)_x$ in the same way.

[21] We mean that f belongs to the path P.

[22] The purpose of the order on vertices is to differenciate without ambiguity e^+ from e^-.

[23] A homomorphism can map a digraph to a graph, cf. Section 2.

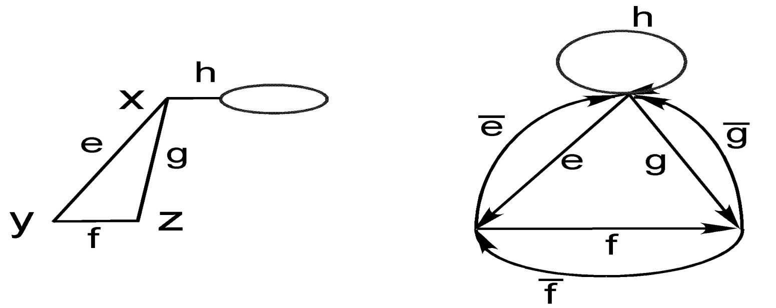

Figure 6. The graph H and the digraph $Sym(H)$ of Example 5.4.

Example 5.4: Figure 6 shows a graph H and the digraph $Sym(H)$ defined from the ordering $x < y < z$. The drawing of the loop h of H recalls that it counts for 1 in the degree of x. For readability, we denote in $Sym(H)$ the arc e^+ by e, the arc e^- by $\overline{e}$ (and similarly for f and g), and the loop h^ℓ by h. As H has no weights, $i.e.$ all weights are 1, $ES(H) = Sym(H)$, with the arc $(e^+, 1)$ identified with e^+, and similarly for the other arcs.

Figure 7 shows the rooted tree $Unf(Sym(H)_x) \upharpoonright 3$ that consists in the first three levels of $Unf(Sym(H)_x)$. Its root is denoted by $\overline{x}$. □

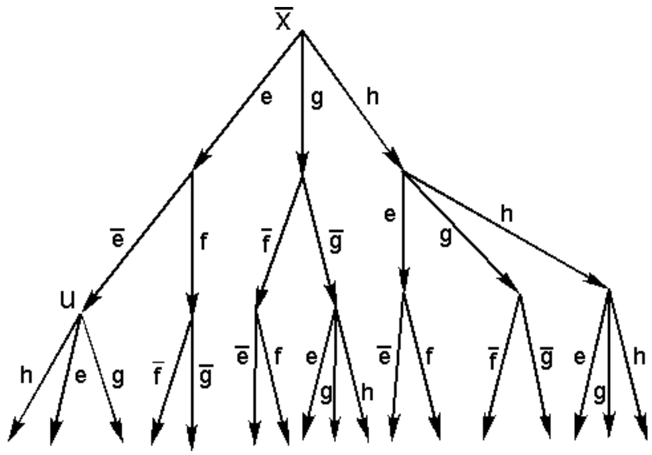

Figure 7. The rooted tree $Unf(Sym(H)_x) \upharpoonright 3$, see Example 5.4.

For defining the universal covering of a weighted graph, we generalize, by the following definition, the condition of Definition 4.6(2) requiring that the walks defining nodes do not traverse twice in a row a same edge or loop. That is, we eliminate from $Unf(ES(H)_x)$ the walks that violate this condition.

Definition 5.5: *The pruning operation.*

(a) Let H be a weighted graph and $x \in V_H$. Then, $Unf(ES(H)_x)$ is a rooted tree, whose root is denoted by $\overline{x}$ rather than by $rt_{Unf(ES(H)_x)}$. The *pruned* rooted tree $Pr(Unf(ES(H)_x))$ is obtained by deleting nodes and arcs as follows:

if a node u of $Unf(ES(H)_x)$ is a walk $(e_1, e_2, \ldots, e_n)$ in $ES(H)$ (that starts from x), $n > 1$, and, for some $f \in E$,

either $e_{n-1} = (f^+, i), e_n = (f^-, 1)$,
or $e_{n-1} = (f^-, i), e_n = (f^+, 1)$,
or $e_{n-1} = (f^\ell, i)$, $e_n = (f^\ell, 1)$,

then, we remove from $Unf(ES(H)_x)$ the arc from $w := (e_1, e_2, \ldots, e_{n-1})$ to u and the subtree issued from u.

(b) We denote by $UC(H, x)$ the rooted tree $Pr(Unf(ES(H)_x))$. □

If H is a graph, *i.e.*, all weights are 1, then $UC(H, x)$ is as in Definition 4.6.

Example 5.6: We continue Example 5.4. Figure 8 shows the rooted tree $Pr(Unf(Sym(H)_x)) \upharpoonright 3 = UC(H, x) \upharpoonright 3$. The first case of pruning removes the subtree $Unf(Sym(H)_x)/u$ where u is the head of the arc labelled by $\overline{e}$ at level 2 in the tree of Figure 7. □

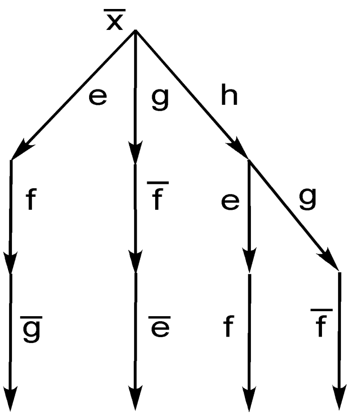

Figure 8. The rooted tree $UC(H, x) \upharpoonright 3 = Pr(Unf(Sym(H)_x)) \upharpoonright 3$. See Example 5.6.

The following theorem is similar to Theorem 3.5. We recall that Unr forgets the root and removes the orientations of a rooted tree.

