Perturbation Results for Distance-edge-monitoring Numbers*

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Abstract. Foucaud et al. recently introduced and initiated the study of a new graph-theoretic concept in the area of network monitoring. Given a graph $G = (V(G), E(G))$, a set $M \subseteq V(G)$ is a distance-edge-monitoring set if for every edge $e \in E(G)$, there is a vertex $x \in M$ and a vertex $y \in V(G)$ such that the edge $e$ belongs to all shortest paths between $x$ and $y$. The smallest size of such a set in $G$ is denoted by $\text{dem}(G)$. Denoted by $G - e$ (resp. $G \setminus u$) the subgraph of $G$ obtained by removing the edge $e$ from $G$ (resp. a vertex $u$ together with all its incident edges from $G$). In this paper, we first show that $\text{dem}(G - e) - \text{dem}(G) \leq 2$ for any graph $G$ and edge $e \in E(G)$. Moreover, the bound is sharp. Next, we construct two graphs $G$ and $H$ to show that $\text{dem}(G) - \text{dem}(G \setminus u)$ and $\text{dem}(H \setminus v) - \text{dem}(H)$ can be arbitrarily large, where $u \in V(G)$ and $v \in V(H)$. We also study the relation between $\text{dem}(H)$ and $\text{dem}(G)$, where $H$ is a subgraph of $G$. In the end, we give an algorithm to judge whether the distance-edge-monitoring set still remain in the resulting graph when any edge of a graph $G$ is deleted.

Keywords: Distance; Perturbation result; Distance-edge-monitoring set.

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1. Introduction

In 2022, Foucaud et al. [10] introduced a new graph-theoretic concept called distance-edge-monitoring set (DEM for short), which means network monitoring using distance probes. Networks are naturally modeled by finite undirected simple connected graphs, whose vertices represent computers and whose edges represent connections between them. When a connection (an edge) fails in the network, we can detect this failure, and thus achieve the purpose of monitoring the network. Probes are made up of vertices we choose in the network. At any given moment, a probe of the network can measure its graph distance to any other vertex of the network. Whenever an edge of the network fails, one of the measured distances changes, so the probes are able to detect the failure of any edge. Probes that measure distances in graphs are present in real-life networks. They are useful in the fundamental task of routing [7, 11] and are also frequently used for problems concerning network verification [11, 3, 4].

In a network, we can put as few detectors as possible to monitor all the edges, a natural question is whether the detectors placed in the original graph are still sufficient and need to be supplemented or reduced when some nodes or edges in the original graph are subjected to external interference and damage, we refer to [8, 9, 14, 15, 18]. This kind of problem is usually called perturbation problem.

Graphs considered are finite, undirected and simple. Let \( G = (V(G), E(G)) \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \), whose cardinality are denoted by \( |V(G)| \) and \( e(G) \), respectively. The neighborhood set of a vertex \( v \in V(G) \) is \( N_G(v) = \{ u \in V(G) | uv \in E(G) \} \). Let \( N_G[v] = N_G(v) \cup \{ v \} \) be the closed neighborhood set of a vertex \( v \). The degree of a vertex \( v \) in \( G \) is denoted \( d(v) = |N_G(v)| \). Let \( \delta(G) \) and \( \Delta(G) \) be the minimum and maximum degree of a graph \( G \), respectively. For any subset \( X \subseteq V(G) \), let \( G[X] \) denote the subgraph of \( G \) induced by \( X \); similarly, for any subset \( F \subseteq E(G) \), let \( G[F] \) denote the subgraph induced by \( F \). We use \( G \setminus X \) to denote the subgraph of \( G \) obtained by removing all the vertices of \( X \) together with the edges incident with them from \( G \); similarly, we use \( G - F \) to denote the subgraph of \( G \) obtained by removing all the edges of \( F \) from \( G \). If \( X = \{ v \} \) and \( F = \{ e \} \), we simply write \( G \setminus v \) and \( G - v \) for \( G - \{ v \} \) and \( G - \{ e \} \), respectively. For an edge \( e \) of \( G \), we denote by \( G + e \) the graph obtained by adding an edge \( e \in E(G) \) to \( G \). The Cartesian product \( G \square H \) of two graphs \( G \) and \( H \) is the graph whose vertex set is \( V(G) \times V(H) \) and whose edge set is the set of pairs \( (u, v)(u', v') \) such that either \( uu' \in E(G) \) and \( v = v' \), or \( vv' \in E(H) \) and \( u = u' \). Let \( G \vee H \) be a join graph of \( G \) and \( H \) with \( V(G \vee H) = V(G) \cup V(H) \) and \( E(G \vee H) = \{ uv | u \in V(G), v \in V(H) \} \cup E(G) \cup E(H) \). We denote by \( d_G(x, y) \) the distance between two vertices \( x \) and \( y \) in graph \( G \). For an edge \( uv \) and a vertex \( w \in V(G) \), the distance between them is defined as \( d_G(uw, w) = \min\{d_G(u, w), d_G(v, w)\} \). A \( x-y \) path with length \( d_G(x, y) \) in \( G \) is a \( x-y \) geodesic. Let \( P_n, C_n \) and \( K_n \) be the path, cycle and complete graph of order \( n \), respectively.

1.1. DEM sets and numbers

Foucaud et al. [10] introduced a new graph-theoretic concept called DEM sets, which is relevant to network monitoring.

Definition 1.1. For a set \( M \) of vertices and an edge \( e \) of a graph \( G \), let \( P(M, e) \) be the set of pairs \( (x, y) \) of \( M \) and a vertex \( y \) of \( V(G) \) such that \( d_G(x, y) \neq d_{G-e}(x, y) \). In other words, \( e \) belongs to all shortest paths between \( x \) and \( y \) in \( G \).
Definition 1.2. For a vertex $x$, let $EM(x)$ be the set of edges $e$ such that there exists a vertex $v$ in $G$ with $(x, v) \in P(\{x\}, e)$, that is $EM(x) = \{e \mid e \in E(G) \text{ and } \exists v \in V(G) \text{ such that } d_G(x, v) \neq d_{G-e}(x, v)\}$ or $EM(x) = \{e \mid e \in E(G) \text{ and } P(\{x\}, e) \neq \emptyset\}$. If $e \in EM(x)$, we say that $e$ is monitored by $x$.

Finding a particular vertex set $M$ and placing a detector on that set to monitor all edge sets in $G$ have practical applications in sensor and network systems.

Definition 1.3. A vertex set $M$ of the graph $G$ is distance-edge-monitoring set (DEM set for short) if every edge $e$ of $G$ is monitored by some vertex of $M$, that is, the set $P(M, e)$ is nonempty. Equivalently, $\cup_{x \in M} EM(x) = E(G)$.

Theorem 1.4. [10] Let $G$ be a connected graph with a vertex $x$ of $G$ and for any $y \in N(x)$, then, we have $xy \in EM(x)$.

One may wonder to know the existence of such an edge detection set $M$. The answer is affirmative. If we take $M = V(G)$, then it follows from Theorem 1.4 that

$$E(G) \subseteq \cup_{x \in V(G)} \cup_{y \in N(x)} \{xy\} \subseteq \cup_{x \in V(G)} EM(x).$$

Therefore, we consider the smallest cardinality of $M$ and give the following parameter.

Definition 1.5. The distance-edge-monitoring number (DEM number for short) $dem(G)$ of a graph $G$ is defined as the smallest size of a distance-edge-monitoring set of $G$, that is

$$dem(G) = \min \{|M| \mid \cup_{x \in M} EM(x) = E(G)\}.$$

Furthermore, for any DEM set $M$ of $G$, $M$ is called a DEM basis if $|M| = dem(G)$.

The vertices of $M$ represent distance probes in a network modeled by $G$. The DEM sets are very effective in network fault tolerance testing. For example, a DEM set can detect a failing edge, and it can correctly locate the failing edge by distance from $x$ to $y$, because the distance from $x$ to $y$ will increases when the edge $e$ fails.

Foucaud et al. [10] showed that $1 \leq dem(G) \leq n - 1$ for any $G$ with order $n$, and graphs with $dem(G) = 1, n - 1$ was characterized in [10].

Theorem 1.6. [10] Let $G$ be a connected graph with at least one edge. Then $dem(G) = 1$ if and only if $G$ is a tree.

Theorem 1.7. [10] $dem(G) = n - 1$ if and only if $G$ is the complete graph of order $n$.

Theorem 1.8. [10] For a vertex $x$ of a graph $G$, the set of edges $EM(x)$ induces a forest.

In a graph $G$, the base graph $G_b$ of a graph $G$ is the graph obtained from $G$ by iteratively removing vertices of degree 1.
Observation 1.1. [10] Let $G$ be a graph and $G_b$ be its base graph. Then we have $\text{dem}(G) = \text{dem}(G_b)$.

A vertex set $M$ is called a vertex cover of $G$ if $M \cap \{u, v\} \neq \emptyset$ for $uv \in E(G)$. The minimum cardinality of a vertex cover $M$ in $G$ is the vertex covering number of $G$, denoted by $\beta(G)$.

Theorem 1.9. [10] In any graph $G$ of order $n$, any vertex cover of $G$ is a DEM set of $G$, and thus $\text{dem}(G) \leq \beta(G)$.

Ji et al. [12] studied the Erdős-Gallai-type problems for distance-edge-monitoring numbers. Yang et al. [16] obtained some upper and lower bounds of $P(M, e), EM(x), \text{dem}(G)$, respectively, and characterized the graphs with $\text{dem}(G) = 3$, and gave some properties of the graph $G$ with $\text{dem}(G) = n - 2$. Yang et al. [17] determined the exact value of distance-edge-monitoring numbers of grid-based pyramids, $M(t)$-graphs and Sierpiński-type graphs.

1.2. Progress and our results

Perturbation problems in graph theory are as follows.

Problem 1. Let $G$ be a graph, and let $e \in E(G)$ and $v \in V(G)$. Let $f(G)$ be a graph parameter.

1. The relation between $f(G)$ and $f(G - e)$;
2. The relation between $f(G)$ and $f(G \setminus v)$.


A graph $H$ is a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. If $V(H) = V(G)$, then $H$ is a spanning subgraph of $G$. If $H$ is a subgraph of a graph $G$, where $H \neq G$, then $H$ is a proper subgraph of $G$. Therefore, if $H$ is a proper subgraph of $G$, then either $V(H) \subset V(G)$ or $E(H) \subset E(G)$.

We first consider the existence of graphs with given values of DEM numbers.

Problem 2. Let $r, s, n$ be three integers with $1 \leq r, s \leq n - 1$.

1. Is there a connected graph $G$ of order $n$ such that $\text{dem}(G) = r$?
2. Let $G$ be a connected graph of order $n$. Is there a connected subgraph $H$ in $G$ such that $\text{dem}(H) = s$ and $\text{dem}(G) = r$?

In Section 2, we give the answers to Problem 2.

