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Perturbation Results for Distance-edge-monitoring Numbers*

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Abstract. Foucaud *et al.* recently introduced and initiated the study of a new graph-theoretic concept in the area of network monitoring. Given a graph $G = (V(G), E(G))$, a set $M \subseteq V(G)$ is a *distance-edge-monitoring set* if for every edge $e \in E(G)$, there is a vertex $x \in M$ and a vertex $y \in V(G)$ such that the edge e belongs to all shortest paths between x and y. The smallest size of such a set in G is denoted by $\operatorname{dem}(G)$. Denoted by $G - e$ (resp. $G \backslash u$) the subgraph of G obtained by removing the edge e from G (resp. a vertex u together with all its incident edges from G). In this paper, we first show that $\operatorname{dem}(G - e) - \operatorname{dem}(G) \leq 2$ for any graph G and edge $e \in E(G)$. Moreover, the bound is sharp. Next, we construct two graphs G and H to show that $\operatorname{dem}(G) - \operatorname{dem}(G \setminus u)$ and $\operatorname{dem}(H \setminus v) - \operatorname{dem}(H)$ can be arbitrarily large, where $u \in V(G)$ and $v \in V(H)$. We also study the relation between $\operatorname{dem}(H)$ and $\operatorname{dem}(G)$, where H is a subgraph of G. In the end, we give an algorithm to judge whether the distance-edge-monitoring set still remain in the resulting graph when any edge of a graph G is deleted.

Keywords: Distance; Perturbation result; Distance-edge-monitoring set.

AMS subject classification 2020: 05C12; 11J83; 35A30; 51K05.

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1. Introduction

In 2022, Foucaud *et al.* [10] introduced a new graph-theoretic concept called *distance-edge-monitoring set* (DEM for short), which means network monitoring using distance probes. Networks are naturally modeled by finite undirected simple connected graphs, whose vertices represent computers and whose edges represent connections between them. When a connection (an edge) fails in the network, we can detect this failure, and thus achieve the purpose of monitoring the network. Probes are made up of vertices we choose in the network. At any given moment, a probe of the network can measure its graph distance to any other vertex of the network. Whenever an edge of the network fails, one of the measured distances changes, so the probes are able to detect the failure of any edge. Probes that measure distances in graphs are present in real-life networks. They are useful in the fundamental task of routing [7, 11] and are also frequently used for problems concerning network verification [1, 3, 4].

In a network, we can put as few detectors as possible to monitor all the edges, a natural question is whether the detectors placed in the original graph are still sufficient and need to be supplemented or reduced when some nodes or edges in the original graph are subjected to external interference and damage, we refer to [8, 9, 14, 15, 18]. This kind of problem is usually called perturbation problem.

Graphs considered are finite, undirected and simple. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, whose cardinality are denoted by $|V(G)|$ and $e(G)$, respectively. The *neighborhood set* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \,|\, uv \in E(G)\}$. Let $N_G[v] = N_G(v) \cup \{v\}$ be the *closed neighborhood set of a vertex* v. The *degree* of a vertex v in G is denoted $d(v) = |N_G(v)|$. Let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degree of a graph G, respectively. For any subset X of $V(G)$, let $G[X]$ denote the subgraph of G induced by X; similarly, for any subset F of $E(G)$, let $G[F]$ denote the subgraph induced by F. We use $G \setminus X$ to denote the subgraph of G obtained by removing all the vertices of X together with the edges incident with them from G; similarly, we use $G - F$ to denote the subgraph of G obtained by removing all the edges of F from G. If $X = \{v\}$ and $F = \{e\}$, we simply write $G \setminus v$ and $G - e$ for $G - \{v\}$ and $G - \{e\}$, respectively. For an edge e of G, we denote by $G + e$ the graph obtained by adding an edge $e \in E(\overline{G})$ to G. The *Cartesian product* $G \square H$ of two graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of pairs $(u, v)(u', v')$ such that either $uu' \in E(G)$ and $v = v'$, or $vv' \in E(H)$ and $u = u'$. Let $G \vee H$ be a *join* graph of G and H with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = \{uv \,|\, u \in V(G),\, v \in V(H)\} \cup E(G) \cup E(H)$. We denote by $d_G(x, y)$ the *distance* between two vertices x and y in graph G. For an edge uv and a vertex $w \in V(G)$, the distance between them is defined as $d_G(uv, w) = \min\{d_G(u, w), d_G(v, w)\}$. A x-y path with length $d_G(x, y)$ in G is a *x-y geodesic*. Let P_n, C_n and K_n be the path, cycle and complete graph of order n, respectively.

1.1. DEM sets and numbers

Foucaud et al. [10] introduced a new graph-theoretic concept called DEM sets, which is relevant to network monitoring.

Definition 1.1. For a set M of vertices and an edge e of a graph G, let $P(M, e)$ be the set of pairs (x, y) with a vertex x of M and a vertex y of $V(G)$ such that $d_G(x, y) \neq d_{G-e}(x, y)$. In other words, e belongs to all shortest paths between x and y in G.

Definition 1.2. For a vertex x, let $EM(x)$ be the set of edges e such that there exists a vertex v in G with $(x, v) \in P(\{x\}, e)$, that is $EM(x) = \{e \,|\, e \in E(G)$ and $\exists v \in V(G)$ such that $d_G(x, v) \neq d_{G-e}(x, v)\}$ or $EM(x) = \{e \,|\, e \in E(G)$and $P(\{x\}, e) \neq \emptyset\}$. If $e \in EM(x)$, we say that e is monitored by x.

Finding a particular vertex set M and placing a detector on that set to monitor all edge sets in G have practical applications in sensor and network systems.

Definition 1.3. A vertex set M of the graph G is *distance-edge-monitoring set* (DEM set for short) if every edge e of G is monitored by some vertex of M, that is, the set $P(M, e)$ is nonempty. Equivalently, $\cup_{x \in M} EM(x) = E(G)$.

Theorem 1.4. [10] Let G be a connected graph with a vertex x of G and for any $y \in N(x)$, then, we have $xy \in EM(x)$.

One may wonder to know the existence of such an edge detection set M. The answer is affirmative. If we take $M = V(G)$, then it follows from Theorem 1.4 that

$$E(G) \subseteq \cup_{x \in V(G)} \cup_{y \in N(x)} \{xy\} \subseteq \cup_{x \in V(G)} EM(x).$$

Therefore, we consider the smallest cardinality of M and give the following parameter.

Definition 1.5. The *distance-edge-monitoring number* (DEM number for short) $\text{dem}(G)$ of a graph G is defined as the smallest size of a distance-edge-monitoring set of G, that is

$$\text{dem}(G) = \min \left\{ |M| \,|\, \cup_{x \in M} EM(x) = E(G) \right\}.$$

Furthermore, for any DEM set M of G, M is called a ***DEM basis*** if $|M| = \text{dem}(G)$.

The vertices of M represent distance probes in a network modeled by G. The DEM sets are very effective in network fault tolerance testing. For example, a DEM set can detect a failing edge, and it can correctly locate the failing edge by distance from x to y, because the distance from x to y will increases when the edge e fails.

Foucaud et al. [10] showed that $1 \leq \text{dem}(G) \leq n - 1$ for any G with order n, and graphs with $\text{dem}(G) = 1, n - 1$ was characterized in [10].

Theorem 1.6. [10] Let G be a connected graph with at least one edge. Then $\text{dem}(G) = 1$ if and only if G is a tree.

Theorem 1.7. [10] $\text{dem}(G) = n - 1$ if and only if G is the complete graph of order n.

Theorem 1.8. [10] For a vertex x of a graph G, the set of edges $EM(x)$ induces a forest.

In a graph G, the *base graph* G_b of a graph G is the graph obtained from G by iteratively removing vertices of degree 1.

Observation 1.1. [10] Let G be a graph and G_b be its base graph. Then we have $\mathrm{dem}(G) = \mathrm{dem}(G_b)$.

A vertex set M is called a *vertex cover* of G if $M \cap \{u, v\} \neq \emptyset$ for $uv \in E(G)$. The minimum cardinality of a vertex cover M in G is the *vertex covering number* of G, denoted by $\beta(G)$.