Theorem 5.7: Let H be a connected and weighted graph.

1) For each $x \in V_H$, the tree $Unr(UC(H, x))$ is a universal covering of H.

2) If $\mu : T \to H$ is a universal covering, then:

(C) For every covering $\kappa : G \to H$, where G is connected and weighted, there is a universal covering $\eta : T \to G$ such that $\mu = \kappa \circ \eta$.

3) Any two universal coverings of H are isomorphic.

4) If $\mu : T \to H$ is a covering such that Condition (C) holds, then T is a tree, hence a universal covering of H.

Proof:

1) By Definition 5.3(c), we have a homomorphism[24] $\iota : ES(H) \to H$. Let $x \in V_H$. We have an unfolding homomorphism $\alpha : Unf(ES(H)_x) \to ES(H)_x$. It maps the root $\overline{x}$ of $Unf(ES(H)_x)$ to x. We will prove that the homomorphism $\iota \circ \alpha : Unf(ES(H)_x) \to H/x$ induces a covering $Unr(UC(H,x)) = Unr(Pr(Unf(ES(H)_x)))$ of H.

We let W be the (unrooted tree) $Unr(Pr(Unf(ES(H)_x)))$. We claim that it is a universal covering of H, with covering homomorphism induced by $\iota \circ \alpha$.

First, we prove that α is surjective on $Pr(Unf(ES(H)_x))$. Let $y \in V_H$. There is a path P in H from x to y. There is a path P' in $Unf(ES(H)_x)$ from $\overline{x}$ to some y' whose image by $\iota \circ \alpha$ is y. This path neither uses an arc of type (f^+, i) after one of type (f^-, j) or vice-versa, nor an arc of type (f^ℓ, i), otherwise P would have an edge occurring twice or a loop.

Hence, the path P' is not deleted by the pruning operation, so that y' is in $Pr(Unf(ES(H)_x))$ and yields y by $\iota \circ \alpha$. Similarly, any edge e of H is on a path P from x with corresponding path P' in $Unf(ES(H)_x)$ and e is the image under $\iota \circ \alpha$ of an arc in P'. Hence, $\iota \circ \alpha$ is surjective on $Pr(Unf(ES(H)_x))$.

Next, we check the condition of Definition 4.14. Let u be a node of $Pr(Unf(ES(H)_x))$. Let e be an edge of H incident to $\alpha(u)$. The arcs of $Unf(ES(H)_x)$ incident to u whose image by α is e are as follows, according to different cases.

Case 1: u is the root. There are $\lambda_H(e)$ such arcs. They are all of type (e^+, i) (cf. the proof of Theorem 3.5(1) for the notion of type), or all of type (e^-, i), or all of type (e^ℓ, i) and they are in $Pr(Unf(ES(H)_x))$. Hence $\lambda_H(e, \alpha(u)) = |\{e' \mid e' \in E_T(u), \iota \circ \alpha(e') = e\}|$.

Case 2: u is not the root and is the head of an arc of type (f^+, i) or (f^-, i) or (f^ℓ, i) where $f \neq e$. We are exactly as in Case 1.

Case 3: u is not the root and is the head of an arc of type (e^+, i) for some i. The arcs of $Unf(ES(H)_x)$ we are considering are the $\lambda_H(e, \alpha(u))$ arcs of types (e^-, j) together with the arc with head u. Hence, we seem to have one arc too much. But the pruning operation eliminates the arc $(e^-, 1)$. Hence, we still have $\lambda_H(e, \alpha(u)) = |\{e' \mid e' \in E_T(u), \iota \circ \alpha(e') = e\}|$.

Case 4: As in Case 3 with an incoming arc of type (e^-, i) or (e^ℓ, i) for some i. The argument is as in Case 3.

Hence, $\iota \circ \alpha$ induces (via the restriction to $Pr(Unf(ES(H)_x))$) a covering from the tree $W := Unr(UC(H,x))$ to H.

2) We prove the assertion for $W := Unr(UC(H,x))$ and $\iota \circ \alpha : W \to H$ as in 1). We let $\kappa : G \to H$ be a covering where G is connected and weighted. Let $x' \in V_G$ be such that $\kappa(x') = x$. For each i, we construct an *i-covering* $\eta_i : UC(H,x) \upharpoonright i \to G_x$, *i.e.*, a homomorphism such that Condition (S') holds for all nodes of $UC(H,x) \upharpoonright i$ at depth less than i. This is similar to the notion of i-unfolding in Definition 3.6. We want that η_{i+1} extends η_i and that $\kappa \circ \eta_i$ is the restriction of $\iota \circ \alpha$ to $UC(H,x) \upharpoonright i$.

For $i = 0$, we define $\eta_0(\overline{x}) := x'$.

[24] See Section 2.2 for homomorphisms from digraphs to graphs.

We now define η_{i+1} extending η_i . Let u be at depth i. We have a weighted surjection from the set $E_W(u)$ to $(Inc_H(x), \lambda_H)$ and a weighted surjection $(Inc_G(\eta_i(u)), \lambda_G)$ to $(Inc_H(x), \lambda_H)$. Lemma 2.1(2) shows that we have a weighted surjection β from $E_W(u)$ to $(Inc_G(\eta_i(u)), \lambda_G)$ such that $\kappa \circ \beta = \iota \circ \alpha$ on $E_W(u)$. Furthermore, we can choose β such that $\beta(e) = \eta_i(e)$ where e is the arc in $UC(H, x))$ with head $\eta_i(u)$. (The node u is not the root of $UC(H, x)$).

If v is the head of an arc with tail u of type (f^+, i) or (f^-, i), then $\beta(v)$ is the end of f different from $\eta_i(u)$; if the type is (f^ℓ, i), then $\beta(v) := \eta_i(u)$.