Proposition 1.10. For any two integers $r, n$ with $1 \leq r \leq n - 1$, there exists a connected graph $G$ of order $n$ such that $\text{dem}(G) = r$. 
Corollary 1.11. Given three integers \( s, t, n \) with \( 1 \leq s \leq t \leq n - 1 \), there exists a connected graph \( H \subseteq G \) such that \( \text{dem}(H) = s \) and \( \text{dem}(G) = t \).

In Section 3, we focus on Problem 1 and study the difference between \( \text{dem}(G - e) \) and \( \text{dem}(G) \).

Theorem 1.12. Let \( G \) be a graph. For any edge \( e \in E(G) \), we have
\[
\text{dem}(G - e) - \text{dem}(G) \leq 2.
\]
Moreover, this bound is sharp.

Let \( G \) be a graph and \( E \subseteq E(G) \). Denote by \( G + E \) the graph with \( V(G + E) = V(G) \) and \( E(G + E) = E(G) \cup E \). We construct graphs with the following properties in Section 3.

Theorem 1.13. For any positive integer \( k \geq 2 \), there exists a graph sequence \( \{G^i | 0 \leq i \leq k\} \), with \( e(G^i) - e(G^0) = i \) and \( V(G^i) = V(G^j) \) for \( 0 \leq i, j \leq k \), such that \( \text{dem}(G^i+1) - \text{dem}(G^0) = i \), where \( 1 \leq i \leq k - 1 \). Furthermore, we have \( \text{dem}(G^0) = 1 \), \( \text{dem}(G^1) = 2 \) and \( \text{dem}(G^i) = i \), where \( 2 \leq i \leq k \).

A feedback edge set of a graph \( G \) is a set of edges such that removing them from \( G \) leaves a forest. The smallest size of a feedback edge set of \( G \) is denoted by \( \text{fes}(G) \) (it is sometimes called the cyclomatic number of \( G \)).

Theorem 1.14. [10] If \( \text{fes}(G) \leq 2 \), then \( \text{dem}(G) \leq \text{fes}(G) + 1 \). Moreover, if \( \text{fes}(G) \leq 1 \), then equality holds.

Theorem 1.14 implies the following corollary, and its proof will be given in Section 3.

Corollary 1.15. Let \( T_n \) be a tree of order \( n \), where \( n \geq 6 \). For edges \( e_1, e_2 \in E(T_n) \), we have
\[
(1) \text{dem}(T_n + e_1) = \text{dem}(T_n) + 1.
\]
\[
(2) \text{dem}(T_n + \{e_1, e_2\}) = 2 \text{ or } 3.
\]

The following result shows that there exists a graph \( G \) and an induced subgraph \( H \) such that the difference \( \text{dem}(G) - \text{dem}(H) \) can be arbitrarily large; see Section 4 for proof details. In addition, we also give an answer to the Problem 1(2).

Theorem 1.16. For any positive integer \( k \), there exist two graphs \( G_1, G_2 \) and their non-spanning subgraphs \( H_1, H_2 \) such that
\[
\text{dem}(G_1) - \text{dem}(H_1) = k \text{ and } \text{dem}(H_2) - \text{dem}(G_2) = k.
\]
Furthermore, \( \text{dem}(G) - \text{dem}(H) \) can be arbitrarily large, even for \( H = G \setminus v \).

Theorem 1.17. For any positive integer \( k \), there exist two graphs \( G, H \) and two vertices \( u \in V(G), v \in V(H) \) such that
\[
(1) \text{dem}(G) - \text{dem}(G \setminus u) \geq k;
\]
\[
(2) \text{dem}(H \setminus v) - \text{dem}(H) \geq k.
\]
For a connected graph $G$ of order $n$, where $n$ is fixed, the difference between $\text{dem}(G)$ and $\text{dem}(G \setminus v)$ can be bounded.

**Proposition 1.18.** For a connected graph $G$ with order $n$ ($n \geq 2$) and $v \in V(G)$, if $G \setminus v$ contains at least one edge, then $\text{dem}(G) - \text{dem}(G \setminus v) \leq n - 2$. Moreover, the equality holds if and only if $G$ is $K_3$.

**Theorem 1.19.** Let $G$ be a connected graph with order $n \geq 4$ and $\text{dem}(G) = 2$. Let $E \subseteq E(G)$. If $\text{dem}(G) = \text{dem}(G - E)$, then $|E| \leq 2n - 6$. Furthermore, the bound is sharp.

For $H \subseteq G$, the **DEM set of $H$ in $G$** is a set $M \subseteq V(H)$ such that $E(H) \subseteq \bigcup_{x \in M} \text{EM}(x)$.

**Definition 1.20.** For $H \subseteq G$, the **restrict-DEM number** $\text{dem}(G|_H)$ of a graph $G$ is defined as the smallest size of a DEM set of $H$ in $G$, that is,

$$\text{dem}(G|_H) = \min \left\{|M| \bigg| E(H) \subseteq \bigcup_{x \in M} \text{EM}(x), M \subseteq V(H) \right\}.$$

**Example 1.21.** Let $G = K_4$ with $V(G) = \{v_0, v_1, v_2, v_3\}$ and $E(G) = \{v_iv_j \mid 0 \leq i < j \leq 3\}$. Let $T_1$ and $T_2$ be the subgraphs of $G$ with $E(T_1) = \{v_0v_1, v_0v_2, v_0v_3\}$ and $E(T_2) = \{v_0v_3, v_3v_1, v_1v_2\}$. Then, $\text{dem}(K_4|_{T_1}) = 1$ and $\text{dem}(K_4|_{T_2}) = 2$. The DEM set of subgraph $T_i$ ($i = 1, 2$) in $K_4$ is shown in Figure 1, where the blue vertices form the set $M$. The reason as follows.

Let $M_1 = \{v_0\}$. Since $v_0v_1, v_0v_2, v_0v_3 \in \text{EM}(v_0)$, it follows that $\text{dem}(K_4|_{T_1}) \leq 1$. Obviously, $\text{dem}(K_4|_{T_1}) \geq 1$, and hence $\text{dem}(K_4|_{T_1}) = 1$. Then, we prove that $\text{dem}(K_4|_{T_2}) = 2$. Since $d_G(v_0, v_1) = d_{G - v_1v_2}(v_0, v_1) = 1$ and $d_G(v_0, v_2) = d_{G - v_1v_2}(v_0, v_2) = 1$, it follows that $v_1v_2 \notin \text{EM}(v_0)$. Similarly, $v_1v_3 \notin \text{EM}(v_0)$. Therefore, $v_1v_2, v_1v_3 \notin \text{EM}(v_0)$. By a similar argument, we have $v_0v_3 \notin \text{EM}(v_1), v_1v_3, v_0v_3 \notin \text{EM}(v_2)$ and $v_1v_2 \notin \text{EM}(v_3)$, and hence $\text{dem}(K_4|_{T_2}) \geq 2$. Let $M = \{v_1, v_3\}$. Then, $v_1v_2, v_1v_3 \in \text{EM}(v_1), v_1v_3, v_0v_3 \in \text{EM}(v_3)$, and hence $\text{dem}(K_4|_{T_2}) \leq 2$. Therefore, we have $\text{dem}(K_4|_{T_2}) = 2$, and so $\text{dem}(K_4|_{T_1}) = i (i = 1, 2)$.

**Theorem 1.22.** Let $T$ be a spanning tree of $K_n$. Then $1 \leq \text{dem}(K_n|_T) \leq \lfloor n/2 \rfloor$. Furthermore, the bound is sharp.

In Section 5, we focus on the following problem and give an algorithm to judge whether the DEM set is still valid in the resulting graph when any edge (or vertex) of a graph $G$ is deleted.
Problem 3. For any graph $G$, if some edges or vertices in $G$ is deleted, we want to know whether the original DEM set can monitor all edges.

2. Results for Problem 2

A kite $K(r, n)$ is a graph obtained from the complete graph $K_{r+1}$ and a path $P_{n-r}$ by attaching a vertex of $K_{r+1}$ and one end-vertex of $P_{n-r}$; see an example of $K(7, 12)$ in Figure 2.

![Figure 2. The graph $K(7, 12)$](image)

We first give the proof of Proposition 1.10.

**Proof of Proposition 1.10** Let $G = K(r, n)$ with $V(G) = \{u_i \mid 0 \leq i \leq n - 1\}$ and $E(G) = \{u_i u_j \mid 0 \leq i < j \leq r\} \cup \{u_r u_s u_{r+s+1} \mid 0 \leq s \leq n - r - 2\}$. From Observation 1.1 and Theorem 1.7 we have $\text{dem}(G) = \text{dem}(G_1) = \text{dem}(K_{r+1}) = r$. In fact, for the above $G$, the path $P_{n-r-1}$ can be replaced by $T_{n-r-1}$, where $T_{n-r-1}$ is any tree of order $n - r - 1$. \[\square\]

Proposition 1.10 shows that Corollary 1.11 is true. For three integers $s, t, n$ with $1 \leq s \leq t \leq n - 1$, let $G = K(t, n)$ and $H = K(s, n) \subseteq G$. From Proposition 1.10 $\text{dem}(G) = t$ and $\text{dem}(H) = s$. Therefore, there exists a connected graph $H \subseteq G$ such that $\text{dem}(H) = s$ and $\text{dem}(G) = t$.

This gives an answer about Problem 2, see Corollary 1.11. One might guess that if $H$ is a subgraph of $G$, then $\text{dem}(H) \leq \text{dem}(G)$, however we will show in the next section that there is no monotonicity for the DEM number.

3. The effect of deleted edge

The following observation is immediate.

**Observation 3.1.** Let $G_1, G_2, \ldots, G_m$ be the connected components of $G$. Then

$$\text{dem}(G) = \text{dem}(G_1) + \cdots + \text{dem}(G_m).$$

Furthermore, we suppose that the DEM number of $K_1$ is 0.
Proposition 3.1. For any $uv \in E(G)$, $uv \notin EM(w)$ for $w \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$ if and only if $uv$ is only monitored by $u$ and $v$.

Proof:
Since $w \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$ and $uv \notin EM(w)$, it follows that $d_G(w, u) = d_G(w, v)$ and $d_G(w, v) = d_G(w, v)$. For any $x \in V(G) - N_G[u] \cup N_G[v]$, the path from $x$ to $u$ must through $w$, where $w \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$. Then $d_G(x, u) = d_G(x, w) + d_G(w, u) = d_G(x, w) + d_G(w, u) = d_G(x, w) + d_G(u, v) = d_G(x, u) + d_G(u, v) = d_G(u, v)$. Similarly, $d_G(x, v) = d_G(w, v)$. For any $x \in V(G) - \{u, v\}$, we have $uv \notin EM(x)$. From Theorem 1.4, $w \in EM(u)$ and $uv \in EM(v)$, and hence $uv$ is only monitored by the vertex in $\{u, v\}$.

Conversely, if $uv$ is only monitored by $u$ and $v$, then $uv \notin EM(w)$ for any $w \in V(G) \setminus \{u, v\}$. Especially, since $(N_G(u) \cup N_G(v)) \setminus \{u, v\} \subseteq V(G) \setminus \{u, v\}$, it follows that $uv \notin EM(w)$ for $w \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$, as desired.