Theorem 1.9. [10] In any graph G of order n, any vertex cover of G is a DEM set of G, and thus $\mathrm{dem}(G) \leq \beta(G)$.

Ji et al. [12] studied the Erdős-Gallai-type problems for distance-edge-monitoring numbers. Yang et al. [16] obtained some upper and lower bounds of $P(M, e)$, $EM(x)$, $\mathrm{dem}(G)$, respectively, and characterized the graphs with $\mathrm{dem}(G) = 3$, and gave some properties of the graph G with $\mathrm{dem}(G) = n - 2$. Yang et al. [17] determined the exact value of distance-edge-monitoring numbers of grid-based pyramids, $M(t)$-graphs and Sierpiński-type graphs.

1.2. Progress and our results

Perturbation problems in graph theory are as follows.

Problem 1. Let G be a graph, and let $e \in E(G)$ and $v \in V(G)$. Let $f(G)$ be a graph parameter.
(1) The relation between $f(G)$ and $f(G - e)$;
(2) The relation between $f(G)$ and $f(G \setminus v)$.

Chartrand et al. [6] studied the perturbation problems on the metric dimension. Monson et al. [14] studied the effects of vertex deletion and edge deletion on the clique partition number in 1996. In 2015, Eroh et al. [9] considered the effect of vertex or edge deletion on the metric dimension of graphs. Wei et al. [15] gave some results on the edge metric dimension of graphs. Delen et al. [8] study the effect of vertex and edge deletion on the independence number of graphs.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \sqsubseteq G$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G. If H is a subgraph of a graph G, where $H \neq G$, then H is a *proper subgraph* of G. Therefore, if H is a proper subgraph of G, then either $V(H) \subset V(G)$ or $E(H) \subset E(G)$.

We first consider the existence of graphs with given values of DEM numbers.

Problem 2. Let r, s, n be three integers with $1 \leq r, s \leq n - 1$.
(1) Is there a connected graph G of order n such that $\mathrm{dem}(G) = r$?
(2) Let G be a connected graph of order n. Is there a connected subgraph H in G such that $\mathrm{dem}(H) = s$ and $\mathrm{dem}(G) = r$?

In Section 2, we give the answers to Problem 2.

Proposition 1.10. For any two integers r, n with $1 \leq r \leq n - 1$, there exists a connected graph G of order n such that $\mathrm{dem}(G) = r$.

Corollary 1.11. Given three integers s, t, n with $1 \leq s \leq t \leq n-1$, there exists a connected graph $H \sqsubseteq G$ such that $\operatorname{dem}(H) = s$ and $\operatorname{dem}(G) = t$.

In Section 3, we focus on Problem 1 (1) and study the difference between $\operatorname{dem}(G-e)$ and $\operatorname{dem}(G)$.

Theorem 1.12. Let G be a graph. For any edge $e \in E(G)$, we have

$$\operatorname{dem}(G-e) - \operatorname{dem}(G) \leq 2.$$

Moreover, this bound is sharp.

Let G be a graph and $E \subseteq E(\overline{G})$. Denote by $G+E$ the graph with $V(G+E) = V(G)$ and $E(G+E) = E(G) \cup E$. We construct graphs with the following properties in Section 3.

Theorem 1.13. For any positive integer $k \geq 2$, there exists a graph sequence $\{G^i \mid 0 \leq i \leq k\}$, with $e(G^i) - e(G^0) = i$ and $V(G^i) = V(G^j)$ for $0 \leq i, j \leq k$, such that $\operatorname{dem}(G^{i+1}) - \operatorname{dem}(G^0) = i$, where $1 \leq i \leq k-1$. Furthermore, we have $\operatorname{dem}(G^0) = 1$, $\operatorname{dem}(G^1) = 2$ and $\operatorname{dem}(G^i) = i$, where $2 \leq i \leq k$.

A *feedback edge set* of a graph G is a set of edges such that removing them from G leaves a forest. The smallest size of a feedback edge set of G is denoted by $\operatorname{fes}(G)$ (it is sometimes called the cyclomatic number of G).

Theorem 1.14. [10] If $\operatorname{fes}(G) \leq 2$, then $\operatorname{dem}(G) \leq \operatorname{fes}(G) + 1$. Moreover, if $\operatorname{fes}(G) \leq 1$, then equality holds.

Theorem 1.14 implies the following corollary, and its proof will be given in Section 3.

Corollary 1.15. Let T_n be a tree of order n, where $n \geq 6$. For edges $e_1, e_2 \in E(\overline{T_n})$, we have
(1) $\operatorname{dem}(T_n + e_1) = \operatorname{dem}(T_n) + 1$.
(2) $\operatorname{dem}(T_n + \{e_1, e_2\}) = 2$ or 3.

The following result shows that there exists a graph G and an induced subgraph H such that the difference $\operatorname{dem}(G) - \operatorname{dem}(H)$ can be arbitrarily large; see Section 4 for proof details. In addition, we also give an answer to the Problem 1 (2).

Theorem 1.16. For any positive integer k, there exist two graphs G_1, G_2 and their non-spanning subgraphs H_1, H_2 such that

$$\operatorname{dem}(G_1) - \operatorname{dem}(H_1) = k \textit{ and } \operatorname{dem}(H_2) - \operatorname{dem}(G_2) = k.$$

Furthermore, $\operatorname{dem}(G) - \operatorname{dem}(H)$ can be arbitrarily large, even for $H = G \setminus v$.

Theorem 1.17. For any positive integer k, there exist two graphs G, H and two vertices $u \in V(G)$, $v \in V(H)$ such that
(1) $\operatorname{dem}(G) - \operatorname{dem}(G \setminus u) \geq k$;
(2) $\operatorname{dem}(H \setminus v) - \operatorname{dem}(H) \geq k$.

For a connected graph G of order n, where n is fixed, the difference between $\operatorname{dem}(G)$ and $\operatorname{dem}(G \setminus v)$ can be bounded.

Proposition 1.18. For a connected graph G with order n $(n \geq 2)$ and $v \in V(G)$, if $G \setminus v$ contains at least one edge, then $\operatorname{dem}(G) - \operatorname{dem}(G \setminus v) \leq n - 2$. Moreover, the equality holds if and only if G is K_3.

Theorem 1.19. Let G be a connected graph with order $n \geq 4$ and $\operatorname{dem}(G) = 2$. Let $E \subseteq E(G)$. If $\operatorname{dem}(G) = \operatorname{dem}(G - E)$, then $|E| \leq 2n - 6$. Furthermore, the bound is sharp.

For $H \sqsubseteq G$, the *DEM set of H in G* is a set $M \subseteq V(H)$ such that $E(H) \subseteq \bigcup_{x \in M} EM(x)$.

Definition 1.20. For $H \sqsubseteq G$, the *restrict-DEM number* $\operatorname{dem}(G|_H)$ of a graph G is defined as the smallest size of a DEM set of H in G, that is,

$$\operatorname{dem}(G|_H) = \min \left\{ |M| \middle| E(H) \subseteq \cup_{x \in M} EM(x), M \subseteq V(H) \right\}.$$

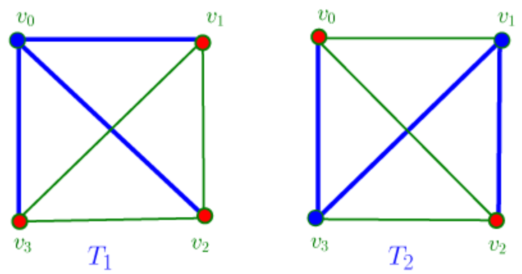

Figure 1. The blue edges are those of trees T_1 and T_2 in K_4.