We define η_{i+1} as η_i extended by all such mappings β. The union of the mappings η_i yields a universal covering

$\eta : W \to G$, where $W := Unr(UC(H, x))$.

If G is a tree, then η is an isomorphism $UC(H, x) \to G$ by Proposition 4.2(3). This completes the proof of 2) and proves 3).

4) As in Theorem 3.5. □

Corollary 5.8: (1) If $\gamma : T \to H$ is a universal covering and $x \in N_T$, then $T_x \simeq UC(H, \gamma(x))$.

(2) If $x, y \in N_T$ and $\gamma(y) = \gamma(x)$, then $T_x \simeq T_y$.

Proof:
(1) Follows from the proof of Theorem 5.7(2).

(2) If $x, y \in N_T$ and $\gamma(y) = \gamma(x)$, then $T_x \simeq UC(H, \gamma(x)) = UC(H, \gamma(x)) \simeq T_y$. □

As in Definition 4.6, we denote by $\boldsymbol{UC}(H)$ *the* universal covering of H, that is the isomorphism class of the trees $Unr(UC(H, x))$.

Example 5.9: Figure 9 shows a weighted graph H and, to the right, the digraph $ES(H)$. Figure 10 shows the first two levels of $Unf(ES(H)_x)$. The dotted arcs are eliminated by pruning. □

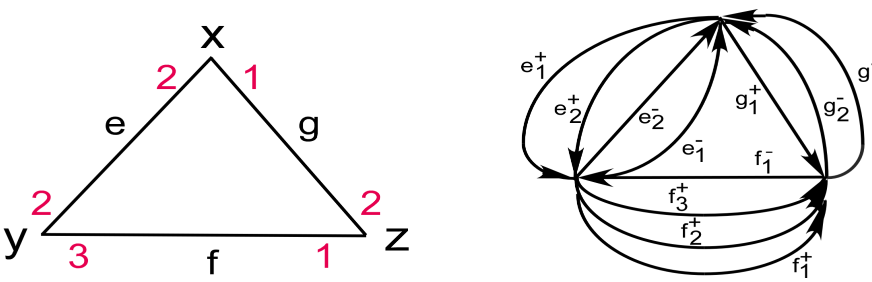

Figure 9. A weighted graph H and the digraph $ES(H)$, see Example 5.9.

If R is a rooted tree, we define $Sym(R)$ by adding to R an "up-going" arc $v \to u$ for each arc $u \to v$. It is nothing but $Sym(Unr(R))$ constructed by Definition 5.3 with all weights equal to 1 and a linear order such that $x < y$ if $x \to y$ in R. We obtain a strongly connected rooted digraph with root rt_R. See Figure 11 for an example.

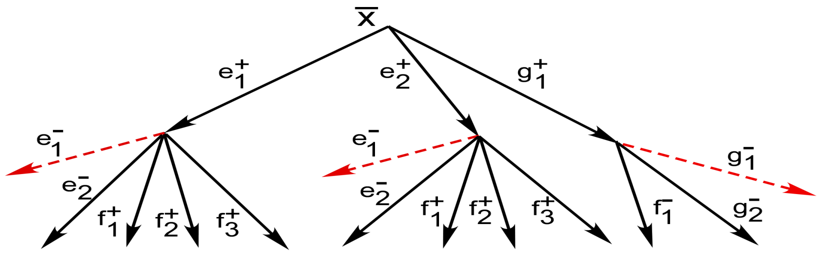

Figure 10. The tree $Unf(ES(H), x) \upharpoonright 2$, cf. Example 4.11.

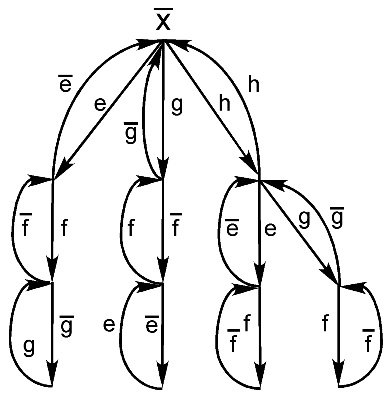

Figure 11. The top part of the digraph $Sym(UC(H, x))$, cf. Examples 5.4 and 5.9.

Proposition 5.10: Let H be a weighted connected graph and $x \in V_H$. We have
$Unf(Sym(UC(H, x))) \simeq Unf(ES(H), x)$.

Proof:
Let $\gamma : UC(H, x) \to H$ be the covering homomorphism[25]. It is actually a homomorphism: $UC(H, x) \to ES(H)$ that is not surjective on arcs because of pruning. We extend γ into a surjective homomorphism $\gamma' : Sym(UC(H, x)) \to ES(H)$ by defining $\gamma'(e)$ for $e \in E_{Sym(UC(H,x))} - E_{UC(H,x)}$ as follows:

if $e : u \to v$ is of the form $(f^+, i) \in E_{ES(H)}$ (this means that we have $f : \gamma(u) - \gamma(v)$ in H and $\gamma(u) < \gamma(v)$), then, $\gamma'(e) := (f^-, 1)$,
if it is of the form (f^-, i), then, $\gamma'(e) := (f^+, 1)$,
if it is of the form (f^ℓ, i), then, $\gamma'(e) := (f^\ell, 1)$.

These "up-going" arcs restablish some arcs deleted by pruning, but not the deleted subtrees.

