

Two Sufficient Conditions for Graphs to Admit Path Factors

Sizhong Zhou, Jiancheng Wu*

School of Science

Jiangsu University of Science and Technology

Zhenjiang, Jiangsu 212100, China

zsz_cumt@163.com, wujiancheng@just.edu.cn

Abstract. Let \mathcal{A} be a set of connected graphs. Then a spanning subgraph A of G is called an \mathcal{A} -factor if each component of A is isomorphic to some member of \mathcal{A} . Especially, when every graph in \mathcal{A} is a path, A is a path factor. For a positive integer $d \geq 2$, we write $\mathcal{P}_{\geq d} = \{P_i | i \geq d\}$. Then a $\mathcal{P}_{\geq d}$ -factor means a path factor in which every component admits at least d vertices. A graph G is called a $(\mathcal{P}_{\geq d}, m)$ -factor deleted graph if $G - E'$ admits a $\mathcal{P}_{\geq d}$ -factor for any $E' \subseteq E(G)$ with $|E'| = m$. A graph G is called a $(\mathcal{P}_{\geq d}, k)$ -factor critical graph if $G - Q$ has a $\mathcal{P}_{\geq d}$ -factor for any $Q \subseteq V(G)$ with $|Q| = k$. In this paper, we present two degree conditions for graphs to be $(\mathcal{P}_{\geq 3}, m)$ -factor deleted graphs and $(\mathcal{P}_{\geq 3}, k)$ -factor critical graphs. Furthermore, we show that the two results are best possible in some sense.

Keywords: graph; degree condition; $\mathcal{P}_{\geq 3}$ -factor; $(\mathcal{P}_{\geq 3}, m)$ -factor deleted graph; $(\mathcal{P}_{\geq 3}, k)$ -factor critical graph.

(2020) Mathematics Subject Classification: 05C70, 05C38

1. Introduction

In our daily life many physical structures can conveniently be simulated by networks. The core issue of network security is the ruggedness and vulnerability of the network, which is also one of the key topics that researchers must consider during the network designing phase. To study the properties of

*Address for correspondence: School of Science, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China.

the network, we use a graph to simulate the network. Vertices of the graph stand for nodes of the network and edges of the graph act as links between the nodes of the network. Henceforth, we replace “network” by the term “graph”. In data transmission networks, the data transmission between two nodes of a network stands for a path between two corresponding vertices of a corresponding graph. Consequently, the availability of data transmission in the network is equivalent to the existence of path factor in the corresponding graph which is generated by the network. Clearly, research on the existence of path factors under specific network structures can help scientists design and construct networks with high data transmission rates. Furthermore, the existence of a path factor deleted graph and a path factor critical graph also plays a key role in data transmission of a network. If some links (resp. nodes) are damaged in the process of data transmission at the moment, the possibility of data transmission between nodes is characterized by whether the corresponding graph of the network is a path factor deleted (resp. critical) graph. In this article, we investigate the existence of path factor deleted graphs and path factor critical graphs which play a key role in studying data transmissions of data transmission networks. We find that there are strong essential connection between some graphic parameters (for instance, degree and connectivity, and so on) and the existence of path factor deleted graphs (or path factor critical graphs), and hence investigations on degree and connectivity, which play an irreplaceable role in the vulnerability of the network and the feasibility of data transmission, can yield theoretical guidance to meet data transmission and network security requirements.

We discuss only finite undirected graphs without loops or multiple edges, otherwise, unless explicitly stated. We use $G = (V(G), E(G))$ to denote a graph, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G . The degree of a vertex x in G , denoted by $d_G(x)$, is the number of edges incident with x in G . The neighborhood of a vertex x in G , denoted by $N_G(x)$, is the set of vertices adjacent to x in G . For any $X \subseteq V(G)$, we write $N_G(X) = \bigcup_{x \in X} N_G(x)$ and denote by $G[X]$ the subgraph of G induced by X . Let $G - X = G[V(G) \setminus X]$. For an edge subset E' of G , we use $G - E'$ to denote the subgraph obtained from G by removing E' . If $d_G(x) = 0$ for some vertex x in G , then x is called an isolated vertex in G . We denote by $I(G)$ the set of isolated vertices of G , and set $i(G) = |I(G)|$. The number of connected components of G is denoted by $\omega(G)$. We denote by $\lambda(G)$ and $\kappa(G)$ the edge connectivity and the vertex connectivity of G , respectively. A vertex subset X of G is said to be independent if $G[X]$ has no edges. The binding number of G is defined by Woodall [15] as

$$bind(G) = \min \left\{ \frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G) \right\}.$$