Then, we give the proof of Theorem 1.12.

Proof of Theorem 1.12. If $G$ is a disconnected graph, then the edge $e$ must be in some connected component $G_1$ of $G$ for any $e \in E(G)$, and hence $e$ can only be monitored by the vertex in $V(G_1)$. Therefore, we just need consider the graph $G$ which is connected. Let $M$ be a DEM set of $G$ with $|M| = dem(G)$ and $e = uv \in E(G)$. If $M$ is also a DEM set of $G - e$, then $dem(G - e) \leq dem(G)$. Otherwise, let $M' = M \cup \{u, v\}$. It suffices to show that $M'$ is a DEM set of $G - e$.

If $G - e$ has two components, say $G_1$ and $G_2$, then $e$ is a cut edge of $G$ and from Observation 3.1, we have $dem(G - e) = dem(G_1) + dem(G_2)$. Without loss of generality, assume that $u \in V(G_1)$ and $v \in V(G_2)$.

Fact 3.2. $dem(G_1) \leq |(M \cap V(G_1)) \cup \{u\}|$ and $dem(G_2) \leq |(M \cap V(G_2)) \cup \{v\}|$.

Proof:
For any edge $e_1 = x_1y_1 \in E(G_1)$, if there exists a vertex $w \in V(G_1) \cap M$ such that $e_1 \in EM(w)$, then we are done. Otherwise, there exists a vertex $w \in V(G_2) \cap M$ such that $d_{G - e_1}(x_1, w) \neq d_G(x_1, w)$ or $d_{G - e_1}(y_1, w) \neq d_G(y_1, w)$. Without loss of generality, we suppose that $d_{G - e_1}(y_1, w) \neq d_G(y_1, w)$ and $d_G(w, x_1) = d_G(w, x_1)$. Since $d_G(y_1, w) = d_G(y_1, x_1) + d_G(x_1, u) + d_G(u, w)$, $d_{G - e_1}(x_1, u) = d_{G - e_1}(x_1, u)$ and $d_{G - e_1}(y_1, x_1) > d_{G - e}(y_1, x_1)$, it follows that

$$d_{G - e_1}(u, y_1) = d_{G - e_1}(u, x_1) + d_{G - e_1}(x_1, y_1) = d_{G - e_1}(u, x_1) + d_{G - e}(x_1, y_1) = d_{G - e}(u, y_1)$$

and hence $d_{G - e_1}(y_1, u) \neq d_{G - e_1}(y_1, u)$. Therefore, $e_1$ is monitored by $(M \cap V(G_1)) \cup \{u\}$ in graph $G - e$. This implies that $dem(G_1) \leq |(M \cap V(G_1)) \cup \{u\}|$. Similarly, we can obtain that $dem(G_2) \leq |(M \cap V(G_2)) \cup \{v\}|$.

From Fact 3.2, we have $dem(G - e) \leq |M'| = |M \cup \{u, v\}| \leq |M| + 2 = dem(G) + 2$. 

□
Suppose that \( G - e \) is connected. If \( M \) is also a DEM set of \( G - e \), then \( \text{dem}(G - e) \leq |M| = \text{dem}(G) \) and we are done. Otherwise, there exists \( e_1 = xy \in E(G - e) \) such that the edge \( e_1 \) is not monitored by \( M \) in \( G - e \). Since \( M \) is a distance-edge-monitoring set of \( G \), it follows that there exists a vertex \( z \in M \) such that \( d_{G-e_1}(x, z) \neq d_G(x, z) \) or \( d_{G-e_1}(y, z) \neq d_G(y, z) \). In addition, since \( e_1 \) is not monitored by \( M \) in \( G - e \), it follows that the distance from \( z \) to \( x \) or \( y \) is not changed after removing the edge \( e_1 \) in \( G - e \), which means that \( d_{G-e_1}(y, z) = d_G(y, z) \) and \( d_{G-e_1}(x, z) = d_G(x, z) \). If \( d_G(e_1, z) = d_G(x, z) \), then the edge \( e \) lies on every \( z - y \) geodesic in \( G \) for \( z \in M \) and \( xy \in EM(z) \) in \( G \), otherwise there exists \( z^* \in M \) and \( xy \in EM(z^*) \) such that \( e \) does not appear in \( z^* - y \) geodesic in \( G \), that is \( d_{G-e}(x, z^*) = d_G(x, z^*) \) and \( d_{G-e_1}(x, z^*) \neq d_G(x, z^*) \), which contradicts to the fact that \( M \) is not the DEM set of graph \( G - e \).

Claim 3.3. If a geodesic in \( G \) from \( z \) to \( y \) traverses the edge \( e \) in the order \( u, v \), then each geodesic in \( G \) from \( z \) to \( y \) traverses \( e \) in the order \( u, v \).

**Proof:**
Assume, to the contrary, that there exists two \( z - y \) geodesics \( P_1^0 \) and \( P_2^0 \), where \( P_1^0 = z \ldots uv \ldots y \) and \( P_2^0 = z \ldots vu \ldots y \). The \( z - y \) geodesic \( P_1^0 \) implies that \( d(u, v) + d(v, y) = d(u, y) \), and the \( z - y \) geodesic \( P_2^0 \) implies that \( d(v, u) + d(u, y) = d(v, y) \), and hence \( d(u, v) = 0 \), a contradiction. \( \square \)

From Claim 3.3 without loss of generality, we may assume that every geodesic in \( G \) from \( z \) to \( y \) traverses the edge \( e \) in the order \( u, v \). Thus, we have \( d_G(z, y) = d_G(z, v) + d_G(v, y) \). We now show that \( xy \) can be monitored by \( v \) in \( G - e \). Note that \( d_{G-e_1}(z, y) \neq d_G(z, y) \), \( d_{G-e}(v, y) = d_G(v, y) \) and \( d_{G-e}(x, y) = d_G(x, y) \). Then \( d_{G-e_1}(v, y) = d_{G-e_1}(v, x) + d_{G-e}(x, y) \). Since \( d_{G-e}(v, y) > d_{G-e_1}(v, y) \), it follows that \( e_1 \) can be monitored by \( v \). Since \( e_1 \in EM(u) \) or \( e_1 \in EM(v) \), it follows that \( M' = M \cup \{u, v\} \) is a distance edge-monitoring-set of \( G - e \), and thus \( \text{dem}(G - e) \leq \text{dem}(G) + 2 \), as desired. \( \square \)

Li et al. [13] got the following result about DEM numbers of \( C_k \Box P_t \).

**Theorem 3.4.** [13] Let \( \ell \) and \( k \) be two integers with \( \ell \geq 3 \) and \( k \geq 2 \). Then

\[
\text{dem}(C_k \Box P_t) = \begin{cases} 
  k & \text{if } k \geq 2\ell + 1, \\
  2\ell & \text{if } k < 2\ell + 1.
\end{cases}
\]

To show the sharpness of Theorem 1.12 we consider the following proposition.

**Proposition 3.5.** There exist two connected graphs \( G_1, G_2 \) of order \( n \) such that \( \text{dem}(G_1 - e) - \text{dem}(G_1) = 2 \) and \( \text{dem}(G_2) - \text{dem}(G_2 - e) = 2 \).

**Proof:**
Firstly, we consider the graph \( G_1 (|V(G_1)| = n \geq 8) \) with vertex set \( V(G_1) = \{v_i | 1 \leq i \leq n - 8\} \cup \{u_i | 1 \leq i \leq 8\} \) and edge set \( E(G_1) = \{u_iv_i | 1 \leq i \leq 8\} \cup \{u_iu_{i+1} | 1 \leq i \leq 7\} \cup \{v_iu_{i+1} | 1 \leq i \leq 7\} \cup \{u_1u_8\} \cup \{u_1u_5\} \cup \{v_1v_8\} \cup \{v_1v_9\} \cup \{v_iv_{i+1} | 9 \leq i \leq n - 9\} \). Let \( G_8^* = G_b(G_1) \).
Obviously, $G^*_8$ is the base graph of $G_1$, which is obtained by removing the all edge in the edge set $\{v_1v_9\} \cup \{v_i v_{i+1} \mid 9 \leq i \leq n-9\}$. The graphs $G^*_8$ and $G^*_8 - u_1u_5$ are shown in Figures 3 and 4 respectively.

Let $M_1 = \{u_2, u_4, v_6, u_7, v_8\}$. Note that $\{u_1u_5, u_5v_5, v_2v_1, u_2u_1, u_2u_2\} \subseteq EM(u_2)$, $\{v_1u_1, u_4u_4, u_4u_5, u_4v_4\} \subseteq EM(u_4)$, $\{v_3u_3, v_2v_1, v_3v_4, v_5v_4, v_5v_2\} \subseteq EM(v_3)$, $\{v_8v_1, u_4v_8, v_8v_7\} \subseteq EM(v_8)$, $\{u_7u_8, u_8u_1, u_6u_7, u_6u_5, u_7v_7\} \subseteq EM(u_7)$ and $\{v_5v_6, v_6v_7, u_6v_6\} \subseteq EM(v_6)$. Therefore, $E(G^*_8) = \cup x \in M_1 EM(x)$, and hence $\text{dem}(G^*_8) \leq |M_1| = 6$.

Let $M$ be a DEM set of $G^*_8$ with the minimum cardinality. For the edge $u_iv_i$, where $2 \leq i \leq 8$ and $i \neq 5$, and any $w \in (N(u_i) \cup N(v_i)) \setminus \{u_i, v_i\}$, we have $d_{G-u_iv_i}(w, u_i) = d_G(w, u_i)$ and $d_{G-u_iv_i}(w, v_i) = d_G(w, v_i)$, and hence $u_iv_i \notin EM(w)$. From Proposition 3.1 the edge $u_iv_i$ ($2 \leq i \leq 8$ and $i \neq 5$) is only monitored by $\{u_i, v_i\}$, and hence $M \cap \{u_i, v_i\} \neq \emptyset$ for $2 \leq i \leq 8$ and $i \neq 5$, and so $\text{dem}(G^*_8) \geq 6$. Therefore, $\text{dem}(G^*_8) = 6$.

Since $G^*_8 - u_1u_5 \cong C_8 \boxdot P_2$, it follows from Theorem 3.4 that $\text{dem}(G^*_8 - u_1u_5) = \text{dem}(C_8 \boxdot P_2) = 8$. From Observation 1.1 $\text{dem}(G_1 - u_1u_5) - \text{dem}(G_1) = \text{dem}(G^*_8 - u_1u_5) - \text{dem}(G^*_8) = 8 - 6 = 2$, as desired.
Next, we consider the graph $G_2$ ($|V(G_2)| = n \geq 6$) with vertex set $V(G_2) = \{v_i \mid 1 \leq i \leq n\}$ and edge set $E(G_2) = \{v_1v_2, v_3v_4, v_5v_6, v_1v_3, v_1v_5, v_2v_4, v_2v_6, v_3v_5, v_4v_6\} \cup \{v_iv_{i+1} \mid 6 \leq i \leq n - 1\}$. Let $G_6'$ be the base graph of $G_2$, that is, $G_b(G_2) = G_6'$. The graphs $G_6'$ and $G_6' - v_1v_3$, are shown in Figure 5 and Figure 6, respectively. From Observation 1.1, $\text{dem}(G_2) = \text{dem}(G_6')$.