Then $\gamma' : Sym(UC(H, x)) \to ES(H)_x$ is an unfolding. It follows from Theorem 3.5 that $Unf(Sym(UC(H, x)))$ is a complete unfolding of $ES(H)_x$. Hence, $Unf(Sym(UC(H, x))) \simeq Unf(Sym(ES(H)_x))$. □

[25]It maps the root $\overline{x}$ of $UC(H, x)$ to x.

Example 5.11: We continue Example 5.4 illustrated in Figure 6. Figure 7 shows $Unf(ES(H)_x)$. Figure 8 shows $Pr(Unf(ES(H)_x)) = UC(H,x)$. Figure 11 shows $Sym(UC(H,x))$ and its up-going arcs. The deleted arc labelled by $\overline{e}$ that reaches u(see Figure 7) is restablished towards $\overline{x}$. The three arcs outgoing from u and labelled by h, e and g are not. □

The following theorem relates universal coverings to complete unfoldings. We will use it for proving Theorem 5.15 from Theorem 3.20.

Theorem 5.12: Let Hbe a weighted graph. For every two vertices $x$and y, we have:

$UC(H,x) \simeq UC(H,y)$ if and only if $Unf(ES(H)_x) \simeq Unf(ES(H)_y)$.

If H is finite, this property is decidable.

Proof:
If $UC(H,x) \simeq UC(H,y)$, we have $Sym(UC(H,x)) \simeq \ Sym(UC(H,y))$. Hence, $Unf(ES(H)_x) \simeq Unf(ES(H)_y)$ by Proposition 5.10.

For the converse, observe that $UC(H,x) := Pr(Unf(ES(H)_x))$, hence is defined by using the definition of nodes as walks.

The definition of $Pr(R)$ where $R := Unf(ES(H)_x)$ uses a mapping s such that any node u of R that is not the root is mapped by s to one of its sons such that $R/s(u) \simeq R/w$ where w is the father of u.

Let $S := \{s(u) \mid u \in N_R, u \neq rt_R\}$. Then $Pr(R)$ is obtained from R by deleting the subtrees R/v for all $v \in S$.

Assume now that R is any rooted tree isomorphic to $Unf(ES(H)_x)$ and that S' is any subset of N_R such that:

each node v in S' is at depth at least 2,

each node $u \neq rt_R$ has a unique son v in S',

and $R/v \simeq R/w$ where w is the father of u that is itself the father of v.

Then, the labelled rooted trees $(Unf(ES(H)_x), S)$ and (R, S') are isomorphic[26]. It follows that $Pr(Unf(ES(H)_x))$ is isomorphic to the tree obtained from R by deleting the subtrees R/v for all $v \in S'$. Hence $UC(H,x)$ can be constructed, *u.t.i*, from any rooted tree isomorphic to $Unf(ES(H)_x)$ and any appropriate set S', without using the concrete description of the nodes of $Unf(ES(H)_x)$ by walks. It follows that $UC(H,x) \simeq UC(H,y)$ if $Unf(ES(H)_x) \simeq Unf(ES(H)_y)$.

The last assertion follows from Theorem 3.14 applied to $Sym(H)$ by using Algorithm 3.18. □

The next proposition defines from a tree a canonical weighted graph of which it is a universal covering.

Proposition 5.13: Let T be a tree and $\sim$ be an equivalence relation on N_T satisfying the following condition:

[26]A node is labelled by 1 if it is in S or in S' and by 0 otherwise.

(N): if $v \sim v'$, w is a neighbour of v, and v has exactly p (p may be ω) neighbours equivalent to w, then v' has exactly p neighbours equivalent to w,

then T is a universal covering of the weighted graph $H := T/\sim$ defined as follows:

- $V_H := N_T/\sim$,
- E_H contains the edge $e : [v]_\sim - [w]_\sim$ if and only if v is adjacent to some vertex in $[w]_\sim$ if and only if, by Condition (N), each vertex of $[v]_\sim$ is adjacent to some vertex in $[w]_\sim$,
- the weight $\lambda(e, [v]_\sim)$ is the number of edges of T linking v and a vertex in some $[w]_\sim$ such that w is adjacent to v.

Proof:
Condition (N) implies that an edge $[v]_\sim - [w]_\sim$ is defined from an edge $v - w$ of T, and that $\lambda(e, [v]_\sim)$ is well-defined. The mapping γ such that $\gamma(v) = [v]_\sim$ and $\gamma(e)$ is the edge $[v]_\sim - [w]_\sim$ if $e : v - w$ is a universal covering of H. If $v \sim w$, then the edge $[v]_\sim - [w]_\sim$ is a loop. $\square$

Corollary 5.14: Every tree is a universal covering of a weighted graph.

Proof:
If T is a tree and $\approx$ is the equivalence relation on N_T defined by $x \approx y$ if and only if $T_x \simeq T_y$, then $H := T/\approx$ is a weighted graph and T is a universal covering of it. $\square$

5.2. Universal coverings of finite weighted graphs

We will call *strongly regular* the universal coverings of finite weighted graphs. These regular trees have not been previously identified to our knowledge. We first extend a result proved by Norris [17, 21] for graphs without weights. Our proof will use Theorem 3.20, the similar result for complete unfoldings, by means of Theorem 5.12.

Theorem 5.15: Let T be a universal covering of a finite weighted graph H with p vertices. For every two nodes x, y of T, we have $T_x \simeq T_y$ if and only if $T_x \upharpoonright (p-1) \simeq T_y \upharpoonright (p-1)$.

Proof:
We prove the property for $T_x := UC(H, x)$ and $T_y := UC(H, y)$. The "only if" direction is clear.