We denote by P_n and K_n the path and the complete graph of order n , respectively. Let G_1 and G_2 be two graphs. Then we denote by $G_1 \cup G_2$ and $G_1 \vee G_2$ the union and the join of G_1 and G_2 , respectively.

Let \mathcal{A} be a set of connected graphs. Then a spanning subgraph A of G is called an \mathcal{A} -factor if each component of A is isomorphic to some member of \mathcal{A} . Especially, when every graph in \mathcal{A} is a path, A is a path factor. For a positive integer $d \geq 2$, we write $\mathcal{P}_{\geq d} = \{P_i | i \geq d\}$. Then a $\mathcal{P}_{\geq d}$ -factor means a path factor in which every component admits at least d vertices. In order to characterize a graph with a $\mathcal{P}_{\geq 3}$ -factor, Kaneko [3] introduced the concept of a sun. A graph H is called a factor-critical

graph if every induced subgraph with $|V(H)| - 1$ vertices of H admits a perfect matching. Assume that H is a factor-critical graph with vertex set $V(H) = \{v_1, v_2, \dots, v_n\}$. By adding new vertices u_1, u_2, \dots, u_n and new edges $v_1u_1, v_2u_2, \dots, v_nu_n$ to H , we acquire a new graph R , which is called a sun. In view of Kaneko, K_1 and K_2 are also suns. A sun with at least six vertices is said to be a big sun. We denote by $sun(G)$ the number of sun components of G .

Las Vergnas [9] characterized a graph admitting a $\mathcal{P}_{\geq 2}$ -factor. Bazgan, Benhamdine, Li and Wońiak [1] claimed that a 1-tough graph G of order at least 3 admits a $\mathcal{P}_{\geq 3}$ -factor. Kaneko [3] derived a criterion for a graph with a $\mathcal{P}_{\geq 3}$ -factor. Kano, Katona and Király [4] gave a simpler proof of Kaneko's result. Kano, Lu and Yu [6] demonstrated that a graph G has a $\mathcal{P}_{\geq 3}$ -factor if $i(G - X) \leq \frac{2}{3}|X|$ for all $X \subseteq V(G)$. Liu [8] showed a result on the existence of $\mathcal{P}_{\geq 3}$ -factors in graphs with given properties. Wang and Zhang [11], Wu [17], Zhou et al [21, 25, 27–29, 31] posed some sufficient conditions for graphs having $\mathcal{P}_{\geq 3}$ -factors. Gao, Wang and Chen [2] obtained some tight bounds for the existence of path factors in graphs. Kano, Lee and Suzuki [5] verified that a connected cubic graph with at least eight vertices admits a $\mathcal{P}_{\geq 8}$ -factor. Wang and Zhang [14], Katerinis and Woodall [7], Zhou, Bian and Sun [22] established some relationships between binding numbers and graph factors. Wu [16], Wang and Zhang [10, 12], Zhou, Pan and Xu [24], Zhou [18, 19], Zhou, Wu and Liu [30] presented some degree conditions for the existence of graph factors in graphs. Some other results on graph factors can be seen at [13, 23, 26].

The following theorem on $\mathcal{P}_{\geq 3}$ -factors are known, which plays an important role in the proofs of our main results.

Theorem 1 ([3, 4]). A graph G has a $\mathcal{P}_{\geq 3}$ -factor if and only if

$$sun(G - X) \leq 2|X|$$

for any vertex subset X of G .