Take $M'_1 = \{v_2, v_3, v_4, v_5\}$. Note that $\{v_1v_2, v_6v_2, v_4v_2\} \subseteq EM(v_2), \{v_1v_3, v_5v_3, v_4v_3\} \subseteq EM(v_3), \{v_6v_4\} \subseteq EM(v_4), \{v_5v_1, v_6v_5\} \subseteq EM(v_5)$, and hence $E(G_6') = \cup_{x \in M'_1} EM(x)$, it follows that $M'_1$ is a DEM set of $G_6'$, and hence $\text{dem}(G_6') \leq |M'_1| = 4$. Let $M'$ be a DEM set of $G_6'$ with the minimum cardinality. For the edge $v_{2i-1}v_{2i}$ ($1 \leq i \leq 3$) and $w \in (N(v_{2i-1}) \cup N(v_{2i})) \setminus \{v_{2i-1}v_{2i}\}$, we have $d_{G_6'}(w, v_{2i-1}) = d_G(w, v_{2i-1})$ and $d_{G_6'}(w, v_{2i}) = d_G(w, v_{2i})$, and so $v_{2i-1}v_{2i} \notin EM(w)$. From Proposition 3.1, the edge $v_{2i-1}v_{2i}$ ($1 \leq i \leq 3$) is monitored by the vertex in $\{v_{2i-1}, v_{2i}\}$, and hence $M' \cap \{v_{2i-1}, v_{2i}\} \neq \emptyset$ ($1 \leq i \leq 3$). All sets $M' \in V(G_6')$ with $|M'| = 3$ are shown in Table 1. Therefore, all sets $M'$ with $|M'| = 3$ are not DEM sets of $G_6'$, and hence $\text{dem}(G_6') \geq 4$. Therefore, we have $\text{dem}(G_6') = 4$.

Table 1. The edges are not monitored by $M'(|M'| = 3)$.

<table>
<thead>
<tr>
<th>$M'$</th>
<th>$E(G_6') - \cup_{x \in M'} EM(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1, v_3, v_6$</td>
<td>$v_2v_4$</td>
</tr>
<tr>
<td>$v_1, v_4, v_5$</td>
<td>$v_2v_6$</td>
</tr>
<tr>
<td>$v_1, v_4, v_6$</td>
<td>$v_3v_5$</td>
</tr>
<tr>
<td>$v_2, v_3, v_5$</td>
<td>$v_4v_6$</td>
</tr>
<tr>
<td>$v_2, v_3, v_6$</td>
<td>$v_1v_5$</td>
</tr>
<tr>
<td>$v_2, v_4, v_5$</td>
<td>$v_1v_3$</td>
</tr>
<tr>
<td>$v_2, v_4, v_6$</td>
<td>$v_1v_3, v_1v_5, v_3v_5$</td>
</tr>
</tbody>
</table>

For the graph $G_6' - v_3v_4$, let $M_3 = \{v_2, v_5\}$. Note that $\{v_1v_2, v_6v_2, v_4v_2, v_1v_3\} \subseteq EM(v_2)$ and $\{v_5v_1, v_6v_5, v_3v_5, v_4v_4\} \subseteq EM(v_5)$. Since $E(G_6' - v_3v_4) = \cup_{x \in M_3} EM(x)$, it follows that $M_3$ is a DEM set of $G_6'$, and hence $\text{dem}(G_6' - v_3v_4) \leq 2$. Since $G_6' - v_3v_4$ is not a tree, it follows from Theorem 1.6 that $\text{dem}(G_6' - v_3v_4) \geq 2$, and so $\text{dem}(G_6' - v_3v_4) = 2$. From Observation 1.1, $\text{dem}(G_2) - \text{dem}(G_2 - v_3v_4) = \text{dem}(G_6') - \text{dem}(G_6' - v_3v_4) = 4 - 2 = 2$, as desired. 

The friendship graph, $Fr(n)$, can be constructed by joining $n$ copies of the complete graph $K_3$ with a common vertex, which is called the universal vertex of $Fr(n)$. Next, we give the proof of Theorem 1.13.

Proof of Theorem 1.13: Let $k, i$ be integers with $1 \leq i \leq k$. The graph $G^i$ is obtained by iteratively adding an edge $u_iv_i$ to the graph $G^{i-1}$. Without loss of generality, let $G^0$ be the graph with $V(G^0) = \{e\} \cup \{u_j \mid 1 \leq j \leq k\} \cup \{v_j \mid 1 \leq j \leq k\}$ and $E(G^0) = \{euj, ev_j \mid 1 \leq j \leq k\}$, and $G^i$ be the graph with $V(G^i) = V(G^{i-1})$ and $E(G^i) = E(G^{i-1}) \cup \{u_iv_i\}$, where $1 \leq i \leq k$. Since $G^0$ is a tree, it
follows from Theorem 1.6 that \( \text{dem}(G^0) = 1 \). Note that the base graph of \( G^1 \) is a complete graph \( K_3 \).

From Observation 1.1 and Theorem 1.7, we have \( \text{dem}(G_1) = \text{dem}(K_3) = 2 \).

Let \( G = G^i \), where \( 2 \leq i \leq k \). Then \( G_i = Fr(i) \). Let \( M = \{u_t \mid 1 \leq t \leq i\} \). From Theorem 1.4, we have \( \{u_t v_t, cu_t \mid 1 \leq t \leq i\} \subseteq \cup_{x \in M} EM(x) \). Since \( 2 = d_G(u_1, v_t) \neq d_{G-cu_t}(u_1, v_t) = 3 \) for \( 2 \leq t \leq i \), it follows that \( cu_t \in EM(u_1) \) for \( 2 \leq t \leq i \). Suppose that \( t = 1 \). Since \( 2 = d_G(u_2, v_1) \neq d_{G-cu_1}(u_2, v_1) = 3 \), it follows that \( cv_1 \in EM(u_2) \), and hence \( E(G) \subseteq \cup_{x \in M} EM(x) \), and so \( \text{dem}(G) \leq i \). Let \( M \) be a DEM set of \( G \) with the minimum cardinality. Note that \( (N(u_j) \cup N(v_j)) \setminus \{u_j, v_j\} = \{c\} \). Since \( d_G(c, u_j) = d_{G-u_jv_j}(c, u_j) \) and \( d_G(c, v_j) = d_{G-u_jv_j}(c, v_j) \), it follows that \( u_j v_j \notin EM(c) \), where \( 1 \leq j \leq k \). From Proposition 3.1, the edge \( u_j v_j \) is only monitored by \( u_j \) or \( v_j \), and hence \( M \cap \{u_j, v_j\} \neq \emptyset \) for \( 1 \leq j \leq k \). Therefore, \( \text{dem}(G) \geq i \), and so \( \text{dem}(G) = i \). Thus, there exists a graph sequence \( \{G^i \mid 0 \leq i \leq k\} \), with \( e(G^i) - e(G^0) = i \) and \( V(G^i) = V(G^j) \) for \( 0 \leq i, j \leq k \), such that \( \text{dem}(G^{i+1}) - \text{dem}(G^0) = i \), where \( 1 \leq i \leq k - 1 \).

Foucaud et al. [10] obtained the following result.

**Theorem 3.6.** [10] Let \( \ell_1 \) and \( \ell_2 \) be two integers with \( \ell_2 \geq 2 \). Then

\[
\text{dem}(P_{\ell_1} \square P_{\ell_2}) = \max\{\ell_1, \ell_2\}
\]

In the end of this section, we give the proof of Corollary 1.15.

**Proof of Corollary 1.15:** For any tree \( T_n \), \( T_n + e_1 \) is an unicyclic graph and \( T_n + \{e_1, e_2\} \) is a tricyclic graph. From Theorems 1.6 and 1.14, we have \( \text{dem}(T_n + e_1) = \text{dem}(T_n) + 1 = 2 \) and \( \text{dem}(T_n + \{e_1, e_2\}) = 2 \) or 3.

4. The effect of deleted vertex

A kipas \( \hat{K}_n \) with \( n \geq 3 \) is the graph on \( n + 1 \) vertices obtained from the join of \( K_1 \) and \( P_n \), where \( V(\hat{K}_n) = \{v_0, v_1, \ldots, v_n\} \) and \( E(\hat{K}_n) = \{v_0 v_i \mid 1 \leq i \leq n\} \cup \{v_i v_{i+1} \mid 1 \leq i \leq n - 1\} \).

**Proposition 4.1.** For \( n \geq 7 \), we have \( \text{dem}(\hat{K}_n) = \lfloor n/2 \rfloor \).

**Proof:**

Let \( P_n \) be the subgraph of \( \hat{K}_n \) with vertex set \( \{v_i \mid 1 \leq i \leq n\} \) and edge set \( \{v_i v_{i+1} \mid 1 \leq i \leq n - 1\} \). First, we prove that \( \text{dem}(\hat{K}_n) \geq \lfloor n/2 \rfloor \). Let \( M \) be a DEM set of \( \hat{K}_n \) with the minimum cardinality. For any vertices \( v_i, v_j \in V(\hat{K}_n) \), we have

\[
d_{\hat{K}_n}(v_i, v_j) = \begin{cases} 
1, & \text{if } i = 0 \text{ or } j = 0 \text{ or } |i - j| = 1; \\
2, & \text{if } 1 \leq i, j \leq n \text{ and } |i - j| \geq 2.
\end{cases}
\]

For any edge \( v_i v_{i+1} \) (\( 2 \leq i \leq n - 2 \)), we have \( (N_G(v_i) \cup N_G(v_{i+1})) \setminus \{v_i, v_{i+1}\} = \{v_{i-1}, v_0, v_{i+2}\} \). Since \( d_G(v_i, v_0) = d_{G-v_i v_{i+1}}(v_i, v_0) = 1, d_G(v_{i+1}, v_0) = d_{G-v_i v_{i+1}}(v_{i+1}, v_0) = 1, d_{G-v_i v_{i+1}}(v_{i-1}) = d_G(v_{i-1}, v_{i-1}) = 1, d_{G-v_i v_{i+1}}(v_{i-1}, v_{i+1}) = d_G(v_{i-1}, v_{i+1}) = 2, d_{G-v_i v_{i+1}}(v_{i+1}, v_{i+2}) = d_G(v_{i+1}, v_{i+2}) = 1, \) and \( d_{G-v_i v_{i+1}}(v_{i+1}, v_{i+2}) = d_G(v_{i+1}, v_{i+2}) = 2 \), it follows that \( v_i v_{i+1} \notin EM(v_{i+2}) \cup EM(v_{i-1}) \cup EM(v_0) \). From Proposition 3.1, the edge \( v_i v_{i+1} \) can only be monitored.
by the vertex in \(\{v_i, v_{i+1}\}\). Similarly, the edge \(v_i v_{i+1}\) is only monitored by the vertex in \(\{v_i, v_{i+1}\}\), where \(i = 1, n-1\). Therefore, \(M \cap \{v_i, v_{i+1}\} \neq \emptyset\) for \(1 \leq i \leq n-1\), that is, \(M\) is a vertex cover set of \(P_n\). Note that the vertex covering number of \(G\) is \(\beta(G)\). Since \(\beta(P_n) = \lfloor n/2 \rfloor\), it follows that \(\text{dem}(\widetilde{K}_n) \geq \lfloor n/2 \rfloor\).