For proving the converse, assume $UC(H, x) \upharpoonright (p-1) \simeq UC(H, y) \upharpoonright (p-1)$. We have also $Sym(UC(H, x) \upharpoonright (p-1)) \simeq Sym(UC(H, y) \upharpoonright (p-1))$. The directed walks of length $p-1$ in $ES(H)$ that start from x are in bijection with the directed paths of length $p-1$ in $Sym(UC(H, x) \upharpoonright (p-1))$ that start from $\overline{x}$, the root of $UC(H, x)$. It follows that $Unf(ES(H)_x) \upharpoonright (p-1) \simeq Unf(ES(H)_y) \upharpoonright (p-1)$. Hence, by Theorem 3.20, we get $Unf(ES(H)_x) \simeq Unf(ES(H)_y)$ and, by Theorem 5.12, $UC(H, x) \simeq UC(H, y)$. $\square$

Example 5.16: Let us consider the graph H of Example 5.4 and Figures 6,7,8 (Section 5.1). Figure 7 shows $Unf(Sym(H)_x)) \upharpoonright 3$, and Figure 8 the result of pruning it. Figure 11 shows the first three levels of $Sym(UC(H, x))$. The directed paths of length 3 in the tree $Unf(Sym(H)_x)$ that start from the root $\overline{x}$ correspond bijectively to the directed walks of length 3 in $Sym(UC(H,x))$ that start from $\overline{x}$. In the proof of Theorem 5.15, we use a similar observation for $ES(H)$ where H is weighted. □

As a consequence of Theorem 5.15, we obtain in [13] a first-order definability result for the strongly regular trees $\boldsymbol{UC}(H)$ similar to that for regular trees following from Theorem 3.20.

Definition 5.17: *Regular unrooted trees.*

We recall that the *subtrees* of a (labelled) rooted tree R are the (labelled) rooted trees R/x for $x \in N_R$. Their nodes are those of R accessible from x. By Definition 3.8, a rooted (labelled) tree R is *regular* if the set of isomorphism classes $[R/x]_{\simeq}$ for $x \in N_R$ is finite. In that case, its cardinality is the *regularity index* $Ind(R)$ of R.

A (labelled) tree T without root is *regular* if the rooted (labelled) tree T_x is regular for some $x \in N_T$.

Proposition 5.18: If a (labelled) tree T is regular, then the rooted (labelled) trees T_y are regular for all $y \in N_T$.

Proof:
Let T_x be regular for $x \in N_T$. If y is a neighbour of x, then the subtrees of T_y are T_y, T_y/x and the subtrees T_x/z for $z \notin \{x,y\}$. Hence there are finitely many up to isomorphism. If y is at distance n of x, there is a path $x - z_1 - \cdots - z_{n-1} - y$ and each rooted tree $T_{z_1}, \ldots, T_{z_{n-1}}, T_y$ is regular by the first observation. □

We may have $Ind(T_y) > Ind(T_x)$, as shown in Example 5.21.

Definition 5.19: *Strongly regular trees.*

A possibly labelled tree T is *strongly regular* if it has finitely many associated rooted trees T_x, *u.t.i*, that is, if the set $\{[T_x]_{\simeq} \mid x \in N_T\}$ is finite. □

We will prove that a strongly regular tree is regular. This is not an immediate consequence of the definition as we do not require that any of the trees T_x is regular. However, all are.

Example 5.20: The rooted tree P such that $N_P := \mathbb{N}$ and $x \leq_P y$ if and only if $y \leq x$ is an *infinite path* P. It is regular, hence, the tree $Unr(P)$ is regular. The rooted trees $Unr(P)_x$ are all regular but pairwise non isomorphic. Hence, $Unr(P)$ is not strongly regular.

Proposition 5.21: Let H be a finite, connected and weighted graph.

(1) Its universal coverings are strongly regular.

(2) For each $x \in V_H$ the rooted tree $UC(H,x)$ is regular.

Proof:
(1) If η: $T \to H$ is a universal covering, then for each node x of T, we have $T_x \simeq UC(H, \eta(x))$ by Corollary 5.8(1). Hence, T is strongly regular.

(2) Let $x \in V_H$. The rooted tree $Unf(ES(H)_x)$ from which we get $UC(H,x)$ by pruning is regular, but this is not enough to conclude.

Let γ: $\mathit{Unf}(ES(H)_x) \to Sym(H)$ be the homomorphism that is the composition of the unfolding $\alpha : \mathit{Unf}(ES(H)_x) \to ES(H)_x$ and $\beta : ES(H) \to Sym(H)$ where β is the identity on vertices.

Let u and u' be nodes of $UC(H,x)$, hence of $\mathit{Unf}(ES(H)_x)$, that are not the root. Let $e : v \to u$ and $e' : v' \to u'$ be the arcs of $\mathit{Unf}(ES(H)_x)$ with heads u and u'. If $\gamma(e) = \gamma(e')$ then $\gamma(u) = \gamma(u')$ and we have $Pr(\mathit{Unf}(ES(H)_x))/u \simeq Pr(\mathit{Unf}(ES(H)_x))/u'$. Note that γ maps $\mathit{Unf}(ES(H)_x)$ to $Sym(H)$. We may have $e = (f^+, i)$ and $e' = (f^+, j)$ so that $\gamma(e) = \gamma(e') = f^+$.

It follows that $UC(H,x)/u \simeq UC(H,x)/u'$ and that the subtrees of $UC(H,x)$ are $UC(H,x)$ itself and those associated as above with the arcs of $Sym(H)$. Hence, there are at most $1 + 2.\,|E_H|$ subtrees *u.t.i*, and $UC(H,x)$ is regular. □

Theorem 5.22: A tree T is strongly regular if and only if it is the universal covering of a finite, connected and weighted graph if and only if it is the universal covering of such a graph without loops.