A graph G is called a $(\mathcal{P}_{\geq d}, m)$ -factor deleted graph if $G - E'$ admits a $\mathcal{P}_{\geq d}$ -factor for any $E' \subseteq E(G)$ with $|E'| = m$. A graph G is called a $(\mathcal{P}_{\geq d}, k)$ -factor critical graph if $G - Q$ has a $\mathcal{P}_{\geq d}$ -factor for any $Q \subseteq V(G)$ with $|Q| = k$. Zhou [20] showed two binding number conditions for graphs to be $(\mathcal{P}_{\geq 3}, m)$ -factor deleted graphs and $(\mathcal{P}_{\geq 3}, k)$ -factor critical graphs.

Theorem 2 ([20]). Let m be a nonnegative integer, and let G be a graph. If $\kappa(G) \geq 2m + 1$ and $bind(G) > \frac{3}{2} - \frac{1}{4m+4}$, then G is a $(\mathcal{P}_{\geq 3}, m)$ -factor deleted graph.

Theorem 3 ([20]). Let k be a nonnegative integer, and let G be a graph with $\kappa(G) \geq k + 2$. If $bind(G) \geq \frac{5+k}{4}$, then G is a $(\mathcal{P}_{\geq 3}, k)$ -factor critical graph.

It is natural and interesting to put forward some new graphic parameter conditions to ensure that a graph is a $(\mathcal{P}_{\geq 3}, m)$ -factor deleted graph or a $(\mathcal{P}_{\geq 3}, k)$ -factor critical graph. In this paper, we show degree conditions for graphs to be $(\mathcal{P}_{\geq 3}, m)$ -factor deleted graphs or $(\mathcal{P}_{\geq 3}, k)$ -factor critical graphs, which are shown in the following.

Theorem 4. Let m and r be two integers with $r \geq 1$ and $0 \leq m \leq 2r + 1$, and let G be a graph of order n with $n \geq 4r + 6m + 4$. If $\kappa(G) \geq r + m$ and G satisfies

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_{2r+1})\} \geq \frac{n}{3}$$

for any independent set $\{x_1, x_2, \dots, x_{2r+1}\}$ of G , then G is a $(\mathcal{P}_{\geq 3}, m)$ -factor deleted graph.

Theorem 5. Let $k \geq 0$ and $r \geq 1$ be two integers, and let G be a graph of order n with $n \geq 4r + k + 4$. If $\kappa(G) \geq r + k$ and G satisfies

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_{2r+1})\} \geq \frac{n + 2k}{3}$$

for any independent set $\{x_1, x_2, \dots, x_{2r+1}\}$ of G , then G is a $(\mathcal{P}_{\geq 3}, k)$ -factor critical graph.

From a computation standpoint, the condition given in each of the two theorems can be checked in polynomial time for fixed r , m and k .

2. The proof of Theorem 4

Proof of Theorem 4. Let $G' = G - E'$ for $E' \subseteq E(G)$ with $|E'| = m$. Then $V(G') = V(G)$ and $E(G') = E(G) \setminus E'$. To prove Theorem 4, it suffices to verify that G' admits a $\mathcal{P}_{\geq 3}$ -factor. Suppose, to the contrary, that G' has no $\mathcal{P}_{\geq 3}$ -factor. Then it follows from Theorem 1 that

$$\text{sun}(G' - X) \geq 2|X| + 1 \quad (1)$$

for some $X \subseteq V(G')$.

Next, we shall consider two cases according to the value of $i(G - X)$ and derive a contradiction in each case.

Case 1. $i(G - X) \geq 2r + 1$.

In this case, $G - X$ admits at least $2r + 1$ isolated vertices $v_1, v_2, \dots, v_{2r+1}$. Thus, we acquire $d_{G-X}(v_i) = 0$ for $1 \leq i \leq 2r + 1$, and so

$$d_G(v_i) \leq d_{G-X}(v_i) + |X| = |X| \quad (2)$$

for $1 \leq i \leq 2r + 1$.