Example 1.21. Let $G = K_4$ with $V(G) = \{v_0, v_1, v_2, v_3\}$ and $E(G) = \{v_i v_j \mid 0 \leq i < j \leq 3\}$. Let T_1 and T_2 be the subgraphs of G with $E(T_1) = \{v_0v_1, v_0v_2, v_0v_3\}$ and $E(T_2) = \{v_0v_3, v_3v_1, v_1v_2\}$. Then, $\operatorname{dem}(K_4|_{T_1}) = 1$ and $\operatorname{dem}(K_4|_{T_2}) = 2$. The DEM set of subgraph T_i $(i = 1, 2)$ in K_4 is shown in Figure 1, where the blue vertices form the set M. The reason as follows.
Let $M_1 = \{v_0\}$. Since $v_0v_1, v_0v_2, v_0v_3 \in EM(v_0)$, it follows that $\operatorname{dem}(K_4|_{T_1}) \leq 1$. Obviously, $\operatorname{dem}(K_4|_{T_1}) \geq 1$, and hence $\operatorname{dem}(K_4|_{T_1}) = 1$. Then, we prove that $\operatorname{dem}(K_4|_{T_2}) = 2$. Since $d_G(v_0, v_1) = d_{G - v_1v_2}(v_0, v_1) = 1$ and $d_G(v_0, v_2) = d_{G - v_1v_2}(v_0, v_2) = 1$, it follows that $v_1v_2 \notin EM(v_0)$. Similarly, $v_1v_3 \notin EM(v_0)$. Therefore, $v_1v_2, v_1v_3 \notin EM(v_0)$. By a similar argument, we have $v_0v_3 \notin EM(v_1)$, $v_1v_3, v_0v_3 \notin EM(v_2)$ and $v_1v_2 \notin EM(v_3)$, and hence $\operatorname{dem}(K_4|_{T_2}) \geq 2$. Let $M = \{v_1, v_3\}$. Then, $v_1v_2, v_1v_3 \in EM(v_1)$, $v_1v_3, v_0v_3 \in EM(v_3)$, and hence $\operatorname{dem}(K_4|_{T_1}) \leq 2$. Therefore, we have $\operatorname{dem}(K_4|_{T_2}) = 2$, and so $\operatorname{dem}(K_4|_{T_i}) = i$ $(i = 1, 2)$.

Theorem 1.22. Let T be a spanning tree of K_n. Then $1 \leq \operatorname{dem}(K_n|_T) \leq \lfloor n/2 \rfloor$. Furthermore, the bound is sharp.

In Section 5, we focus on the following problem and give an algorithm to judge whether the DEM set is still valid in the resulting graph when any edge (or vertex) of a graph G is deleted.

Problem 3. For any graph G, if some edges or vertices in G is deleted, we want to know whether the original DEM set can monitor all edges.

2. Results for Problem 2

A *kite* $K(r,n)$ is a graph obtained from the complete graph K_{r+1} and a path P_{n-r} by attaching a vertex of K_{r+1} and one end-vertex of P_{n-r}; see an example of $K(7,12)$ in Figure 2.

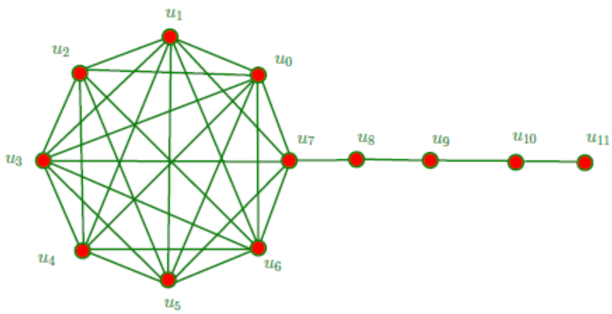

Figure 2. The graph $K(7,12)$

We first give the proof of Proposition 1.10.

Proof of Proposition 1.10: Let $G = K(r,n)$ with $V(G) = \{u_i \,|\, 0 \leq i \leq n-1\}$ and $E(G) = \{u_iu_j \,|\, 0 \leq i < j \leq r\} \cup \{u_{r+s}u_{r+s+1} \,|\, 0 \leq s \leq n-r-2\}$. From Observation 1.1 and Theorem 1.7, we have $\text{dem}(G) = \text{dem}(G_b) = \text{dem}(K_{r+1}) = r$. In fact, for the above G, the path P_{n-r-1} can be replaced by T_{n-r-1}, where T_{n-r-1} is any tree of order $n-r-1$. □

Proposition 1.10 shows that Corollary 1.11 is true. For three integers s,t,n with $1 \leq s \leq t \leq n-1$, let $G = K(t,n)$ and $H = K(s,n) \sqsubseteq G$. From Proposition 1.10, $\text{dem}(G) = t$ and $\text{dem}(H) = s$. Therefore, there exists a connected graph $H \sqsubseteq G$ such that $\text{dem}(H) = s$ and $\text{dem}(G) = t$.

This gives an answer about Problem 2, see Corollary 1.11. One might guess that if H is a subgraph of G, then $\text{dem}(H) \leq \text{dem}(G)$, however we will show in the next section that there is no monotonicity for the DEM number.

3. The effect of deleted edge

The following observation is immediate.

Observation 3.1. Let $G_1, G_2, ..., G_m$ be the connected components of G. Then

$$\text{dem}(G) = \text{dem}\,(G_1) + \cdots + \text{dem}\,(G_m)\,.$$

Furthermore, we suppose that the DEM number of K_1 is 0.

Proposition 3.1. For any $uv \in E(G)$, $uv \notin EM(w)$ for $w \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$ if and only if uv is only monitored by u and v.

Proof:
Since $w \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$ and $uv \notin EM(w)$, it follows that $d_G(w, u) = d_{G-uv}(w, u)$ and $d_G(w, v) = d_{G-uv}(w, v)$. For any $x \in V(G) - N_G[u] \cup N_G[v]$, the path from x to u must through w_1, where $w_1 \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$. Then $d_G(x, u) = d_G(x, w_1) + d_G(w_1, u) = d_G(x, w_1) + d_{G-uv}(w_1, u) = d_{G-uv}(x, w_1) + d_{G-uv}(w_1, u) = d_{G-uv}(x, u)$. Similarly, $d_G(x, v) = d_{G-uv}(x, v)$. For any $x \in V(G) - \{u, v\}$, we have $uv \notin EM(x)$. From Theorem 1.4, $uv \in EM(u)$ and $uv \in EM(v)$, and hence uv is only monitored by the vertex in $\{u, v\}$.

Conversely, if uv is only monitored by u and v, then $uv \notin EM(w)$ for any $w \in V(G) \setminus \{u, v\}$, Especially, since $(N_G(u) \cup N_G(v)) \setminus \{u, v\} \subseteq V(G) \setminus \{u, v\}$, it follows that $uv \notin EM(w)$ for $w \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$, as desired. □

Then, we give the proof of Theorem 1.12.

Proof of Theorem 1.12: If G is a disconnected graph, then the edge e must be in some connected component G_1 of G for any $e \in E(G)$, and hence e can only be monitored by the vertex in $V(G_1)$. Therefore, we just need consider the graph G which is connected. Let M be a DEM set of G with $|M| = \operatorname{dem}(G)$ and $e = uv \in E(G)$. If M is also a DEM set of $G - e$, then $\operatorname{dem}(G - e) \leq \operatorname{dem}(G)$. Otherwise, let $M' = M \cup \{u, v\}$. It suffices to show that M' is a DEM set of $G - e$.

If $G - e$ has two components, say G_1 and G_2, then e is a cut edge of G and from Observation 3.1, we have $\operatorname{dem}(G - e) = \operatorname{dem}(G_1) + \operatorname{dem}(G_2)$. Without loss of generality, assume that $u \in V(G_1)$ and $v \in V(G_2)$.

Fact 3.2. $\operatorname{dem}(G_1) \leq |(M \cap V(G_1)) \cup \{u\}|$ and $\operatorname{dem}(G_2) \leq |(M \cap V(G_2)) \cup \{v\}|$.