Next, we prove that \(\text{dem}(\widetilde{K}_n) \leq \lfloor n/2 \rfloor\). Let \(M = \{v_i \mid i \equiv 0 \pmod{2}, 1 \leq i \leq n\}\). For any edge \(e \in E(P_n) \cup \{v_0v_i \mid i \equiv 0 \pmod{2}, 1 \leq i \leq n\}\), it follows from Theorem 1.4 that \(e\) is monitored by the vertex in \(M\). In addition, for any edge \(v_0v_i \in \{v_0v_i \mid i \equiv 1 \pmod{2}, 1 \leq i \leq n\}\), since \(n \geq 7\), it follows that there exists \(j\) such that \(d_G(v_i, v_j) = 2\) and \(d_{G-v_0v_i}(v_i, v_j) = 3\), where \(j = i + 3\) for \(1 \leq i \leq n-4\) and \(j = 2\) for \(n-3 \leq i \leq n\), and hence \(v_0v_i \in EM(v_j)\). Since any edge \(v_0v_i \in E(\widetilde{K}_n)\) can be monitored by the vertex in \(M\), it follows that \(\text{dem}(\widetilde{K}_n) \leq \lfloor n/2 \rfloor\), and hence \(\text{dem}(\widetilde{K}_n) = \lfloor n/2 \rfloor\).

\[\square\]

**Proof of Theorem 1.16** Note that \(\widetilde{K}_{2k+2} = K_1 \cup P_{2k+2}\), where \(V(K_1) = \{v_0\}\). From Theorem 1.6 we have \(\text{dem}(P_{2k+2}) = 1\). From Lemma 4.1 we have \(\text{dem}(\widetilde{K}_{2k+2}) = k + 1\), and hence \(\text{dem}(\widetilde{K}_{2k+2}) - \text{dem}(\widetilde{K}_{2k+2} - v_0) = \text{dem}(P_{2k+2}) = k\). Let \(G_1 = \widetilde{K}_{2k+2}\) and \(H_1 = P_{2k+2}\). Then \(\text{dem}(G_1) - \text{dem}(H_1) = \text{dem}(\widetilde{K}_{2k+2}) - \text{dem}(P_{2k+2}) = k\), where \(H_1\) is not a spanning subgraph of \(G_1\).

Let \(G_{2k+3}\) be a graph with vertex set \(V(G_{2k+3}) = \{u_i \mid 1 \leq i \leq k+1\} \cup \{v_i \mid 0 \leq i \leq k+1\}\) and edge set \(E(G_{2k+3}) = \{v_0u_i \mid 1 \leq i \leq k+1\} \cup \{u_i v_i \mid 1 \leq i \leq k+1\}\). Obviously, we have \(G_{2k+3} \setminus v_0 \cong (k+1)K_2\). From Observation 3.1 and Theorem 1.7 we have \(\text{dem}(G_{2k+3} - v_0) = \text{dem}((k+1)K_2) = (k+1)\text{dem}(K_2) = k + 1\). Since \(G_{2k+3}\) is a tree, it follows from Theorem 1.6 that \(\text{dem}(G_{2k+3}) = 1\), and hence \(\text{dem}(G_{2k+1} \setminus v_0) - \text{dem}(G_{2k+1}) = k\). Let \(G_2 = G_{2k+1}\) and \(H_2 = (k+1)K_2\). Then \(\text{dem}(H_2) - \text{dem}(G_2) = \text{dem}((k+1)K_2) - \text{dem}(G_{2k+1}) = k\), where \(H_2\) is not a spanning subgraph of \(G_2\), as desired.

\[\square\]

Note that \(G_{2k+3} \setminus v_0 \cong (k+1)K_2\) is disconnected graph. For the connected graphs, we can also show that there is a connected subgraph \(H\) such that \(\text{dem}(H) - \text{dem}(G)\) can be arbitrarily large; see Theorem 4.5.

The *conical graph* \(C(\ell, k)\) is a graph obtained by taking adjacency from a center vertex \(c\) to the first layer of Cartesian product of \(P_{\ell}\) and \(C_k\), where \(\ell \geq 1\) and \(k \geq 3\).

Let the vertex set \(V(C(\ell, k)) = \{c\} \cup \{u^j_i \mid 1 \leq i \leq k, 1 \leq j \leq \ell\}\) and the edge set \(E(C(\ell, k)) = (\cup_{i=1}^\ell E(C_i)) \cup (\cup_{j=1}^k E(P_j))\), where \(E(C_i) = \{u^j_k u^j_i \mid 1 \leq j \leq k \} \cup \{u^j_i u^j_{i+1} \mid 1 \leq j \leq k \} \cup \{u^j_{i+1} u^j_i \mid 1 \leq j \leq k \} \cup \{u^j_i u^j_{i+1} \mid 1 \leq j \leq k \}\). The conical graph \(C(3, 8)\) is shown in Figure 7.

For \(\ell = 1\), the graph \(C(1, k)\) is the wheel graph \(W_k\), which is formed by connecting a single vertex \(c\) to all the vertices of cycle \(C_k\). It is clear that \(|V(C(\ell, k))| = k\ell + 1\) and \(e(C(\ell, k)) = 2k\ell\).

**Lemma 4.2.** Let \(n \geq 3\) be an integer. For \(v \in V(C_n)\), we have

\[
|EM(v) \cap E(C_n)| = \begin{cases} 
n - 1 & \text{if } n \text{ is odd}, 
n - 2 & \text{if } n \text{ is even}. 
\end{cases}
\]
Proof:
Let $G = C_n$ be cycle with $V(G) = \{v_i \mid 1 \leq i \leq n\}$ and $E(G) = \{v_iv_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_nv_1\}$. Without loss of generality, let $v = v_1$. Suppose that $n$ is odd and $e_1 = v_{[n/2]+1}v_{[n/2]+2}$. Since $d_G(v_1, v_{[n/2]+1}) = d_G-e_i(v_1, v_{[n/2]+1})$ and $d_G(v_1, v_{[n/2]+2}) = d_G-e_i(v_1, v_{[n/2]+2})$, it follows that $e_1 \notin EM(v_1)$. For any $e \in \{v_iv_{i+1} \mid 1 \leq i \leq \lfloor n/2 \rfloor\}$, since $d_G(v_1, v_{i+1}) \neq d_G-e_i(v_1, v_{i+1})$, it follows that $e \in EM(v_1)$. For any $e \in \{v_iv_{i+1} \mid \lceil n/2 \rceil + 1 \leq i \leq n-1\}$, since $d_G(v_1, v_i) \neq d_G-e(v_1, v_i)$, it follows that $e \in EM(v_1)$. From Theorem 1.4, we have $v_nv_1 \in EM(v_1)$. Therefore, $EM(v_1) = \{v_1v_2, v_2v_3, \ldots, v_{[n/2]}v_{[n/2]+1}, v_1v_n, v_nv_{n-1}, \ldots, v_{[n/2]+3}v_{[n/2]+2}\}$, and hence $|EM(v) \cap E(C_n)| = n - 1$. Similarly, if $n$ is even, then $|EM(v) \cap E(C_n)| = n - 2$. \hfill \Box

Theorem 4.3. For $k \geq 9$ and $\ell \geq 2$, we have 

$$\text{dem}(C(\ell, k)) = \begin{cases} 
\sum_{i=1}^{\ell} \lfloor k/(4i - 2) \rfloor, & \text{if } \ell \leq a_k; \\
\sum_{i=1}^{a_k} \lfloor k/(4i - 2) \rfloor + 2(\ell - a_k), & \text{if } \ell \geq a_k + 1,
\end{cases}$$

where $a_k = \lfloor k/4 + (1 + (-1)^{k+1})/8 \rfloor$.

Proof:
Let $G = C(\ell, k)$ with $V(G) = \{c\} \cup \{u_i^j \mid 1 \leq i \leq k, 1 \leq j \leq \ell\}$ and $E(G) = (\cup_{i=1}^{\ell} E(C_i)) \cup (\cup_{i=1}^{k} E(P_i))$, where $E(C_i) = \{u_i^ju_{i+1}^j \mid 1 \leq j \leq k-1\}$, $E(P_i) = \{cu_1^i\} \cup \{u_1^j u_{i+1}^{j+1} \mid 1 \leq j \leq \ell - 1\}$. Let $M$ be a DEM set of $G$ with $M = \text{dem}(C(\ell, k))$.

Fact 4.4. For any vertex $v \in V(C_i)$, we have 

$$|EM(v) \cap E(C_i)| = \begin{cases} 
4i - 2, & \text{if } 1 \leq i \leq a_k; \\
 k - 2, & \text{if } k \text{ is even and } i \geq a_k + 1;
 k - 1, & \text{if } k \text{ is odd and } i \geq a_k + 1,
\end{cases}$$

and $|EM(v) \cap E(C_j)| = 0$, where $1 \leq j \neq i \leq \ell$. Furthermore, we have $|EM(c) \cap E(C_i)| = 0$. 

Figure 7. The conical graph $C(3, 8)$
Proof:
For any vertices $u_s^i, u_t^i \in V(C_i)$, where $1 \leq s, t \leq k$ and $1 \leq i \leq \ell$, since there exists a path $u_s^i \ldots u_t^i \in \mathcal{G} u_t^i$ from $u_s^i$ to $u_t^i$, it follows that $d_G(u_s^i, u_t^i) \leq 2i$. It is easy to see that there are two possible types of shortest paths $P_{u_s^i u_t^i}$ from $u_s^i$ to $u_t^i$:

**Type 1:** If $d_G(u_s^i, u_t^i) \geq 2i$, then $P_{u_s^i u_t^i} = u_s^i \ldots u_s^i u_{i-1}^i$.

**Type 2:** If $d_G(u_s^i, u_t^i) < 2i$, then the shortest path $P_{u_s^i u_t^i} \subseteq C_i$, where $C_i$ is subgraph of $G$.

Therefore, $d_G(u_s^i, u_t^i) = \min\{2i, d_G(u_s^i, u_t^i)\}$. Suppose that $1 \leq i \leq a_k$. For $s-2i+1 \leq t \leq s-1$, since $d_G(u_s^i, u_t^i) < 2i$, it follows that $d_G(u_s^i, u_t^i) = \min\{2i, d_G(u_s^i, u_t^i)\} = d_G(u_s^i, u_t^i)$, and hence $d_G(u_s^i, u_t^i) = d_G(u_s^i, u_t^i) = d_G(u_s^i, u_t^i) = d_G(u_s^i, u_t^i) = d_G(u_s^i, u_t^i) = d_G(u_s^i, u_t^i) = d_G(u_s^i, u_t^i)$. Thus, $d_G(u_s^i, u_t^i) = d_G(u_s^i, u_t^i) + 1 = d_G(u_s^i, u_t^i)$, and so $u_s^i u_t^i \in C_i$.