Proof:
If T is the universal covering of a finite, connected and weighted graph, then it is strongly regular by Proposition 5.21.

Conversely, let T be a strongly regular tree. Let $\sim$ be the equivalence relation on N_T such that $x \sim y$ if and only if $T_x \simeq T_y$. This equivalence relation satisfies Condition (N) of Proposition 5.13 and has finitely many classes. Hence, by this proposition, T is a universal covering of the finite weighted graph $H := T/\sim$.

Finally we show how to replace H by H' without loops. A loop of weight p arises in H if a node has p neighbours equivalent to it. To avoid loops, we define on T a proper 2-coloring. We define $\sim'$ such that $x \sim' y$ if and only if $T_x \simeq T_y$ and x and y have the same color. Then T is a universal covering of the finite weighted graph $H' := T/\sim'$ that has no loop[27]. □

Examples 5.23: 1) Let T consist of a biinfinite path B (cf. Example 4.9(2)), where each node x has, in addition, an incident pendent edge $x - x'$ for some new node x'. The rooted trees T_x for $x \in N_B$ are all isomorphic, and so are the trees $T_{x'}$. The quotient graph is the edge $[x]_\sim - [x']_\sim$ together with a loop at $[x]_\sim$ of weight 2, that yields trees isomorphic to T_x. The two other half-edges have weight 1.

2) For the tree of Example 4.20(1), we get an edge with weights 3 and 4.

Remark 5.24: 1) Finite weighted graphs can be used as finite descriptions of strongly regular trees, even of infinite degree. The construction of Theorem 5.22 defines a minimal and canonical one.

2) By Theorem 5.22, a strongly regular tree is the universal covering of a finite minimal weighted graph H. It is not necessarily that of a finite graph G, otherwise such a graph G would cover H, and Example 4.25 shows that this may be not possible.

Corollary 5.25: Every node-labelled strongly regular tree is regular.

Proof:
Immediate from Theorem 5.22 and Proposition 5.21. □

[27] Note that H' is a connected component of $H \times K_2$ defined in Definition 3.21.

6. Common coverings of finite graphs

Our aim is to examine the theorem by Leighton [18] that we stated in Theorem 4.10. Its proof is quite difficult. Alternative no more easier proofs have been given that use tools from combinatorics, topology and group theory [1, 5, 19, 20, 24, 25]. We will give an easy proof for particular cases, including that of k-regular graphs proved in [3].

Theorem 6.1: If two finite connected graphs G and H are coverings of a same graph M, they have a common covering by a graph K having at most $4\,|V_G|\,.\,|V_H|$ vertices. The graph K has at most $|V_G|\,.\,|V_H|$ vertices if M is loop-free.

Proof:
We first assume that G, H and M are loop-free. (They may have parallel edges).

Let $\alpha : G \to M$ and $\beta : H \to M$ be coverings. It follows that, if $x - x'$ is an edge of G, then $\alpha(x) \neq \alpha(x')$, and similarly for β. We construct K as follows:

$V_K := \{(x, y) \in V_G \times V_H \mid \alpha(x) = \beta(y)\}$.
$E_K := \{(e, f) \in E_G \times E_H \mid \alpha(e) = \beta(f)\}$.

An edge (e, f) of K links (x, y) and (x', y') if $e : x - x'$ and $f : y - y'$. We cannot not have (e, f) linking also (x, y') and (x', y) because this would mean that $\beta(y) = \alpha(x) = \alpha(x')$ contradicting a previous remark.

We define $\gamma : K \to G$ as the first projection, *i.e.*, $\gamma(x, y) := x$ and $\gamma(e, f) := e$. Similarly $\eta : K \to H$ is the second projection. It is clear that γ and η are homomorphisms.

We prove that γ is surjective. Let $x \in V_G$. There is $y \in V_H$ such that $\beta(y) = \alpha(x)$ because β is surjective. Hence $(x, y) \in V_K$ and $\gamma(x, y) = x$. Let $e : x - x'$ be an edge of G. Then $\alpha(e) : \alpha(x) - \alpha(x')$ is an edge of M. There is in H an edge $f : y - y'$ such that $\beta(f) = \alpha(e)$. We have $\beta(f) : \beta(y) - \beta(y')$, $\beta(y) = \alpha(x), \beta(y') = \alpha(x')$. Then, η is surjective too.

It remains to prove that γ and η are coverings. We prove that for γ.

Consider $(x, y) \in V_K$ and its image x in G by γ. Let $e_1, \ldots, e_p$ be the edges of G incident with x. Let $f_1, \ldots, f_q$ be the edges of H incident with y. The edges of M incident with $\alpha(x)$ are $\alpha(e_1), \ldots, \alpha(e_p)$ that are pairwise distinct. Those incident with $\beta(y)$ are $\beta(f_1), \ldots, \beta(f_q)$, also pairwise distinct. But $\alpha(x) = \beta(y)$, hence, $q = p$, and we can renumber these edges so that $\alpha(e_i) = \beta(f_i)$ for each i. The edges of K incident with (x, y) are thus (e_i, f_i) for $i = 1, .., p$. Hence γ is a covering as wanted. We have $|V_K| \leq |V_G|\,.\,|V_H|$.

If K is not connected, then each of its connected components is a covering as wanted.

We now consider the case where G, H and M may have loops. It follows from Lemma 4.22 that we have coverings $\alpha' : G \times K_2 \to M \times K_2$ and $\beta' : H \times K_2 \to M \times K_2$. As $G \times K_2$, $H \times K_2$ and $M \times K_2$ have no loops, the previous proof yields coverings $\gamma : K \to G \times K_2$ and $\eta : K \to H \times K_2$. As $G \times K_2$ and $H \times K_2$ cover G and H respectively, we have (by Proposition 4.16) coverings $\gamma' : K \to G$ and $\eta' : K \to H$ where K has at most $4\,|V_G|\,.\,|V_H|$ vertices. $\square$

This theorem does not apply to the two graphs of Example 4.4.