Proof:
For any edge $e_1 = x_1 y_1 \in E(G_1)$, if there exists a vertex $w \in V(G_1) \cap M$ such that $e_1 \in EM(w)$, then we are done. Otherwise, there exists a vertex $w \in V(G_2) \cap M$ such that $d_{G-e_1}(x_1, w) \neq d_G(x_1, w)$ or $d_{G-e_1}(y_1, w) \neq d_G(y_1, w)$. Without loss of generality, we suppose that $d_{G-e_1}(y_1, w) \neq d_G(y_1, w)$ and $d_G(w, e_1) = d_G(w, x_1)$. Since $d_G(y_1, w) = d_G(y_1, x_1) + d_G(x_1, u) + d_G(u, w)$, $d_{G-\{e,e_1\}}(x_1, u) = d_{G-e_1}(x_1, u)$ and $d_{G-\{e,e_1\}}(y_1, x_1) > d_{G-e}(y_1, x_1)$, it follows that

$$
\begin{aligned}
d_{G-\{e,e_1\}}(u, y_1) &= d_{G-\{e,e_1\}}(u, x_1) + d_{G-\{e,e_1\}}(x_1, y_1) \\
&= d_{G-\{e,e_1\}}(u, x_1) + d_{G-e}(x_1, y_1) \\
&> d_{G-e}(u, x_1) + d_{G-e}(x_1, y_1) \\
&= d_{G-e}(u, y_1)
\end{aligned}
$$

and hence $d_{G-\{e,e_1\}}(y_1, u) \neq d_{G-e_1}(y_1, u)$. Therefore, e_1 is monitored by $(M \cap V(G_1)) \cup \{u\}$ in graph $G - e$. This implies that $\operatorname{dem}(G_1) \leq |(M \cap V(G_1)) \cup \{u\}|$. Similarly, we can obtain that $\operatorname{dem}(G_2) \leq |(M \cap V(G_2)) \cup \{v\}|$. □

From Fact 3.2, we have $\operatorname{dem}(G - e) \leq |M'| = |M \cup \{u, v\}| \leq |M| + 2 = \operatorname{dem}(G) + 2$.

Suppose that $G - e$ is connected. If M is also a DEM set of $G - e$, then $\operatorname{dem}(G - e) \leq |M| = \operatorname{dem}(G)$ and we are done. Otherwise, there exists $e_1 = xy \in E(G - e)$ such that the edge e_1 is not monitored by M in $G-e$. Since M is a distance- edge-monitoring set of G, it follows that there exists a vertex $z \in M$ such that $d_{G-e_1}(x, z) \neq d_G(x, z)$ or $d_{G-e_1}(y, z) \neq d_G(y, z)$. In addition, since e_1 is not monitored by M in $G-e$, it follows that the distance from z to x or y is not changed after removing the edge e_1 in $G-e$, which means that $d_{G-\{e,e_1\}}(y, z) = d_{G-e}(y, z)$ and $d_{G-\{e,e_1\}}(x, z) = d_{G-e}(x, z)$. If $d_G(e_1, z) = d_G(x, z)$, then the edge e lies on every $z-y$ geodesic in G for $z \in M$ and $xy \in EM(z)$ in G, otherwise there exists $z^* \in M$ and $xy \in EM(z^*)$ such that e does not appear in $z^* - y$ geodesic in G, that is $d_{G-e}(x, z^*) = d_G(x, z^*)$ and $d_{G-\{e,e_1\}}(x, z^*) \neq d_G(x, z^*)$, which contradicts to the fact that M is not the DEM set of graph $G - e$.

Claim 3.3. If a geodesic in G from z to y traverses the edge e in the order u, v, then each geodesic in G from z to y traverses e in the order u, v.

Proof:
Assume, to the contrary, that there exists two $z - y$ geodesics P_1^g and P_2^g, where $P_1^g = z \dots uv \dots y$ and $P_2^g = z \dots vu \dots y$. The $z-y$ geodesic P_1^g implies that $d(u, v) + d(v, y) = d(u, y)$, and the $z-y$ geodesic P_2^g implies that $d(v, u) + d(u, y) = d(v, y)$, and hence $d(u, v) = 0$, a contradiction. □

From Claim 3.3, without loss of generality, we may assume that every geodesic in G from z to y traverses the edge e in the order u, v. Thus, we have $d_G(z, y) = d_G(z, v) + d_G(v, y)$. We now show that xy can be monitored by v in $G - e$. Note that $d_{G-e_1}(z, y) \neq d_G(z, y)$, $d_{G-e}(v, y) = d_G(v, y)$ and $d_{G-e}(x, y) = d_G(x, y)$. Then $d_{G-\{e,e_1\}}(v, y) = d_{G-\{e,e_1\}}(v, x) + d_{G-\{e,e_1\}}(x, y)$ $= d_{G-e_1}(v, x) + d_{G-e_1}(x, y) > d_G(v, x) + d_G(x, y) = d_{G-e}(v, x) + d_{G-e}(x, y) \geq d_{G-e}(v, y)$. Since $d_{G-e}(v, y) > d_{G-\{e,e_1\}}(v, y)$, it follows that e_1 can be monitored by v. Since $e_1 \in EM(u)$ or $e_1 \in EM(v)$, it follows that $M' = M \cup \{u, v\}$ is a distance edge-monitoring-set of $G - e$, and thus $\operatorname{dem}(G - e) \leq \operatorname{dem}(G) + 2$, as desired. □

Li et al. [13] got the following result about DEM numbers of $C_k \square P_\ell$.

Theorem 3.4. [13] Let ℓ and k be two integers with $\ell \geq 3$ and $k \geq 2$. Then

$$\operatorname{dem}(C_k \square P_\ell) = \begin{cases} k & \text{if } k \geq 2\ell + 1, \\ 2\ell & \text{if } k < 2\ell + 1. \end{cases}$$

To show the sharpness of Theorem 1.12, we consider the following proposition.

Proposition 3.5. There exist two connected graphs G_1, G_2 of order n such that $\operatorname{dem}(G_1 - e) - \operatorname{dem}(G_1) = 2$ and $\operatorname{dem}(G_2) - \operatorname{dem}(G_2 - e) = 2$.

Proof:
Firstly, we consider the graph G_1 ($|V(G_1)| = n \geq 8$) with vertex set $V(G_1) = \{v_i | 1 \leq i \leq n - 8\} \cup \{u_i | 1 \leq i \leq 8\}$ and edge set $E(G_1) = \{u_i v_i \,|\, 1 \leq i \leq 8\} \cup \{u_i u_{i+1} \,|\, 1 \leq i \leq 7\} \cup \{v_i v_{i+1} \,|\, 1 \leq i \leq 7\} \cup \{u_1 u_8\} \cup \{u_1 u_5\} \cup \{v_1 v_8\} \cup \{v_1 v_9\} \cup \{v_i v_{i+1} \,|\, 9 \leq i \leq n - 9\}$. Let $G_8^* = G_b(G_1)$.

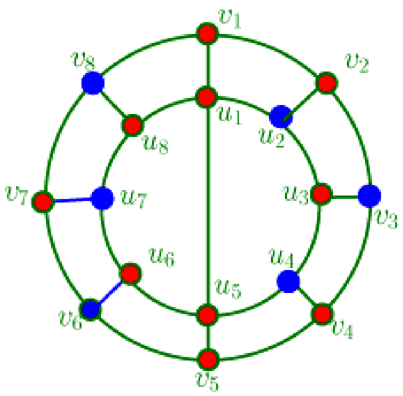

Figure 3. $\mathrm{dem}(G_8^*) = 6$

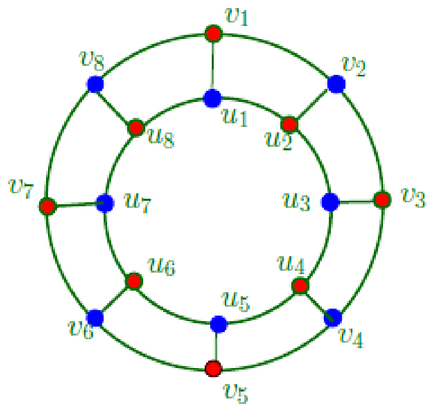

Figure 4. $\mathrm{dem}(G_8^* - u_1u_5) = 8$

Obviously, G_8^* is the base graph of G_1, which is obtained by removing the all edge in the edge set $\{v_1v_9\} \cup \{v_iv_{i+1} \,|\, 9 \leq i \leq n-9\}$. The graphs G_8^* and $G_8^* - u_1u_5$ are shown in Figures 3 and 4, respectively.