Similarly, for any edge $u_s^i u_t^i \in E(C_i)$, where $s \leq t \leq s+2i-2$, since $d_G(u_s^i, u_t^i) \neq d_G(u_s^i, u_t^i)$, it follows that $u_s^i u_t^i \in E(M(u_s^i))$. Therefore, $\{u_s^i u_t^i \mid s-2i+1 \leq t \leq s+2i-2\} \subseteq E(M(u_s^i))$, where the subscripts are taken modulo $k$, that is, $u_{s+1}^i = u_s^i$.

For any $u_s^i u_t^i \in E(C_i)$, where $s \leq t \leq s+2i-2$, since $d_G(u_s^i, u_t^i) \neq d_G(u_s^i, u_t^i)$, it follows that $u_s^i u_t^i \in E(M(u_s^i))$. Therefore, $\{u_s^i u_t^i \mid s-2i+1 \leq t \leq s+2i-2\} \subseteq E(M(u_s^i))$, where the subscripts are taken modulo $k$.

Suppose that $i \geq a_k + 1$. For $i \geq a_k + 1$ and $1 \leq s, t \leq k$, if $u_s^i, u_t^i \in V(C_i)$, then $d_G(u_s^i, u_t^i) = d_G(u_s^i, u_t^i)$. For any cycle $C_i$ with length $k$, if $k$ is even, then it follows from Lemma 4.2 that $|EM(v) \cap E(C_i)| = k-1$. If $k$ is odd, then it follows from Lemma 4.2 that $|EM(v) \cap E(C_i)| = k-1$, as desired.

For any vertex $u_s^i \in V(C_i)$ and any edge $e = u_s^i u_t^i \in E(C_i)$, where $1 \leq j \neq i \leq \ell$ and $1 \leq m \leq k$, since $d_G(e(u_s^i, u_t^i) = d_G(u_s^i, u_t^i)$ and $d_G(e(u_s^i, u_t^i) = d_G(u_s^i, u_t^i)$, it follows that $e \notin EM(u_s^i)$, and hence $|EM(u_s^i) \cap E(C_i)| = 0$.

For any edge $e_3 = u_s^i u_t^i \in E(C_i)$, we have $d_G(c, u_s^i) = d_G(c, u_t^i) = i$, and hence $d_G(c, u_s^i) = d_G(c, u_t^i) = i$, and so $e \notin EM(u_s^i)$. Therefore, $|EM(u_s^i) \cap E(C_i)| = 0$.

Suppose that $\ell \geq a_k + 1$. Since $e(C_i) = k$, it follows from Fact 4.4 that $|EM(v) \cap E(C_i)| \geq 2$ for $a_k + 1 \leq i \leq \ell$ and $|EM(v) \cap E(C_i)| \geq \lceil k/(4i-2) \rceil$ for $1 \leq i \leq a_k$, and so $\text{dem}(G) \geq \sum_{i=1}^{a_k} \lceil k/(4i-2) \rceil + 2(\ell - a_k)$.

Let $M = \bigcup_{i=1}^{a_k} M_i$, where

\[
M_i = \begin{cases} 
\{u_{i-1}^i \mid 1 \leq j \leq k, j \equiv 1 \mod (4i-2)\}, & \text{if } i \leq a_k; \\
\{u_i^i, u_{i/2-k}^i\}, & \text{if } a_k + 1 \leq i \leq \ell.
\end{cases}
\]
Therefore, \( e \in \bigcup_{x \in M} EM(x) \subseteq \bigcup_{x \in S} EM(x) \) for any edge \( e \in E(C_i) \), where \( 1 \leq i \leq \ell \). It suffices to prove that \( e \in \bigcup_{x \in M} EM(x) \) for each edge in \( E(P_i) \), where \( 1 \leq i \leq k \). For some vertex \( u_1^i \in M_1 \) and any \( u_j^i \in V(G) \), where \( 1 \leq i \neq j \leq k \), if \( j \in \{1, 2, \ldots, i - 3, i + 3, \ldots, k\} \), where the subscripts are taken modulo \( n \), then \( d_G(u_1^i, u_j^i) \neq d_G(u_1^i, u_j^i) \), and hence \( cu_j^i \in EM(u_1^i) \). Similarly, for \( 1 \leq i \neq j \leq k \), \( d_G(u_1^i, u_j^i) \neq d_G(u_1^i, u_j^i) \) for \( j \in \{1, 2, \ldots, i - 3, i + 3, \ldots, k\} \), it follows that \( u_j^i u_j^{i-1} \in EM(u_1^i) \), and hence \( EM(u_1^i) = \{u_1^i u_i^{i+1}, u_1^i u_{i-1}^i, cu_j^i \} \cup \{j \in \{1, 2, \ldots, i - 3, i + 3, \ldots, k\} \} \cup \{u_j^{p-1} u_j^p \mid 2 \leq p \leq \ell - 1, j \in \{1, 2, \ldots, i - 3, i + 3, \ldots, k\} \} \). Without loss of generality, let \( u_1^i \in M \). Then \( \bigcup_{i \in \{1,4,\ldots,k-2\}} E(P_i) \subseteq EM(u_1^i) \). Since \( EM(u_1^i) = \{u_1^i u_i^{i+1}, u_1^i u_{i-1}^i, cu_j^i \} \cup \{j \in \{1, 2, \ldots, i - 3, i + 3, \ldots, k\} \} \cup \{u_j^{p-1} u_j^p \mid 2 \leq p \leq \ell - 1, j \in \{1, 2, \ldots, i - 3, i + 3, \ldots, k\} \} \) and \( k \geq 9 \), it follows that \( E(P_2) \subseteq EM(u_{k-3}^1) \), \( E(P_3) \subseteq EM(u_{k-3}^1) \), \( E(P_2) \subseteq EM(u_{k-2}^1) \), \( E(P_3) \subseteq EM(u_{k-2}^1) \) and \( E(P_{k-1}) \subseteq EM(u_k^1) \).

For any edge \( e \in E(P_i) \), where \( i \in \{2, 3, k, k - 1\} \), if \( k \) is even, then \( e \in EM(u_k^1) \cup EM(u_{k-3}^1) \); if \( k \) is odd, then \( e \in EM(u_k^1) \cup EM(u_{k-2}^1) \). Therefore, \( e \in \bigcup_{x \in S} EM(x) \) for any \( e \in E(P_i) \), where \( 1 \leq i \leq k \), and so \( \text{dem}(G) \leq \sum_{i=1}^{a_k} \lfloor k/(4i - 2) \rfloor + 2(\ell - a_k) \). Thus, \( \text{dem}(G) = \sum_{i=1}^{a_k} \lfloor k/(4i - 2) \rfloor + 2(\ell - a_k) \). For \( \ell \leq a_k + 1 \), it is similar to the case that \( \ell \geq a_k + 1 \), as desired. □

**Theorem 4.5.** For any positive integer \( k \geq 9 \), there exists a connected graph \( G \) such that such that
\[
\text{dem}(G \setminus v) - \text{dem}(G) = \lfloor k/2 \rfloor - \lfloor k/6 \rfloor,
\]
where \( v \in V(G) \).

**Proof:**

Let \( G = C(2, k) \), where \( \ell = 2 \) and \( k \geq 5 \). Note that \( G \setminus v_0 = C_k \square K_2 \). From Theorem 3.4, we have \( \text{dem}(G \setminus v_0) = \text{dem}(C_k \square K_2) = k \). From Theorem 4.3, we have \( \text{dem}(C(2, k)) = \sum_{i=1}^{2} \lfloor k/(4i - 2) \rfloor = \lfloor k/2 \rfloor + \lfloor k/6 \rfloor \), and hence \( \text{dem}(G - v) - \text{dem}(G) = k - \lfloor k/2 \rfloor - \lfloor k/6 \rfloor = \lfloor k/2 \rfloor - \lfloor k/6 \rfloor \), as desired. □

Let \( G = C(\ell, k) \) and \( H = C_k \square P_\ell \). From Theorems 4.3 and 3.4 if \( \ell \gg k \), then \( \text{dem}(G) / \text{dem}(H) \approx 1 \). From Theorems 4.3 and 3.4 if \( k = 402 \) and \( \ell = 100 \), then \( \text{dem}(G) / \text{dem}(H) \approx 0.561453 \).

**Corollary 4.6.** There exist two connected graphs \( H \) and \( G \) such that
\[
\frac{\text{dem}(G)}{\text{dem}(H)} \approx 0.561453,
\]
where \( H \) is an induced subgraph of \( G \).

**Proof of Theorem 1.17.** Let \( G = \tilde{K}_{2k+2} \). From Proposition 1.16 there exists a vertex \( u \in V(G) \) such that \( \text{dem}(G) - \text{dem}(G \setminus u) = k \). Note that \( G \setminus u = P_{2k+2} \) is a connected graph. In addition, let \( H = C(2, t) \), \( k = \lceil t/2 \rceil - \lceil t/6 \rceil \) and \( v \in V(H) \). From Theorem 4.5, we have \( \text{dem}(C(2, t) \setminus v) - \text{dem}(C(2, t)) = k \), where \( C(2, t) \setminus v = C_t \square K_2 \) is a connected graph.

In fact, \( G \setminus v \) is a subgraph of \( G \). From Theorem 1.17, for any positive integer \( k \geq 3 \), there exists a graph \( G \) such that \( \text{dem}(G \setminus v) - \text{dem}(G) \geq k \). □
Claim 4.8. Since \( \text{dem}(uv) \geq 1.6 \) that \( uv \) case that \( H_2 \subseteq G \). From Theorem 1.8, we have \( \text{dem}(H_2) = 1 \), and hence \( \text{dem}(H_2) \leq \text{dem}(G) \), and so Corollary 4.7 holds.

Corollary 4.7. There exists a connected graph \( G \) and two non-spanning subgraphs \( H_1, H_2 \subseteq G \) such that \( \text{dem}(H_1) \geq \text{dem}(G) \) and \( \text{dem}(H_2) \leq \text{dem}(G) \).