A k-*regular* graph has all its vertices of degree k. It may have loops. A loop contributes 1 to the degree of its vertex.

Proposition 6.2: Let G and H be finite connected graphs. They have a common finite cover in the following cases.

(1) They have the same degree matrix (up to a permutation of rows and columns), that is symmetric.

(2) They are k-regular.

(3) Each of them has exactly one cycle, no loops, and they have isomorphic universal covers.

Proof:
(1) The degree matrix of G and H is the adjacency matrix (counting loops and parallel edges) of a graph M covered by G and H. Theorem 6.1 is applicable. Note that M has loops if and only if some values on the diagonal of the adjacency matrix are not null.

(2) The graphs G and H cover the graph with one vertex and k loops. Theorem 6.1 is applicable, which gives the result proved in [3]. It is a special case of (1).

(3) The graph G is the union of a cycle $x_1 - x_2 - \cdots - x_n - x_1$ and pairwise disjoint trees T_i, each of them having node x_i and no node x_j, for $j \neq i$. Its universal cover U is the union of a biinfinite path $\cdots - z_0 - z_1 - z_2 - \cdots - z_n - z_{n+1} - z_{n+2} - \cdots - z_{2n} - z_{2n+1} - \ldots$ and, similarly, of pairwise disjoint trees U_i containing nodes z_i. The covering homorphism α maps each z_{p+kn} to x_p for $p \in [n]$ and $k \in \mathbb{Z}$ and, isomorphically, each tree U_{p+kn} to T_p.

Hence, U can be seen, up to isomorphism, as a periodic biinfinite sequence of at most n finite trees. From H, we have a similar description. A binifinite sequence of the form $X^{\mathbb{Z}} = Y^{\mathbb{Z}}$ where X has length n and Y has length p is equal to $(X^p Y'^n)^{\mathbb{Z}}$ for a circular shift Y' of Y. From the sequence $X^p Y'^n$ one can build a common cover of G and H.

We can alternatively apply Theorem 6.1. We observe that X and Y' are respectively S^q and S^m for some sequence S, hence, we can define a loop-free graph M with one cycle covered by G and H if S has length at least 2. If S has length 1, one can define such a graph M with a loop of weight 2. $\square$

By Lemma 4.22, it suffices to prove Theorem 4.10 for finite bipartite graphs, because if two finite graphs G, H have a common universal cover T, then T covers also $G \times K_2$ and $H \times K_2$ that are finite and bipartite. A common finite cover of $G \times K_2$ and $H \times K_2$ is also one of G and H. The proof of [18] uses this observation. In order to indicate why its proof is difficult, we explain informally why a natural proof generalizing that of Theorem 6.1 fails.

Definition 6.3: *Quotients of strongly regular labelled trees.*

(a) Let T be a tree. It is bipartite with bipartition (N_T^1, N_T^2) of its nodes. Let α be a labelling of $N_T^1 \cup N_T^2 \cup E_T$. We let $\sim$ be an equivalence relation on $N_T^1 \cup N_T^2 \cup E_T$ such that each equivalence class is included in N_T^1, or in N_T^2 or in E_T, and two equivalent vertices or edges have the same label. We require that if e and e' are equivalent edges, then $e : x - y$, $e' : x' - y'$ for some x, y, x', y' such that $x \sim x'$ and $y \sim y'$. Furthermore, we modify as follows the condition of Definition 4.11:

If x and y are equivalent vertices, then, $\sim$ defines a bijection $E_T(x) \cap [e]_\sim \to E_T(y) \cap [e]_\sim$.

We obtain a quotient graph $T/\sim$ and a covering $T \to T/\sim$ that preserves labels.

(b) Let $\gamma : T \to G$ be a universal covering of a finite bipartite graph G. We label T as follows. A node $x \in N_T$ is labelled by $\gamma(x)$ and an edge $e \in E_T$ by $\gamma(e)$. The labelled tree T_γ is strongly regular. Assume now that $\eta : T \to H$ is a universal covering where H is also a finite and bipartite graph. We define a labelled tree $T_{\gamma,\eta}$ that combines the labels of T_γ and of T_η: a node $x \in N_T$ is labelled by $(\gamma(x), \eta(x))$ and an edge e is labelled by $(\gamma(e), \eta(e))$. □

Letting G, H, γ, η and $T_{\gamma,\eta}$ be as in this definition:

Proposition 6.4: If $T_{\gamma,\eta}$ is strongly regular, there exists a finite bipartite graph K that is a covering of both G and H.

Proof:
Let $\approx$ be the equivalence relation on $N_{T_{\gamma,\eta}}$ such that $x \approx y$ if and only if $(T_{\gamma,\eta})_x \simeq (T_{\gamma,\eta})_y$. Two equivalent nodes have the same label that is a pair in $V_G \times V_H$. (However, Example 6.5 below shows that two nodes may have the same label in $T_{\gamma,\eta}$ without being equivalent for $\approx$).

Without assuming that $T_{\gamma,\eta}$ is strongly regular, we first examine the neighbourhood of a node x. Its incident edges have labels $(e_1, f_1), \ldots, (e_p, f_p)$ and respective other ends $z_1, \ldots, z_p$. In G, the vertex $\gamma(x)$ has incident edges $e_1, \ldots, e_p$ and respective other ends $\gamma(z_1), \ldots, \gamma(z_p)$. In H, the vertex $\eta(x)$ has incident edges $f_1, \ldots, f_p$ and respective other ends $\eta(z_1), \ldots, \eta(z_p)$.