Let $M_1 = \{u_2, u_4, v_3, v_6, u_7, v_8\}$. Note that $\{u_1u_5, u_5v_5, u_2v_2, u_2u_1, u_2u_3\} \subseteq EM(u_2)$, $\{v_1u_1, u_4u_3, u_4u_5, u_4v_4\} \subseteq EM(u_4)$, $\{v_3u_3, v_2v_3, v_4v_3, v_5v_4, v_2v_1\} \subseteq EM(v_3)$, $\{v_8v_1, u_8v_8, v_8v_7\} \subseteq EM(v_8)$, $\{u_7u_8, u_8u_1, u_6u_7, u_6u_5, u_7v_7\} \subseteq EM(u_7)$ and $\{v_5v_6, v_6v_7, u_6v_6\} \in EM(v_6)$. Therefore, $E(G_8^*) = \cup_{x \in M_1} EM(x)$, and hence $\mathrm{dem}(G_8^*) \leq |M_1| = 6$.

Let M be a DEM set of G_8^* with the minimum cardinality. For the edge u_iv_i, where $2 \leq i \leq 8$ and $i \neq 5$, and any $w \in (N(u_i) \cup N(v_i)) \setminus \{u_i, v_i\}$, we have $d_{G-u_iv_i}(w, u_i) = d_G(w, u_i)$ and $d_{G-u_iv_i}(w, v_i) = d_G(w, v_i)$, and hence $u_iv_i \notin EM(w)$. From Proposition 3.1, the edge u_iv_i ($2 \leq i \leq 8$ and $i \neq 5$) is only monitored by $\{u_i, v_i\}$, and hence $M \cap \{u_i, v_i\} \neq \emptyset$ for $2 \leq i \leq 8$ and $i \neq 5$, and so $\mathrm{dem}(G_8^*) \geq 6$. Therefore, $\mathrm{dem}(G_8^*) = 6$.

Since $G_8^* - u_1u_5 \cong C_8 \Box P_2$, it follows from Theorem 3.4 that $\mathrm{dem}(G_8^* - u_1u_5) = \mathrm{dem}(C_8 \Box P_2) = 8$. From Observation 1.1, $\mathrm{dem}(G_1 - u_1u_5) - \mathrm{dem}(G_1) = \mathrm{dem}(G_8^* - u_1u_5) - \mathrm{dem}(G_8^*) = 8 - 6 = 2$, as desired.

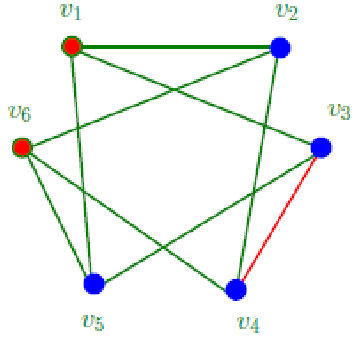

Figure 5. $\mathrm{dem}(G_6') = 4$

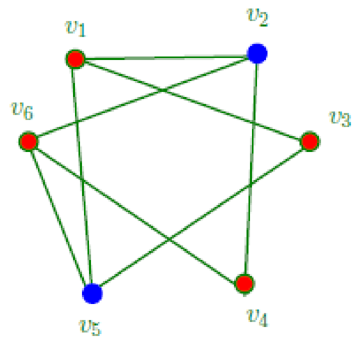

Figure 6. $\mathrm{dem}(G_6' - v_3v_4) = 2$

Next, we consider the graph G_2 ($|V(G_2)| = n \geq 6$) with vertex set $V(G_2) = \{v_i \mid 1 \leq i \leq n\}$ and edge set $E(G_2) = \{v_1v_2, v_3v_4, v_5v_6, v_1v_3, v_1v_5, v_2v_4, v_2v_6, v_3v_5, v_4v_6\} \cup \{v_iv_{i+1} \mid 6 \leq i \leq n-1\}$. Let G_6' be the base graph of G_2, that is, $G_b(G_2) = G_6'$. The graphs G_6' and $G_6' - v_1v_3$, are shown in Figure 5 and Figure 6, respectively. From Observation 1.1, $\operatorname{dem}(G_2) = \operatorname{dem}(G_6')$.

Take $M_1' = \{v_2, v_3, v_4, v_5\}$. Note that $\{v_1v_2, v_6v_2, v_4v_2\} \subseteq EM(v_2)$, $\{v_1v_3, v_5v_3, v_4v_3\} \subseteq EM(v_3)$, $\{v_6v_4\} \subseteq EM(v_4)$, $\{v_5v_1, v_6v_5\} \subseteq EM(v_5)$, and hence $E(G_6') = \cup_{x\in M_1'} EM(x)$, it follows that M_1' is a DEM set of G_6', and hence $\operatorname{dem}(G_6') \leq |M_1'| = 4$. Let M' be a DEM set of G_6' with the minimum cardinality. For the edge $v_{2i-1}v_{2i}$ $(1 \leq i \leq 3)$ and $w \in (N(v_{2i-1}) \cup N(v_{2i})) \setminus \{v_{2i-1}v_{2i}\}$, we have $d_{G-v_{2i-1}v_{2i}}(w, v_{2i-1}) = d_G(w, v_{2i-1})$ and $d_{G-v_{2i-1}v_{2i}}(w, v_{2i}) = d_G(w, v_{2i})$, and so $v_{2i-1}v_{2i} \notin EM(w)$. From Proposition 3.1, the edge $v_{2i-1}v_{2i}$ $(1 \leq i \leq 3)$ is monitored by the vertex in $\{v_{2i-1}, v_{2i}\}$, and hence $M' \cap \{v_{2i-1}, v_{2i}\} \neq \emptyset$ $(1 \leq i \leq 3)$. All sets $M' \in V(G_6')$ with $|M'| = 3$ are shown in Table 1. Therefore, all sets M' with $|M'| = 3$ are not DEM sets of G_6', and hence $\operatorname{dem}(G_6') \geq 4$. Therefore, we have $\operatorname{dem}(G_6') = 4$.

Table 1. The edges are not monitored by $M'(|M'| = 3)$.

M'	$E(G_6') - \cup_{x\in M'} EM(x)$
v_1, v_3, v_6	v_2v_4
v_1, v_4, v_5	v_2v_6
v_1, v_4, v_6	v_3v_5
v_2, v_3, v_5	v_4v_6
v_2, v_3, v_6	v_1v_5
v_2, v_4, v_5	v_1v_3
v_1, v_3, v_5	v_2v_6, v_2v_4, v_4v_6
v_2, v_4, v_6	v_1v_3, v_1v_5, v_3v_5

For the graph $G_6' - v_3v_4$, let $M_3 = \{v_2, v_5\}$. Note that $\{v_1v_2, v_6v_2, v_4v_2, v_1v_3\} \subseteq EM(v_2)$ and $\{v_5v_1, v_6v_5, v_3v_5, v_6v_4\} \subseteq EM(v_5)$. Since $E(G_6' - v_3v_4) = \cup_{x\in M_3} EM(x)$, it follows that M_3 is a DEM set of G_6', and hence $\operatorname{dem}(G_6' - v_3v_4) \leq 2$. Since $G_6' - v_3v_4$ is not a tree, it follows from Theorem 1.6 that $\operatorname{dem}(G_6' - v_3v_4) \geq 2$, and so $\operatorname{dem}(G_6' - v_3v_4) = 2$. From Observation 1.1, $\operatorname{dem}(G_2) - \operatorname{dem}(G_2 - v_3v_4) = \operatorname{dem}(G_6') - \operatorname{dem}(G_6' - v_3v_4) = 4 - 2 = 2$, as desired. □

The *friendship graph*, $Fr(n)$, can be constructed by joining n copies of the complete graph K_3 with a common vertex, which is called the *universal vertex* of $Fr(n)$. Next, we give the proof of Theorem 1.13.