Proof of Proposition 4.18: For any graph \( G \) with order \( n \) and \( G \setminus v \) with at least one edge, we have \( \text{dem}(G) \leq n - 1 \) and \( \text{dem}(G \setminus v) \geq 1 \), and hence \( \text{dem}(G) - \text{dem}(G \setminus v) \leq n - 2 \). Furthermore, let \( G = K_3 \), then \( \text{dem}(G) - \text{dem}(G \setminus v) = n - 2 \), and hence the upper bound is sharp. Conversely, since \( \text{dem}(G) - \text{dem}(G \setminus v) = n - 2 \), it follows that \( \text{dem}(G) = n - 1 \) and \( \text{dem}(G \setminus v) = 1 \). From Theorem 1.6, \( G \setminus v \) is a tree. Suppose that \( |V(G)| \geq 4 \). Since \( \text{dem}(G) = n - 1 \), it follows from Theorem 1.6 that \( G = K_n \), and hence \( G \setminus v = K_{n-1} \) which contradicts to the fact that \( G \setminus v \) is a tree. Suppose that \( |V(G)| \leq 3 \). Since \( \text{dem}(G) = n - 1 \), it follows that \( G = K_n \), where \( n \leq 3 \). If \( G = K_2 \), then \( G \setminus v = K_1 \), which contradicts to the fact that \( G \setminus v \) contains at least one edge. Therefore, \( G = K_3 \), as desired.

Next, we consider the subgraph \( H \) of \( G \). If \( H \) is a proper subgraph of \( G \) satisfying \( \text{dem}(H) \leq \text{dem}(G) \), then what is the relation between \( H \) and \( G \)? A natural question is what is the maximum number of edges we can delete from \( G \) without changing the number of distance-edge monitoring? We give a partial answer as follows.

Recall that the base graph \( G_b \) is a subgraph of \( G \) with \( \text{dem}(G) = \text{dem}(G_b) \). Therefore, we can give a lower bound for the edge set \( E \) such that \( \text{dem}(G) = \text{dem}(G - E) \).

Observation 4.1. Let \( G \) be a connected graph, and let \( E_1 = E(G) - E(G_b) \). For \( E \subseteq E(G) \), if \( \text{dem}(G) = \text{dem}(G - E) \) and \( G - E \) is a connected graph with order at least 2, then \( |E| \geq |E_1| \).

Proof of Theorem 4.19. Let \( E \subseteq E(G) \) satisfying \( \text{dem}(G) = \text{dem}(G - E) \) and \( M = \{ u, v \} \) be a DEM set of \( G \) with \( |M| = \text{dem}(G) = 2 \). From Theorem 1.8, we have \( |EM(u)| \leq n - 1 \) and \( |EM(v)| \leq n - 1 \). If \( uv \in E(G) \), then \( e(G) \leq 2(n - 1) - 1. \) Since \( \text{dem}(G) = \text{dem}(G - E) = 2 \), it follows from Theorem 1.6 that \( G - E \) must contain a cycle, and hence \( |E| \leq 2(n - 1) - 1 - 3 = 2n - 6 \).

Suppose that \( uv \notin E(G) \). If \( |EM(u) \cap EM(v)| \geq 1 \), then \( |E| \leq 2n - 6 \), which is similar to the case that \( uv \in E(G) \). If \( EM(u) \cap EM(v) = \emptyset \), it follows from Theorem 1.8 that \( e(G) \leq 2(n - 1) \). Since \( \text{dem}(G - E) = 2 \), it then follows from Theorem 1.6 that \( G - E \) must contain a cycle, and hence \( |E| \leq 2(n - 1) - 3 = 2n - 5 \). Furthermore, we give the following claim.

Claim 4.8. \( |E| \leq 2n - 6 \).

Proof: Assume, to the contrary, that \( |E| = 2n - 5 \). Since \( \text{dem}(G - E) = 2 \), it follows from Theorem 1.6 that \( G - E = C_3 \). Without loss of generality, let \( V(G - E) = \{ v_1, v_2, v_3 \} \). In addition, from Theorem 1.8, the subgraph induced by the edge set \( EM(u) \) and \( EM(v) \) are the spanning trees of \( G \). If \( u, v \in \{ v_1, v_2, v_3 \} \), then \( uv \in E(G) \), a contradiction. Thus, \( u \notin \{ v_1, v_2, v_3 \} \) or \( v \notin \{ v_1, v_2, v_3 \} \). Without loss generality, suppose that \( u \notin \{ v_1, v_2, v_3 \} \).
If \( d_G(u) = 1 \), then \(|N(u)| = 1\). Let \( N(u) = \{w\} \). Since the subgraph induced by the edge set \( EM(u) \) and \( EM(v) \) are the spanning trees of \( G \), it follows that \( uv \in EM(u) \cap EM(v) \), which contradicts to the fact that \( EM(u) \cap EM(v) = \emptyset \). Therefore, \( d_G(u) \geq 2 \). Since the subgraph induced by the edge set \( EM(v) \) is a spanning tree of \( G \), it follows that there exists a vertex \( u_1 \in N(u) \) such that \( uu_1 \in EM(v) \). From Theorem 1.4 we have \( uu_1 \in EM(u) \), and hence \( uu_1 \in EM(u) \cap EM(v) \), which contradicts to the fact that \( EM(u) \cap EM(v) = \emptyset \). 

From Claim 4.8 we have \(|E| \leq 2n - 6\). Furthermore, let \( G = (n - 2)K_1 \cup K_2 \) with vertex set \( V(G) = \{v_i \mid 1 \leq i \leq n\} \) and edge set \( E(G) = \{v_1v_2\} \cup \{v_1v_i, v_2v_i \mid 3 \leq i \leq n\} \). Then, \( \text{dem}(G) = 2 \). Let \( E = \{v_1v_i, v_2v_i \mid 4 \leq i \leq n\} \subseteq E(G) \). From Observation 3.1 \( \text{dem}(G - E) = \text{dem}(K_3) + (n - 1) \text{dem}(K_1) = 2 \), and hence there exists an edge set \( E_1 \) such that \( \text{dem}(G - E) = 2 \) and \(|E| = 2n - 6\), as desired.

In the end of this section, we give the proof of Theorem 1.22 as follows.

**Proof of Theorem 1.22.** Let \( G = K_n \) with vertex set \( \{v_i \mid 1 \leq i \leq n\} \) and edge set \( \{v_i v_j \mid 1 \leq i < j \leq n\} \). Let \( T \) be a spanning tree in \( K_n \). For any edge \( uv \in E(T) \) and vertex \( w \in ((N_G(u) \cup N_G(v)) \setminus \{u, v\}) \cap V(T) \), we have \( d_G(w, u) = d_G(w, v) = 1 \) and \( d_G - uv(w, u) = d_G - uv(w, v) = 1 \), and hence \( uv \notin EM(w) \). From Proposition 3.1 any edge \( uv \in E(T) \) is only monitored by \( u \) or \( v \), and hence \( \text{dem}(K_n | T) \geq \beta(T) \). From Theorem 1.9 \( \text{dem}(K_n | T) \leq \beta(T) \), and hence \( \text{dem}(K_n | T) = \beta(T) \). Since \( T \) is tree with order \( n \), it follows that \( T \) is a bipartite graph. Without loss of generality, let \( V(T) = U \cup V \ (|U| \leq |V|) \), which is a bipartite partition of \( V(T) \). From the pigeonhole principle, we have \(|U| \leq \lfloor \frac{n}{2} \rfloor \). For any \( uv \in E(T) \), we have \( \{u, v\} \cap U = \emptyset \), and hence \( \beta(T) \leq \lfloor \frac{n}{2} \rfloor \). In addition, \( T \) contains at least one edge, and hence \( \beta(T) \geq 1 \), and so \( 1 \leq \beta(T) \leq \lfloor \frac{n}{2} \rfloor \).

Suppose that \( T = S_n \) with vertex set \( \{v_i \mid 1 \leq i \leq n\} \) and edge set \( E(S_n) = \{v_i v_j \mid 2 \leq i \leq n\} \). Then \( \{v_i\} \) is the vertex cover set of \( S_n \), and hence \( \beta(T) = 1 \), and so the lower bound is sharp. Suppose that \( T = P_n \) with vertex set \( \{v_i \mid 1 \leq i \leq n\} \) and edge set \( E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n - 1\} \). Then, \( \{v_i \mid i \equiv 0 \pmod{2}, 1 \leq i \leq n\} \) is a minimum vertex cover set of \( P_n \), and hence \( \beta(T) = \lfloor \frac{n}{2} \rfloor \), and so the upper bound is sharp.

Similar to Theorem 1.22 we have the following corollary.

**Corollary 4.9.** Let \( H \) be a subgraph of \( G \) with \(|V(H)| = p\). Then,

\[
1 \leq \text{dem}(G|_H) \leq p - 1.
\]

Furthermore, the bounds are sharp.

**Theorem 4.10.** If \( H \) is a connected induced subgraph of graph \( G \), then

\[
\text{dem}(G) - \text{dem}(G|_H) \leq |V(G)| - |V(H)|.
\]

Moreover, if \( G \) and \( H \) are both complete graphs, then the bound is sharp.
Proof:
For any graph $G$ and $H$, where $H$ is an induced subgraph of $G$. Let $M_1 \subseteq V(H)$ be a restrict-DEM set of $H$ in $G$ with $|M_1| = \text{dem}(G|_H)$. Let $M = (V(G) - V(H)) \cup M_1$. We will prove that $\text{dem}(G) \leq |M|$. For any edge $uv \in E(G)$, if $u$ or $v$ in $V(G) - V(H)$, then it follows from Theorem 1.4 that $e$ is monitored by the vertex in $V(G) \setminus V(H)$. For any edge $e \in E(H)$, since $M_1$ is a restrict-DEM set, it follows that $e$ is monitored by the vertex in $M_1$, and hence $M$ is a DEM set in $G$. Since $|M| = |M_1| + (|V(G)| - |V(H)|) = |V(G)| - |V(H)| + \text{dem}(G|_H)$, it follows that $\text{dem}(G) \leq |M| = \text{dem}(G|_H) + (|V(G)| - |V(H)|)$, and so $\text{dem}(G) = \text{dem}(G|_H) \leq (|V(G)| - |V(H)|)$. Furthermore, let $G = K_n$ and $H = K_m (3 \leq m \leq n)$. From Theorem 1.7, $\text{dem}(G) = n - 1$. For any $uv \in E(H)$ and $w \in (N(u) \cup N(v)) \setminus \{u, v\}$, we have $d_G(w, u) = d_G(w, v) = 1$ and $d_{G-uv}(w, u) = d_{G-uv}(w, v) = 1$, and hence $uv \notin EM(w)$. From Proposition 3.1, any edge $uv \in E(H)$ is only monitored by $u$ or $v$, and so $\text{dem}(K_n|_{K_m}) \geq \beta(K_m) = m - 1$. From Theorem 1.9, $\text{dem}(K_n|_{K_m}) \leq \beta(K_m) = m - 1$, and hence $\text{dem}(K_n|_{K_m}) = m - 1$. Therefore, $\text{dem}(G) - \text{dem}(G|_H) = (n - 1) - (m - 1) = |V(G)| - |V(H)|$, as desired. \qed

5. Perturbation results for some known graphs

Firstly, we study the change of DEM numbers for some well-known graphs when any edge (or vertex) of the graph is deleted.