Obviously, $\{v_1, v_2, \dots, v_{2r+1}\}$ is an independent set of G . Then using (2) and the degree condition of Theorem 4, we infer

$$|X| \geq \max\{d_G(v_1), d_G(v_2), \dots, d_G(v_{2r+1})\} \geq \frac{n}{3}. \quad (3)$$

It follows from (1) and (3) that

$$n \geq |X| + \text{sun}(G' - X) \geq |X| + 2|X| + 1 = 3|X| + 1 \geq 3 \cdot \frac{n}{3} + 1 = n + 1,$$

which is a contradiction.

Case 2. $i(G - X) \leq 2r$.

Subcase 2.1. X is not a vertex cut set of G .

Clearly, $\omega(G - X) = \omega(G) = 1$. If $|X| \geq \frac{m+1}{2}$, then we have

$$\text{sun}(G' - X) = \text{sun}(G - X - E') \leq \omega(G - X - E') \leq \omega(G - X) + m = m + 1 \leq 2|X|,$$

which contradicts (1).

If $1 \leq |X| < \frac{m+1}{2}$, then it follows from $m \leq 2r + 1$ that

$$\begin{aligned} \lambda(G - X) &\geq \kappa(G - X) \geq \kappa(G) - |X| > (r + m) - \frac{m+1}{2} \\ &\geq \left(\frac{m-1}{2} + m \right) - \frac{m+1}{2} = m - 1. \end{aligned}$$

By the integrality of $\lambda(G - X)$, we get

$$\lambda(G - X) \geq m. \quad (4)$$

For $e \in E'$, we admit by (4)

$$\lambda(G' - X + \{e\}) = \lambda(G - X - E' \setminus \{e\}) \geq \lambda(G - X) - (m - 1) \geq m - (m - 1) = 1,$$

which implies $\omega(G' - X + \{e\}) = 1$. Hence, we derive

$$\omega(G' - X) \leq \omega(G' - X + \{e\}) + 1 = 2. \quad (5)$$

Using (1), (5) and $1 \leq |X| < \frac{m+1}{2}$, we deduce

$$3 \leq 2|X| + 1 \leq \text{sun}(G' - X) \leq \omega(G' - X) \leq 2,$$

which is a contradiction.

If $|X| = 0$, then from (1), we have

$$\text{sun}(G') = \text{sun}(G' - X) \geq 2|X| + 1 = 1. \quad (6)$$

Note that $\kappa(G) \geq r + m$ and $|E'| = m$. Then $G' = G - E'$ is r -connected, and so $1 = \omega(G') \geq \text{sun}(G')$. Combining this with (6), we get

$$\text{sun}(G') = 1. \quad (7)$$

In view of (7) and $n \geq 4r + 6m + 4$, we know that G' is a big sun. Let R denote the factor-critical graph of G' with $|V(R)| = \frac{n}{2}$. Then G' has at least $2r + 3m + 2$ vertices with degree 1. Note that $G' = G - E'$ with $|E'| = m$. Hence, G admits at least $2r + m + 2$ vertices with degree 1. Then we may select an independent set $\{v_1, v_2, \dots, v_{2r+1}\} \subseteq V(G) \setminus V(R)$ such that $d_G(v_i) = 1$ for $1 \leq i \leq 2r + 1$. According to the degree condition of Theorem 4, we infer

$$1 = \max\{d_G(v_1), d_G(v_2), \dots, d_G(v_{2r+1})\} \geq \frac{n}{3},$$

that is,

$$n \leq 3,$$

which contradicts $n \geq 4r + 6m + 4$.

Subcase 2.2. X is a vertex cut set of G .

In this subcase, $\omega(G - X) \geq 2$ and $|X| \geq r + m$. In terms of (1), we obtain

$$\begin{aligned} \text{sun}(G - X) &\geq \text{sun}(G - X - E') - 2m \\ &= \text{sun}(G' - X) - 2m \geq 2|X| + 1 - 2m \geq 2(r + m) + 1 - 2m \\ &= 2r + 1, \end{aligned}$$

which implies that there exist t sun components in $G - X$, denoted by H_1, H_2, \dots, H_t , where $t \geq 2r + 1$. We choose $v_i \in V(H_i)$ with $d_{H_i}(v_i) \leq 1$ for $1 \leq i \leq 2r + 1$. Clearly, $\{v_1, v_2, \dots, v_{2r+1}\}$ is an independent set of G . Then it follows from the degree condition of Theorem 4 that