Proof of Theorem 1.13: Let k, i be integers with $1 \leq i \leq k$. The graph G^i is obtained by iteratively adding an edge u_iv_i to the graph G^{i-1}. Without loss of generality, let G^0 be the graph with $V(G^0) = \{c\} \cup \{u_j \mid 1 \leq j \leq k\} \cup \{v_j \mid 1 \leq j \leq k\}$ and $E(G^0) = \{cu_j, cv_j \mid 1 \leq j \leq k\}$, and G^i be the graph with $V(G^i) = V(G^{i-1})$ and $E(G^i) = E(G^{i-1}) \cup \{u_iv_i\}$, where $1 \leq i \leq k$. Since G^0 is a tree, it follows from Theorem 1.6 that $\operatorname{dem}(G^0)=1$. Note that the base graph of G^1 is a complete graph K_3. From Observation 1.1 and Theorem 1.7, we have $\operatorname{dem}(G_1)=\operatorname{dem}(K_3)=2$.

Let $G=G^i$, where $2\leq i\leq k$. Then $G_b=Fr(i)$. Let $M=\{u_t\,|\,1\leq t\leq i\}$. From Theorem 1.4, we have $\{u_tv_t, cu_t\,|\,1\leq t\leq i\}\subseteq\cup_{x\in M}EM(x)$. Since $2=d_G(u_1,v_t)\neq d_{G-cv_t}(u_1,v_t)=3$ for $2\leq t\leq i$, it follows that $cv_t\in EM(u_1)$ for $2\leq t\leq i$. Suppose that $t=1$. Since $2=d_G(u_2,v_1)\neq d_{G-cv_1}(u_2,v_1)=3$, it follows that $cv_1\in EM(u_2)$, and hence $E(G)\subseteq\cup_{x\in M}EM(x)$, and so $\operatorname{dem}(G)\leq i$. Let M be a DEM set of G with the minimum cardinality. Note that $(N(u_j)\cup N(v_j))\setminus\{u_j,v_j\}=\{c\}$. Since $d_G(c,u_j)=d_{G-u_jv_j}(c,u_j)$ and $d_G(c,v_j)=d_{G-u_jv_j}(c,v_j)$ it follows that $u_jv_j\notin EM(c)$, where $1\leq j\leq k$. From Proposition 3.1, the edge u_jv_j is only monitored by u_j or v_j, and hence $M\cap\{u_j,v_j\}\neq\emptyset$ for $1\leq j\leq k$, Therefore, $\operatorname{dem}(G)\geq i$, and so $\operatorname{dem}(G)=i$. Thus, there exists a graph sequence $\{G^i\,|\,0\leq i\leq k\}$, with $e(G^i)-e(G^0)=i$ and $V(G^i)=V(G^j)$ for $0\leq i,j\leq k$, such that $\operatorname{dem}(G^{i+1})-\operatorname{dem}(G^0)=i$, where $1\leq i\leq k-1$. □

Foucaud et al. [10] obtained the following result.

Theorem 3.6. [10] Let ℓ_1 and ℓ_2 be two integers with $\ell\geq 2$ and $\ell_2\geq 2$. Then

$$\operatorname{dem}(P_{\ell_1}\Box P_{\ell_2})=\max\{\ell_1,\ell_2\}$$

In the end of this section, we give the proof of Corollary 1.15.

Proof of Corollary 1.15: For any tree T_n, T_n+e_1 is an unicyclic graph and $T_n+\{e_1,e_2\}$ is a tricyclic graph. From Theorems 1.6 and 1.14, we have $\operatorname{dem}(T_n+e_1)=\operatorname{dem}(T_n)+1=2$ and $\operatorname{dem}(T_n+\{e_1,e_2\})=2$ or 3. □

4. The effect of deleted vertex

A *kipas* $\widehat{K}_n$ with $n\geq 3$ is the graph on $n+1$ vertices obtained from the join of K_1 and P_n, where $V(\widehat{K}_n)=\{v_0,v_1,\ldots,v_n\}$ and $E(\widehat{K}_n)=\{v_0v_i\,|\,1\leq i\leq n\}\cup\{v_iv_{i+1}\,|\,1\leq i\leq n-1\}$.

Proposition 4.1. For $n\geq 7$, we have $\operatorname{dem}(\widehat{K}_n)=\lfloor n/2\rfloor$.

Proof:
Let P_n be the subgraph of $\widehat{K}_n$ with vertex set $\{v_i\,|\,1\leq i\leq n\}$ and edge set $\{v_iv_{i+1}\,|\,1\leq i\leq n-1\}$. First, we prove that $\operatorname{dem}(\widehat{K}_n)\geq\lfloor n/2\rfloor$. Let M be a DEM set of $\widehat{K}_n$ with the minimum cardinality. For any vertices $v_i,v_j\in V(\widehat{K}_n)$, we have

$$d_{\widehat{K}_n}(v_i,v_j)=\begin{cases}1, & \text{if } i=0 \text{ or } j=0 \text{ or } |i-j|=1;\\ 2, & \text{if } 1\leq i,j\leq n \text{ and } |i-j|\geq 2.\end{cases}$$

For any edge v_iv_{i+1} ($2\leq i\leq n-2$), we have $(N_G(v_i)\cup N_G(v_{i+1}))\setminus\{v_i,v_{i+1}\}=\{v_{i-1},v_0,v_{i+2}\}$. Since $d_G(v_i,v_0)=d_{G-v_iv_{i+1}}(v_i,v_0)=1$, $d_G(v_{i+1},v_0)=d_{G-v_iv_{i+1}}(v_{i+1},v_0)=1$, $d_{G-v_iv_{i+1}}(v_i,v_{i-1})=d_G(v_i,v_{i-1})=1$, $d_{G-v_iv_{i+1}}(v_{i+1},v_{i-1})=d_G(v_{i+1},v_{i-1})=2$, $d_{G-v_iv_{i+1}}(v_{i+1},v_{i+2})=d_G(v_{i+1},v_{i+2})=1$, and $d_{G-v_iv_{i+1}}(v_i,v_{i+2})=d_G(v_i,v_{i+2})=2$, it follows that $v_iv_{i+1}\notin EM(v_{i+2})\cup EM(v_{i-1})\cup EM(v_0)$. From Proposition 3.1, the edge v_iv_{i+1} can only be monitored by the vertex in $\{v_i, v_{i+1}\}$. Similarly, the edge v_iv_{i+1} is only monitored by the vertex in $\{v_i, v_{i+1}\}$, where $i = 1, n-1$. Therefore, $M \cap \{v_i, v_{i+1}\} \neq \emptyset$ for $1 \leq i \leq n-1$, that is, M is a vertex cover set of P_n. Note that the vertex covering number of G is $\beta(G)$. Since $\beta(P_n) = \lfloor n/2 \rfloor$, it follows that $\operatorname{dem}(\widehat{K}_n) \geq \lfloor n/2 \rfloor$.

Next, we prove that $\operatorname{dem}(\widehat{K}_n) \leq \lfloor n/2 \rfloor$. Let $M = \{v_i \,|\, i \equiv 0 \pmod 2, 1 \leq i \leq n\}$. For any edge $e \in E(P_n) \cup \{v_0v_i \,|\, i \equiv 0 \pmod 2, 1 \leq i \leq n\}$, it follows from Theorem 1.4 that e is monitored by the vertex in M. In addition, for any edge $v_0v_i \in \{v_0v_i \,|\, i \equiv 1 \pmod 2, 1 \leq i \leq n\}$, since $n \geq 7$, it follows that there exists j such that $d_G(v_i, v_j) = 2$ and $d_{G-v_0v_i}(v_i, v_j) = 3$, where $j = i+3$ for $1 \leq i \leq n-4$ and $j = 2$ for $n-3 \leq i \leq n$, and hence $v_0v_i \in EM(v_j)$. Since any edge $v_0v_i \in E(\widehat{K}_n)$ can be monitored by the vertex in M, it follows that $\operatorname{dem}(\widehat{K}_n) \leq \lfloor n/2 \rfloor$, and hence $\operatorname{dem}(\widehat{K}_n) = \lfloor n/2 \rfloor$. □

Proof of Theorem 1.16 Note that $\widehat{K}_{2k+2} = K_1 \vee P_{2k+2}$, where $V(K_1) = \{v_0\}$. From Theorem 1.6, we have $\operatorname{dem}(P_{2k+2}) = 1$. From Lemma 4.1, we have $\operatorname{dem}(\widehat{K}_{2k+2}) = k+1$, and hence $\operatorname{dem}(\widehat{K}_{2k+2}) - \operatorname{dem}(\widehat{K}_{2k+2} - v_0) = \operatorname{dem}(\widehat{K}_{2k+2}) - \operatorname{dem}(P_{2k+2}) = k$. Let $G_1 = \widehat{K}_{2k+2}$ and $H_1 = P_{2k+2}$. Then $\operatorname{dem}(G_1) - \operatorname{dem}(H_1) = \operatorname{dem}(\widehat{K}_{2k+2}) - \operatorname{dem}(P_{2k+2}) = k$, where H_1 is not a spanning subgraph of G_1.