5.1. Deleting one edge or vertex from some known graphs

Let $NV_i(G) = \{v \mid d_G(v) = i, v \in V(G)\}$ and $NE_{a,b}(G) = \{uv \mid uv \in E(G), d_G(u) = a, d_G(v) = b\}$. Note that $E(P_n) = NE_{1,2}(P_n) \cup NE_{2,2}(P_n)$. If $e \in NE_{1,2}(P_n)$, then $\text{dem}(P_n - e) = 1$, and hence $\text{dem}(P_n) - \text{dem}(P_n - e) = 0$. If $e \in NE_{2,2}(P_n)$, then $\text{dem}(P_n - e) = 2$, and hence $\text{dem}(P_n) - \text{dem}(P_n - e) = -1$.

**Corollary 5.1.** Let $P_n$ be a path of order $n$, where $n \geq 2$. For any $e \in E(P_n)$, we have

$$
\text{dem}(P_n - e) = \begin{cases} 
\text{dem}(P_n), & \text{if } e \in NE_{1,2}(P_n); \\
\text{dem}(P_n) + 1, & \text{if } e \in NE_{2,2}(P_n).
\end{cases}
$$

Foucaud et al. [10] obtained the DEM numbers of complete bipartite graph $K_{\ell_1,\ell_2}$.

**Theorem 5.2.** [10] Let $\ell_1$ and $\ell_2$ be two integers with $\ell_1 \geq 1$ and $\ell_2 \geq 1$. Then

$$
\text{dem}(K_{\ell_1,\ell_2}) = \min\{\ell_1, \ell_2\}.
$$

The following corollary is immediate.

**Corollary 5.3.** Let $n \geq 3$ be an integer. Then,

(i) for any edge $e \in E(C_n)$, $\text{dem}(C_n - e) = \text{dem}(C_n) - 1 = 1$;
(ii) for any edge $e \in E(K_n)$, $\text{dem}(K_n - e) = \text{dem}(K_n) - 1 = n - 2$;
(iii) for any edge $e \in E(K_{n,n})$, $\text{dem}(K_{n,n} - e) = \text{dem}(K_{n,n}) = n$. 

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Proof:

(i) From Theorem 1.6 we have $\text{dem}(C_n - e) = \text{dem}(P_n) = 1$. Since $\text{dem}(C_n) = 2$, it follows that $\text{dem}(C_n - e) = \text{dem}(C_n) - 1 = 1$.

(ii) Let $G = K_n$. From Theorem 1.7 we have $\text{dem}(G) = n - 1$ and $\text{dem}(G - uv) \leq n - 2$ for any edge $uv \in E(G)$. Then, we prove that $\text{dem}(G - uv) \geq n - 2$. Suppose that $\text{dem}(G - uv) \leq n - 3$. Let $M$ be a DEM set of $G - uv$ with the minimum cardinality. For any edge $xy \in E(G - uv)$ and vertex $w \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}$, we have $d_G(w, x) = d_{G - xy}(w, x)$ and $d_G(w, y) = d_{G - xy}(w, y)$, and hence $xy \notin EM(w)$. From Proposition 3.1, any edge $xy \in E(G - uv)$ is only monitored by $x$ and $y$, and hence $M \cap \{x, y\} \neq \emptyset$. If $u, v \in M$, then there exist two vertices $v_1, v_2 \in V(G) \setminus \{u, v\}$ such that $v_1, v_2 \notin M$, and hence $v_1v_2 \notin \cup_{x \in M} EM(x)$, which contradicts to the fact that $\cup_{x \in M} EM(x) = E(G - uv)$. Suppose that $u$ or $v \notin M$. Without loss of generality, let $u \notin M$. Then there exists a vertex $v_1 \in V(G) \setminus \{u, v\}$ such that $v_1 \notin M$, and hence $uv_1 \notin \cup_{x \in M} EM(x)$, which contradicts to the fact that $\cup_{x \in M} EM(x) = E(G - uv)$. Therefore, $\text{dem}(G - uv) \leq n - 2$, and so $\text{dem}(G - uv) = n - 2$. This implies that $\text{dem}(K_n - e) = \text{dem}(K_n) - 1 = n - 2$.

(iii) Let $G = K_{n,n}$ with vertex set $V(G) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq n\}$ and edge set $E(G) = \{uv_i | 1 \leq i, j \leq n\}$. From Theorem 5.2, we have $\text{dem}(K_{n,n}) = n$. Without loss of generality, let $G_1 = G - u_1v_1$. Firstly, we prove that $\text{dem}(G_1) \geq n$. Suppose that $M$ is a DEM set of $G_1$ with $|M| = n - 1$. Then, there exists two vertices $u_p, v_q \notin M$ with $u_pv_q \in E(G_1)$. For any $u \in M \cap \{u_i | 1 \leq i \leq n\}$, we have $d_{G_1 - u_pv_q}(u, u_p) = d_{G_1}(u, u_p) = 2$ and $d_{G_1 - u_pv_q}(u, v_q) = d_{G_1}(u, v_q) = 1$, and hence $u_pv_q \notin EM(u)$. Similarly, $u_pv_q \notin EM(v)$ for any $v \in M \cap \{u_i | 1 \leq i \leq n\}$, and hence $u_pv_q \notin \cup_{x \in M} EM(x)$, which contradicts to the fact that $\cup_{x \in M} EM(x) = E(G_1)$. Therefore, $\text{dem}(G_1) \geq n$. Then, we prove that $\text{dem}(G_1) \leq |M| = n$. Let $M = \{u_i | 1 \leq i \leq n\}$. Then $M$ is a vertex cover set of $G_1$. From Theorem 1.9, $\text{dem}(G_1) \leq |M| = n$, and hence $\text{dem}(G_1) = n$. Therefore, $\text{dem}(K_{n,n} - e) = \text{dem}(K_{n,n}) = n$. □

There exist three graphs $G_1, G_2, G_3$ and $v \in V(G)$ such that $\text{dem}(G_1) > \text{dem}(G_1 - v), \text{dem}(G_2) = \text{dem}(G_2 - v), \text{dem}(G_3) < \text{dem}(G_3 - v)$, respectively.

The following corollary is immediate.

Corollary 5.4. Let $n \geq 3$ be an integer. For any $v \in v(G)$, we have

(i) $\text{dem}(C_n \setminus v) = \text{dem}(C_n) - 1 = 1$;
(ii) $\text{dem}(K_n \setminus v) = \text{dem}(K_n) - 1 = n - 2$;
(iii) $\text{dem}(K_{n,n} \setminus v) = \text{dem}(K_{n,n}) - 1 = n - 1$.

Proposition 5.5. Let $P_n$ be a path with vertex set $\{v_i | 1 \leq i \leq n\}$ and edge set $\{v_iv_{i+1} | 1 \leq i \leq n - 1\}$, where $n \geq 5$. For any $v \in V(P_n)$, we have

$$\text{dem}(P_n \setminus v) = \begin{cases} \text{dem}(P_n), & \text{if } v \in \{v_1, v_2, v_{n-1}, v_n\}; \\
\text{dem}(P_n) + 1, & \text{if } v \in \{v_i | 3 \leq i \leq n - 2\}. \end{cases}$$
Proof:
From Theorem 1.6, we have dem\((P_n) = 1\). If \(v \in \{v_1, v_n\}\), then dem\((P_n \setminus v) = 1\). If \(v \in \{v_2, v_{n-1}\}\), then dem\((P_n \setminus v) = dem(P_{n-1}) + dem(K_1) = 1\), where \(V(K_1) = \{v_1\}\) or \(V(K_1) = \{v_n\}\), and hence dem\((P_n) - dem(P_n \setminus v) = 0\). If \(v \in \{v_i | 3 \leq i \leq n - 2\}\), then dem\((P_n \setminus v) = 2\), and hence dem\((P_n) - dem(P_n \setminus v) = -1\). \(\Box\)

Proposition 5.5 shows that there exists a graph \(G\) and \(v \in V(G)\) such that dem\((G) = dem(G \setminus v)\) or dem\((G) < dem(G \setminus v)\).

5.2. Whether the DEM set is still applicable?

Foucaud et al. proved in \([10]\) that the problem DEM SET is \(NP\)-complete. For any graph \(G\), a natural question is whether the original DEM set can monitor all edges if some edges or vertices in \(G\) are deleted. We design the Algorithm 1 and the time complexity is polynomial.

**WHETHER THE DEM SET IS STILL APPLICABLE?**

**Instance:** A graph \(G = (V, E)\), an edge \(e \in E(G)\) and a DEM set \(M\) of \(G\).

**Question:** Whether \(M\) is still a DEM set for the graph \(G - e\)?

Given a graph \(G\), a DEM set \(M\), and an edge \(e \in E(G)\), our goal is to determine whether the original DEM set \(M\) is still valid in the resulting graph \(G - e\). The algorithm is shown in Algorithm 1.

**Algorithm 1** The algorithm for determining \(M\) is or not monitor set for \(G - e\)

**Input:** A graph \(G\), \(M \subseteq V(G)\) and \(e \in E(G)\);

**Output:** \(E(G - e) \subseteq \cup_{x \in M} EM(x)\) is TRUE or FALSE;

1: \(M_1 \leftarrow E(G - e)\)

2: for each vertex \(v \in M\) do

3: \(M_1 \leftarrow M_1 - EM(v)\)

4: end for

5: if \(M_1 = \emptyset\) then return \(E(G - e) \subseteq \cup_{x \in M} EM(x)\) is TRUE;

6: else return \(E(G - e) \subseteq \cup_{x \in M} EM(x)\) is FALSE;

7: end if

The algorithm of how to compute the edge set \(EM(x)\) from \(G\) is polynomial by the breadth-first spanning tree algorithm. Hence the time complexity of Algorithm 1 is polynomial.

6. Conclusion

In this paper, we studied the effect of deleting edges and vertices in a graph \(G\) on the DEM number. We obtained that dem\((G - e) - dem(G) \leq 2\) for any graph \(G\) and \(e \in E(G)\). Furthermore, the bound is sharp. In addition, we can find a graph \(H\) and \(v \in V(H)\) such that dem\((H \setminus v) - dem(H)\) can be
arbitrarily large. This fact gives an answer to the monotonicity of the DEM number. This means that there exist two graphs \( H \) and \( G \) with \( H \sqsubseteq G \) such that \( \text{dem}(H) \geq \text{dem}(G) \).

It is interesting to consider the following problems for future work.

1. Characterize the graphs \( \text{dem}(H) \geq \text{dem}(G) \) if \( H \sqsubseteq G \).
2. For a graph \( G \) and \( E \subseteq E(G) \), what is the maximum value of \(|E|\) such that \( \text{dem}(G) = \text{dem}(G - E) \)?
3. For any \( \epsilon > 0 \), whether the ratio \( \frac{\text{dem}(G)}{\text{dem}(H)} \leq \epsilon \) holds, where \( H \) is an induced subgraph of \( G \).

In addition, it would be interesting to study distance-edge monitoring sets in further standard graph classes, including circulant graphs, graph products, or line graphs. In addition, characterizing the graphs with \( \text{dem}(G) = n - 2 \) would be of interest, as well as clarifying further the relation of the parameter \( \text{dem}(G) \) to other standard graph parameters, such as arboricity, vertex cover number and feedback edge set number.

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**References**