$$\max\{d_G(v_1), d_G(v_2), \dots, d_G(v_{2r+1})\} \geq \frac{n}{3}. \quad (8)$$

Without loss of generality, we may let $d_G(v_1) \geq \frac{n}{3}$ by (8). Thus, we infer

$$d_{G[X]}(v_1) = d_G(v_1) - d_{H_1}(v_1) \geq \frac{n}{3} - 1,$$

and so

$$|X| \geq d_{G[X]}(v_1) \geq \frac{n}{3} - 1. \quad (9)$$

In light of (1), (9), $r \geq 1$, $i(G - X) \leq 2r$ and $n \geq 4r + 6m + 4$, we deduce

$$\begin{aligned} n &\geq |X| + 2 \cdot \text{sun}(G - X) - i(G - X) \\ &\geq |X| + 2(\text{sun}(G - X - E') - 2m) - i(G - X) \\ &= |X| + 2(\text{sun}(G' - X) - 2m) - i(G - X) \\ &\geq |X| + 2(2|X| + 1 - 2m) - 2r \\ &= 5|X| - 4m - 2r + 2 \\ &\geq 5\left(\frac{n}{3} - 1\right) - 4m - 2r + 2 \\ &= n + \frac{2n}{3} - 4m - 2r - 3 \\ &\geq n + \frac{2(4r + 6m + 4)}{3} - 4m - 2r - 3 \\ &= n + \frac{2r}{3} - \frac{1}{3} \\ &\geq n + \frac{1}{3} \\ &> n, \end{aligned}$$

which is a contradiction. We complete the proof of Theorem 4. □

Remark 1. We now show that

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_{2r+1})\} \geq \frac{n}{3}$$

in Theorem 4 cannot be replaced by

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_{2r+1})\} \geq \frac{n-1}{3}.$$

Let $m \geq 0$ and $r \geq 1$ be two integers, and t be a sufficiently large integer. We construct a graph $G = K_{rt+m} \vee ((2rt+1)K_1 \cup (mK_2))$. Then G is $(rt+m)$ -connected, $n = 3rt + 3m + 1$ and

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_{2r+1})\} \geq rt + m = \frac{n-1}{3}$$

for any independent set $\{x_1, x_2, \dots, x_{2r+1}\}$ of G . Let $E' = E(mK_2)$ and $G' = G - E'$. Choose $X = V(K_{rt+m})$. Then we obtain

$$\text{sun}(G' - X) = 2rt + 2m + 1 = 2|X| + 1 > 2|X|.$$

By virtue of Theorem 1, G' has no $\mathcal{P}_{\geq 3}$ -factor, and so G is not a $(\mathcal{P}_{\geq 3}, m)$ -factor deleted graph.

But we do not know whether the condition $\kappa(G) \geq r + m$ in Theorem 4 is best possible or not. Naturally, it is interesting to further study the above problem.

3. The proof of Theorem 5

Proof of Theorem 5. Let $G' = G - Q$ for $Q \subseteq V(G)$ with $|Q| = k$. To verify Theorem 5, it suffices to claim that G' has a $\mathcal{P}_{\geq 3}$ -factor. Suppose, to the contrary, that G' has no $\mathcal{P}_{\geq 3}$ -factor. Then by Theorem 1, we derive

$$\text{sun}(G' - X) \geq 2|X| + 1 \tag{1}$$

for some $X \subseteq V(G')$.

Claim 1. $|X| \geq r$.