Let G_{2k+3} be a graph with vertex set $V(G_{2k+3}) = \{u_i \,|\, 1 \leq i \leq k+1\} \cup \{v_i \,|\, 0 \leq i \leq k+1\}$ and edge set $E(G_{2k+3}) = \{v_0u_i \,|\, 1 \leq i \leq k+1\} \cup \{u_iv_i \,|\, 1 \leq i \leq k+1\}$. Obviously, we have $G_{2k+3} \setminus v_0 \cong (k+1)K_2$. From Observation 3.1 and Theorem 1.7, we have $\operatorname{dem}(G_{2k+3} - v_0) = \operatorname{dem}((k+1)K_2) = (k+1)\operatorname{dem}(K_2) = k+1$. Since G_{2k+3} is a tree, it follows from Theorem 1.6 that $\operatorname{dem}(G_{2k+3}) = 1$, and hence $\operatorname{dem}(G_{2k+1} \setminus v_0) - \operatorname{dem}(G_{2k+1}) = k$. Let $G_2 = G_{2k+1}$ and $H_2 = (k+1)K_2$. Then $\operatorname{dem}(H_2) - \operatorname{dem}(G_2) = \operatorname{dem}((k+1)K_2) - \operatorname{dem}(G_{2k+1}) = k$, where H_2 is not a spanning subgraph of G_2, as desired. □

Note that $G_{2k+3} \setminus v_0 \cong (k+1)K_2$ is disconnected graph. For the connected graphs, we can also show that there is a connected subgraph H such that $\operatorname{dem}(H) - \operatorname{dem}(G)$ can be arbitrarily large; see Theorem 4.5.

The *conical graph* $C(\ell, k)$ is a graph obtained by taking adjacency from a center vertex c to the first layer of Cartesian product of P_ℓ and C_k, where $\ell \geq 1$ and $k \geq 3$.

Let the vertex set $V(C(\ell, k)) = \{c\} \cup \{u_i^j \,|\, 1 \leq i \leq k, 1 \leq j \leq \ell\}$ and the edge set $E(C(\ell, k)) = (\cup_{i=1}^{\ell} E(C_i)) \cup (\cup_{i=1}^{k} E(P_i))$, where $E(C_i) = \{u_k^i u_1^i\} \cup \{u_j^i u_{j+1}^i \,|\, 1 \leq j \leq k-1\}$ $(1 \leq i \leq \ell)$, $E(P_i) = \{cu_i^1\} \cup \{u_i^j u_i^{j+1} \,|\, 1 \leq j \leq \ell-1\}$ $(1 \leq i \leq k)$. The conical graph $C(3, 8)$ is shown in Figure 7.

For $\ell = 1$, the graph $C(1, k)$ is the wheel graph W_k, which is formed by connecting a single vertex c to all the vertices of cycle C_k. It is clear that $|V(C(\ell, k))| = k\ell + 1$ and $e(C(\ell, k)) = 2k\ell$.

Lemma 4.2. Let $n \geq 3$ be an integer. For $v \in V(C_n)$, we have

$$|EM(v) \cap E(C_n)| = \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n-2 & \text{if } n \text{ is even.} \end{cases}$$

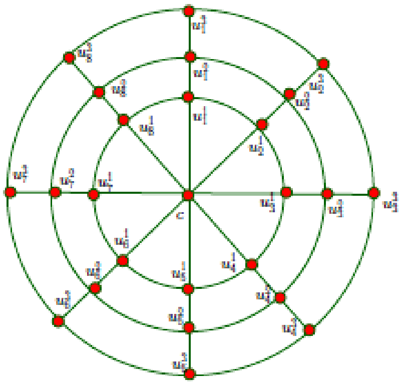

Figure 7. The conical graph $C(3,8)$

Proof:
Let $G = C_n$ be cycle with $V(G) = \{v_i \,|\, 1 \leq i \leq n\}$ and $E(G) = \{v_i v_{i+1} \,|\, 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. Without loss of generality, let $v = v_1$. Suppose that n is odd and $e_1 = v_{\lfloor n/2 \rfloor+1} v_{\lfloor n/2 \rfloor+2}$. Since $d_G(v_1, v_{\lfloor n/2 \rfloor+1}) = d_{G-e_1}(v_1, v_{\lfloor n/2 \rfloor+1})$ and $d_G(v_1, v_{\lfloor n/2 \rfloor+2}) = d_{G-e_1}(v_1, v_{\lfloor n/2 \rfloor+2})$, it follows that $e_1 \notin EM(v_1)$. For any $e \in \{v_i v_{i+1} \,|\, 1 \leq i \leq \lfloor n/2 \rfloor\}$, since $d_G(v_1, v_{i+1}) \neq d_{G-e}(v_1, v_{i+1})$, it follows that $e \in EM(v_1)$. For any $e \in \{v_i v_{i+1} \,|\, \lfloor n/2 \rfloor + 2 \leq i \leq n-1\}$, since $d_G(v_1, v_i) \neq d_{G-e}(v_1, v_i)$, it follows that $e \in EM(v_1)$. From Theorem 1.4, we have $v_n v_1 \in EM(v_1)$. Therefore, $EM(v_1) = \{v_1 v_2, v_2 v_3, \ldots, v_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor+1}, v_1 v_n, v_n v_{n-1}, \ldots, v_{\lfloor n/2 \rfloor+3} v_{\lfloor n/2 \rfloor+2}\}$, and hence $|EM(v) \cap E(C_n)| = n-1$. Similarly, if n is even, then $|EM(v) \cap E(C_n)| = n-2$. □

Theorem 4.3. For $k \geq 9$ and $\ell \geq 2$, we have

$$\mathrm{dem}(C(\ell,k)) = \begin{cases} \sum_{i=1}^{\ell} \lceil k/(4i-2) \rceil, & \text{if } \ell \leq a_k; \\ \sum_{i=1}^{a_k} \lceil k/(4i-2) \rceil + 2(\ell - a_k), & \text{if } \ell \geq a_k + 1, \end{cases}$$

where $a_k = \lfloor k/4 + (1 + (-1)^{k+1})/8 \rfloor$.

Proof:
Let $G = C(\ell,k)$ with $V(G) = \{c\} \cup \{u_i^j \,|\, 1 \leq i \leq k, 1 \leq j \leq \ell\}$ and $E(G) = (\cup_{i=1}^{\ell} E(C_i)) \cup (\cup_{i=1}^{k} E(P_i))$, where $E(C_i) = \{u_k^i u_1^i\} \cup \{u_j^i u_{j+1}^i \,|\, 1 \leq j \leq k-1\}$, $E(P_i) = \{c u_i^1\} \cup \{u_i^j u_i^{j+1} \,|\, 1 \leq j \leq \ell - 1\}$. Let M be a DEM set of G with $M = \mathrm{dem}(C(\ell,k))$.

Fact 4.4. For any vertex $v \in V(C_i)$, we have

$$|EM(v) \cap E(C_i)| = \begin{cases} 4i-2, & \text{if } 1 \leq i \leq a_k; \\ k-2, & \text{if } k \text{ is even and } i \geq a_k + 1; \\ k-1, & \text{if } k \text{ is odd and } i \geq a_k + 1. \end{cases}$$

and $|EM(v) \cap E(C_j)| = 0$, where $1 \leq j \neq i \leq \ell$. Furthermore, we have $|EM(c) \cap E(C_i)| = 0$.