If $x' \approx x$, then, since $(T_{\gamma,\eta})_x \simeq (T_{\gamma,\eta})_{x'}$, the edges incident to x' have labels $(e_1, f_1), \ldots, (e_p, f_p)$ and respective other ends $z'_1, \ldots, z'_p$. Consider an isomorphism α: $(T_{\gamma,\eta})_x \to (T_{\gamma,\eta})_{x'}$. Since the edge labels $(e_1, f_1), \ldots, (e_p, f_p)$ are pairwise distinct, it maps z_i to z'_i for each i. Hence, it is an isomorphism $(T_{\gamma,\eta})_{z_i} \to (T_{\gamma,\eta})_{z'_i}$ and $z_i \approx z'_i$. It follows that we get a quotient graph $K := T_{\gamma,\eta}/\approx$ that inherits the labels of $T_{\gamma,\eta}/\approx$.

A vertex $[x]_\approx$ has label $(\gamma(x), \eta(x))$. An edge of K coming from $g : x - y$ in $T_{\gamma,\eta}$ (it links $[x]_\approx$ and $[y]_\approx$ in K) has label $(\gamma(g), \eta(g)) \in E_G \times E_H$. This is well-defined by the above remarks about neighbourhoods in $T_{\gamma,\eta}$.

We claim that K is a covering of both G and H. We let $\kappa : V_K \cup E_K \to V_G \cup E_G$ be defined as follows: $\kappa([x]_\approx) := \gamma(x)$, the first component of the label of $[x]_\approx$; if $m : [x]_\approx - [y]_\approx$ is an edge of K coming from $g : x - y$ in $T_{\gamma,\eta}$, we define $\kappa(m) := \gamma(g)$.

Claim: $\kappa : K \to G$ is a covering.

Proof: κ is a surjective homomorphism. To prove that it is a covering, we consider a vertex $[x]_\approx$ of K where x is a node in $T_{\gamma,\eta}$. We recapitulate the above observations.

The edges of $T_{\gamma,\eta}$ incident with x are $g_1, \ldots, g_p$ with respective ends $y_1, \ldots, y_p$ and labels (e_1, f_1) $, \ldots, (e_p, f_p)$. The edges of G incident with $\gamma(x)$ are $e_1, \ldots, e_p$. We get edges $[x]_\approx - [y_i]_\approx$ in K, each with label (e_i, f_i). They yield by κ the edges $e_1, \ldots, e_p$. Hence, κ is a bijection of $E_K([x]_\approx)$ to $E_G(x)$. □

Similarly, we have a covering $K \to H$.

Finally, if $T_{\gamma,\eta}$ is strongly regular, the equivalence $\approx$ has finitely many classes and K is finite. □

We do not obtain a proof of Theorem 4.10 because the tree $T_{\gamma,\eta}$ constructed from two covering homomorphisms of finite graphs G and H is not necessarily strongly regular.

Example 6.5: *A tree $T_{\gamma,\eta}$ that is not strongly regular.*

We let G be the bipartite graph such that $V^1_G = \{a\}$, $V^2_G = \{b\}, E_G = \{1, 2, 3, 4\}$, and H similarly be such that $V^1_H = \{c\}$, $V^2_H = \{d\}, E_H = \{5, 6, 7, 8\}$. They both have two vertices and four parallel edges. Let $\gamma : T \to G$ be a universal covering of G.

We choose adjacent nodes r and s of T such that $\gamma(r) = a, \gamma(s) = b$ and $\gamma(e) = 1$ where $e : r - s$. We get a labelled tree T_γ. We will enrich its labelling so as to obtain a tree $T_{\gamma,\eta}$ for some covering $\eta : T \to H$.

For this purpose, we replace each node label a of T_γ by (a, c), each label b of T_γ by (b, d), each edge label 1 by (1,5) and each label 2 by (2,6). Then for each edge in the rooted tree $T_r - T_r/s$ (obtained by deleting T_r/s from T_r), we replace 3 by (3,7) and 4 by (4,8); for each edge in the subtree $T_s - T_s/r$, we replace 3 by (3,8) and 4 by (4,7).

We get a labelled tree $T_{\gamma,\eta}$ related to a universal covering $\eta : T \to H$.

It is clear that $T_{\gamma,\eta}$ is not strongly regular because the edge labels (3,7) are present in the part $T_r - T_r/s$, but not in the other part $T_s - T_s/r$, and these two parts are infinite. □

Questions 6.6: Does Theorem 4.10 extend to finite weighted graphs?

It does in a somewhat trivial way for graphs whose weights are all ω . Let G and H be two such connected weighted graphs. Let K be their product with $V_K := V_G \times V_H$ and $(x, y) - (x', y')$ in K if and only if $x - x'$ and $y - y'$ in G and H respectively. Since $\omega + \omega = \omega$ the two projections $\pi_1 : V_K \to V_G$ and $\pi_2 : V_K = V_H$ are coverings.

The next case to consider would be when weights are 1 or ω.

7. Conclusion

We have generalized the notions of regular trees studied in [10, 11], in [4, 12, 15] and in [2, 7, 8, 21] having motivations in program semantics by attaching weights to the arcs or edges of the digraphs or graphs of which we consider complete unfoldings or universal coverings. In particular, infinite weights yield trees with nodes of infinite degree. Our finite weighted graphs offer effective descriptions and yield decidability results.

The new notion of a *strongly regular tree* defined as a universal covering of a finite weighted graph is investigated in the companion article [13].

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