Proof:

Assume $|X| \leq r - 1$. Since $\kappa(G) \geq r + k$ and $|Q| = k$, $G' - X = G - Q - X$ is connected, which implies $\omega(G' - X) = 1$. By (1), we get

$$1 \leq 2|X| + 1 \leq \text{sun}(G' - X) \leq \omega(G' - X) = 1,$$

which implies $|X| = 0$ and $\text{sun}(G' - X) = \text{sun}(G') = 1$. Then by $|V(G')| = n - k \geq 4r + 4$, we see that G' is a big sun. Let R be the factor-critical graph of G' with $|V(R)| = \frac{1}{2}|V(G')| = \frac{1}{2}(n - k)$. Then by $n \geq 4r + k + 4$, G' admits an independent set $\{v_1, v_2, \dots, v_{2r+1}\} \subseteq V(G') \setminus V(R)$ with $d_{G'}(v_i) = 1$ for $1 \leq i \leq 2r + 1$.

According to the degree condition of Theorem 5, we obtain

$$\begin{aligned} k + 1 &= |Q| + 1 = |Q| + \max\{d_{G'}(v_1), d_{G'}(v_2), \dots, d_{G'}(v_{2r+1})\} \\ &\geq \max\{d_G(v_1), d_G(v_2), \dots, d_G(v_{2r+1})\} \\ &\geq \frac{n + 2k}{3}, \end{aligned}$$

that is,

$$n \leq k + 3,$$

which contradicts $n \geq 4r + k + 4$. Claim 1 is proved. \square

Claim 2. $i(G' - X) \leq 2r$.

Proof:

Let $i(G' - X) \geq 2r + 1$. Then there exist at least $2r + 1$ isolated vertices $v_1, v_2, \dots, v_{2r+1}$ in $G' - X$, namely, $d_{G'-X}(v_i) = 0$ for $1 \leq i \leq 2r + 1$. Hence, we have

$$d_G(v_i) \leq d_{G-Q}(v_i) + |Q| = d_{G'}(v_i) + k \leq d_{G'-X}(v_i) + |X| + k = |X| + k \quad (2)$$

for $1 \leq i \leq 2r + 1$.

In terms of (2) and the degree condition of Theorem 5, we obtain

$$|X| + k \geq \max\{d_G(v_1), d_G(v_2), \dots, d_G(v_{2r+1})\} \geq \frac{n + 2k}{3},$$

which implies

$$|X| \geq \frac{n - k}{3}. \quad (3)$$

It follows from (1) and (3) that

$$\begin{aligned} n &\geq |Q| + |X| + \text{sun}(G' - X) \geq k + |X| + 2|X| + 1 \\ &= 3|X| + k + 1 \geq 3 \cdot \frac{n - k}{3} + k + 1 = n + 1, \end{aligned}$$

which is a contradiction. We complete the proof of Claim 2. \square

Using (1) and Claim 1, we derive

$$\text{sun}(G' - X) \geq 2|X| + 1 \geq 2r + 1,$$

which implies that $G' - X$ admits t sun components H_1, H_2, \dots, H_t , where $t \geq 2r + 1$. Choose $v_i \in V(H_i)$ such that $d_{H_i}(v_i) \leq 1$ for $1 \leq i \leq 2r + 1$. Obviously, $\{v_1, v_2, \dots, v_{2r+1}\}$ is an independent set of G . By the degree condition of Theorem 5, we have

$$\max\{d_G(v_1), d_G(v_2), \dots, d_G(v_{2r+1})\} \geq \frac{n + 2k}{3}. \quad (4)$$

Without loss of generality, let $d_G(v_1) \geq \frac{n+2k}{3}$ by (4). Thus, we derive

$$k + |X| = |Q| + |X| \geq d_{G[Q \cup X]}(v_1) = d_G(v_1) - d_{H_1}(v_1) \geq \frac{n + 2k}{3} - 1,$$

that is,

$$|X| \geq \frac{n - k}{3} - 1. \quad (5)$$

According to (1), (5), Claim 2, $r \geq 1$ and $n \geq 4r + k + 4$, we obtain

$$\begin{aligned}
n &\geq |Q| + |X| + 2 \cdot sun(G' - X) - i(G' - X) \\
&\geq k + |X| + 2(2|X| + 1) - 2r \\
&= 5|X| + k - 2r + 2 \\
&\geq 5 \left(\frac{n-k}{3} - 1 \right) + k - 2r + 2 \\
&= n + \frac{2n}{3} - \frac{2k}{3} - 2r - 3 \\
&\geq n + \frac{2(4r+k+4)}{3} - \frac{2k}{3} - 2r - 3 \\
&= n + \frac{2r}{3} - \frac{1}{3} \\
&\geq n + \frac{1}{3} \\
&> n,
\end{aligned}$$

which is a contradiction. The proof of Theorem 5 is complete. \square

Remark 2. We now show that

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_{2r+1})\} \geq \frac{n+2k}{3}$$

in Theorem 5 cannot be replaced by

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_{2r+1})\} \geq \frac{n+2k-1}{3}.$$

Let $k \geq 0$ and $r \geq 1$ be two integers, and t be a sufficiently large integer. We construct a graph $G = K_{rt+2k+1} \vee ((2rt+2k+3)K_1)$. Then G is $(rt+2k+1)$ -connected, $n = 3rt+4k+4 = 3(rt+2k+1) - 2k+1$ and

$$\max\{d_G(x_1), d_G(x_2), \dots, d_G(x_{2r+1})\} = rt+2k+1 = \frac{n+2k-1}{3}$$

for any independent subset $\{x_1, x_2, \dots, x_{2r+1}\}$ of G . Let $Q \subseteq V(K_{rt+2k+1})$ with $|Q| = k$, and $G' = G - Q$. Select $X = V(K_{rt+k+1}) \subseteq V(K_{rt+2k+1}) \setminus Q$. Then we possess

$$sun(G' - X) = 2rt + 2k + 3 = 2(rt + k + 1) + 1 = 2|X| + 1 > 2|X|.$$

Using Theorem 1, G' has no $\mathcal{P}_{\geq 3}$ -factor, and so G is not a $(\mathcal{P}_{\geq 3}, k)$ -factor critical graph.

But we do not know whether the condition $\kappa(G) \geq r + k$ in Theorem 5 is best possible or not. Naturally, it is interesting to further study the above problem.

Data availability statement

My manuscript has no associated data.

Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

Acknowledgments

The authors would like to express their gratitude to the anonymous reviewers for their helpful comments and valuable suggestions in improving this paper. This work was supported by the Natural Science Foundation of Shandong Province, China (ZR2023MA078).

References

- [1] Bazgan C, Benhamdine A, Li H, Woźniak M. Partitioning vertices of 1-tough graph into paths, *Theoretical Computer Science* 2001. 263(1-2):255–261. doi:10.1016/S0304-3975(00)00247-4.
- [2] Gao W, Wang W, Chen Y. Tight bounds for the existence of path factors in network vulnerability parameter settings, *International Journal of Intelligent Systems* 2021. 36:1133–1158. doi:10.1002/int.22335.
- [3] Kaneko A. A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, *Journal of Combinatorial Theory, Series B* 2003. 88(2):195–218. doi:10.1016/S0095-8956(03)00027-3.
- [4] Kano M, Katona GY, Király Z. Packing paths of length at least two, *Discrete Mathematics* 2004. 283(1-3):129–135. doi:10.1016/j.disc.2004.01.016.
- [5] Kano M, Lee C, Suzuki K. Path and cycle factors of cubic bipartite graphs, *Discussiones Mathematicae Graph Theory* 2008. 28(3):551–556. doi:10.7151/dmgt.1426.
- [6] Kano M, Lu H, Yu Q. Component factors with large components in graphs, *Applied Mathematics Letters* 2010. 23(4):385–389. doi:10.1016/j.aml.2009.11.003.
- [7] Katerinis P, Woodall D. Binding numbers of graphs and the existence of k -factors, *The Quarterly Journal of Mathematics Oxford* 1987. 38:221–228. doi:10.1093/qmath/38.2.221.
- [8] Liu H. Binding number for path-factor uniform graphs, *Proceedings of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science* 2022. 23(1):25–32.
- [9] Las Vergnas M. An extension of Tutte's 1-factor theorem, *Discrete Mathematics* 1978. 23:241–255.
- [10] Wang S, Zhang W. Degree conditions for the existence of a $\{P_2, P_5\}$ -factor in a graph, *RAIRO-Operations Research* 2023. 57(4):2231–2237. doi:10.1051/ro/2023111.
- [11] Wang S, Zhang W. Isolated toughness for path factors in networks, *RAIRO-Operations Research* 2022. 56(4):2613–2619. doi:10.1051/ro/2022123.
- [12] Wang S, Zhang W. Independence number, minimum degree and path-factors in graphs, *Proceedings of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science* 2022. 23(3):229–234.
- [13] Wang S, Zhang W. On k -orthogonal factorizations in networks, *RAIRO-Operations Research* 2021. 55(2):969–977.
- [14] Wang S, Zhang W. Some results on star-factor deleted graphs, *Filomat* 2024. 38(3):1101–1107.

- [15] Woodall D. The binding number of a graph and its Anderson number, *Journal of Combinatorial Theory, Series B* 1973. 15:225–255. doi:10.1016/0095-8956(73)90038-5.
- [16] Wu J. A sufficient condition for the existence of fractional (g, f, n) -critical covered graphs, *Filomat* 2024. 38(6):2177–2183.
- [17] Wu J. Path-factor critical covered graphs and path-factor uniform graphs, *RAIRO-Operations Research* 2022. 56(6):4317–4325. doi:10.1051/ro/2022208.
- [18] Zhou S. A neighborhood union condition for fractional (a, b, k) -critical covered graphs, *Discrete Applied Mathematics* 2022. 323:343–348. doi:10.1016/j.dam.2021.05.022.
- [19] Zhou S. Degree conditions and path factors with inclusion or exclusion properties, *Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie* 2023. 66(1):3–14.
- [20] Zhou S. Remarks on path factors in graphs, *RAIRO-Operations Research* 2020. 54(6):1827–1834. doi:10.1051/ro/2019111.
- [21] Zhou S. Some results on path-factor critical avoidable graphs, *Discussiones Mathematicae Graph Theory* 2023. 43(1):233–244. doi:10.7151/dmgt.2364.
- [22] Zhou S, Bian Q, Sun Z. Two sufficient conditions for component factors in graphs, *Discussiones Mathematicae Graph Theory* 2023. 43(3):761–766.
- [23] Zhou S, Liu H. Two sufficient conditions for odd $[1, b]$ -factors in graphs, *Linear Algebra and its Applications* 2023. 661:149–162. doi:10.1016/j.laa.2022.12.018.
- [24] Zhou S, Pan Q, Xu L. Isolated toughness for fractional $(2, b, k)$ -critical covered graphs, *Proceedings of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science* 2023. 24(1):11–18.
- [25] Zhou S, Sun Z, Bian Q. Isolated toughness and path-factor uniform graphs (II), *Indian Journal of Pure and Applied Mathematics* 2023. 54(3):689–696.
- [26] Zhou S, Sun Z, Liu H. \mathcal{D} -index and \mathcal{Q} -index for spanning trees with leaf degree at most k in graphs, *Discrete Mathematics* 2024. 347(5):113927. doi:10.1016/j.disc.2024.113927.
- [27] Zhou S, Sun Z, Liu H. Some sufficient conditions for path-factor uniform graphs, *Aequationes Mathematicae* 2023. 97(3):489–500.
- [28] Zhou S, Sun Z, Yang F. A result on $P_{\geq 3}$ -factor uniform graphs, *Proceedings of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science* 2022. 23(1):3–8.
- [29] Zhou S, Wu J, Bian Q. On path-factor critical deleted (or covered) graphs, *Aequationes Mathematicae* 2022. 96(4):795–802. doi:10.1007/s00010-021-00852-4.
- [30] Zhou S, Wu J, Liu H. Independence number and connectivity for fractional (a, b, k) -critical covered graphs, *RAIRO-Operations Research* 2022. 56(4):2535–2542. doi:10.1051/ro/2022119.
- [31] Zhou S, Zhang Y, Sun Z. The A_α -spectral radius for path-factors in graphs, *Discrete Mathematics* 2024. 347:113940. doi:10.1016/j.disc.2024.113940.