Proof:
For any vertices $u_s^i, u_t^i \in V(C_i)$, where $1 \leq s,t \leq k$ and $1 \leq i \leq \ell$, since there exists a path $u_s^i \ldots, u_s^1 c u_t^1 \ldots u_t^{i-1} u_t^i$ from u_s^i to u_t^i, it follows that $d_G(u_s^i, u_t^i) \leq 2i$. It is easy to see that there are two possible types of shortest paths $P_{u_s^i u_t^i}$ from u_s^i to u_t^i:

Type 1: If $d_{C_i}(u_s^i, u_t^i) \geq 2i$, then $P_{u_s^i u_t^i} = u_s^i \ldots, u_s^1 c u_t^1 \ldots u_t^{i-1} u_t^i$;

Type 2: If $d_{C_i}(u_s^i, u_t^i) < 2i$, then the shortest path $P_{u_s^i u_t^i} \sqsubseteq C_i$, where C_i is subgraph of G.

Therefore, $d_G(u_s^i, u_t^i) = \min\{2i, d_{C_i}(u_s^i, u_t^i)\}$. Suppose that $1 \leq i \leq a_k$. For $s-2i+1 \leq t \leq s-1$, since $d_{C_i}(u_s^i, u_t^i) < 2i$, it follows that $d_G(u_s^i, u_t^i) = \min\{2i, d_{C_i}(u_s^i, u_t^i)\} = d_{C_i}(u_s^i, u_t^i)$, and hence $d_G(u_s^i, u_t^i u_{t+1}^i) = d_{C_i}(u_s^i, u_{t+1}^i)$. Thus, $d_G(u_s^i, u_t^i u_{t+1}^i) = d_{G-u_t^i u_{t+1}^i}(u_s^i, u_t^i u_{t+1}^i)$ and $d_{G-u_t^i u_{t+1}^i}(u_s^i, u_t^i) > d_{C_i}(u_s^i, u_{t+1}^i) + 1 = d_G(u_s^i, u_t^i)$, and so $u_t^i u_{t+1}^i \in EM(u_s^i)$.

Similarly, for any edge $u_t^i u_{t+1}^i \in E(C_i)$, where $s \leq t \leq s+2i-2$, since $d_G(u_s^i, u_{t+1}^i)) \neq d_{G-u_t^i u_{t+1}^i}(u_s^i, u_{t+1}^i)$, it follows that $u_t^i u_{t+1}^i \in EM(u_s^i)$. Therefore, $\{u_t^i u_{t+1}^i \,|\, s-2i+1 \leq t \leq s+2i-2\} \subseteq EM(u_s^i)$, where the subscripts are taken modulo k, that is, $u_{k+1}^i = u_1^i$.

For any $u_j^i u_{j+1}^i \in E(C_i) - \big(\{u_t^i u_{t+1}^i \,|\, s-2i+1 \leq t \leq s+2i-2\} \cup \{u_{s-2i}^i u_{s-2i-1}^i, u_{s+2i+1}^i u_{s+2i}^i\}\big)$, since $d(u_s^i, u_j^i) = d(u_s^i, u_{j+1}^i) = 2i$ and $d_{G-u_j^i u_{j+1}^i}(u_s^i, u_j^i) = d_{G-u_j^i u_{j+1}^i}(u_s^i, u_{j+1}^i) = 2i$, it follows that $u_j^i u_{j+1}^i \notin EM(u_i)$. In addition, let $e_1 = u_{s-2i}^i u_{s-2i-1}^i$ and $e_2 = u_{s+2i-1}^i u_{s+2i}^i$. Then, $d_G(u_s^i, e_1) = d_G(u_s^i, u_{s-2i-1}^i) = 2i$ and $d_G(u_s^i, e_2) = d_G(u_s^i, u_{s+2i-1}^i) = 2i$. Since $d_{G-e_1}(u_s^i, u_{s-2i}^i) = d_G(u_s^i, u_{s-2i}^i) = 2i$ and $d_{G-e_2}(u_s^i, u_{s+2i}^i) = d_G(u_s^i, u_{s+2i}^i) = 2i$, it follows that $e_1, e_2 \notin EM(u_i)$, and hence $|EM(u_s^i) \cap E(C_i)| = |\{u_t^i u_{t+1}^i \,|\, s-2i+1 \leq t \leq s+2i-2\}| = 4i-2$ for any vertex $u_s^i \in V(C_i)$, where the subscripts are taken modulo k.

Suppose that $i \geq a_k + 1$. For $i \geq a_k + 1$ and $1 \leq s,t \leq k$, if $u_s^i, u_t^i \in V(C_i)$, then $d_G(u_s^i, u_t^i) = d_{C_i}(u_s^i, u_t^i)$. For any cycle C_i with length k, if k is even, then it follows from Lemma 4.2 that $|EM(v) \cap E(C_i)| = k-2$. If k is odd, then it follows from Lemma 4.2 that $|EM(v) \cap E(C_i)| = k-1$, as desired.

For any vertex $u_s^i \in V(C_i)$ and any edge $e = u_m^j u_{m+1}^j \in E(C_j)$, where $1 \leq j \neq i \leq \ell$ and $1 \leq m \leq k$, since $d_{G-e}(u_s^i, u_m^j) = d_G(u_s^i, u_m^j)$ and $d_{G-e}(u_s^i, u_{m+1}^j) = d_G(u_s^i, u_{m+1}^j)$, it follows that $e \notin EM(u_s^i)$, and hence $|EM(u_s^i) \cap E(C_j)| = 0$.

For any edge $e_3 = u_s^i u_{s+1}^i \in E(C_i)$, we have $d_G(c, u_s^i) = d_G(c, u_{s+1}^i) = i$, and hence $d_{G-e_3}(c, u_s^i) = d_{G-e_3}(c, u_{s+1}^i) = i$, and so $e_3 \notin EM(c)$. Therefore, $|EM(c) \cap E(C_i)| = 0$. □

Suppose that $\ell \geq a_k + 1$. Since $e(C_i) = k$, it follows from Fact 4.4 that $|M \cap E(C_i)| \geq 2$ for $a_k + 1 \leq i \leq \ell$ and $|M \cap E(C_i)| \geq \lceil k/(4i-2) \rceil$ for $1 \leq i \leq a_k$, and so $\mathrm{dem}(G) \geq \sum_{i=1}^{a_k} \lceil k/(4i-2) \rceil + 2(\ell - a_k)$.

Let $M = \cup_{i=1}^{i=k} M_i$, where

$$M_i = \begin{cases} \{u_j^i \,|\, 1 \leq j \leq k, j \equiv 1 \bmod (4i-2)\}, & \text{if } i \leq a_k; \\ \{u_1^i, u_{\lceil k/2 \rceil}^i\}, & \text{if } a_k + 1 \leq i \leq \ell. \end{cases}$$

Therefore, $e \in \cup_{x\in M_i} EM(x) \subseteq \cup_{x\in M} EM(x)$ for any edge $e \in E(C_i)$, where $1 \leq i \leq \ell$. It suffices to prove that $e \in \cup_{x\in M} EM(x)$ for each edge in $E(P_i)$, where $1 \leq i \leq k$. For some vertex $u_i^1 \in M_1$ and any $u_j^1 \in V(G)$, where $1 \leq i \neq j \leq k$, if $j \in \{1, 2, \ldots, i-3, i+3, \ldots, k\}$, where the subscripts are taken modulo n, then $d_G(u_i^1, u_j^1) \neq d_{G-cu_j^1}(u_i^1, u_j^1)$, and hence $cu_j^1 \in EM(u_i^1)$. Similarly, for $1 \leq t \leq \ell-1$, since $d_G(u_i^1, u_j^t) \neq d_{G-u_j^t u_j^{t-1}}(u_i^1, u_j^t)$ for $j \in \{1, 2, \ldots, i-3, i+3, \ldots, k\}$, it follows that $u_j^t u_j^{t-1} \in EM(u_i^1)$, and hence $EM(u_i^1) = \{u_i^1 u_{i+1}^1, u_i^1 u_{i-1}^1, cu_i^1\} \cup \{cu_j^1 \,|\, j \in \{1, 2, \ldots, i-3, i+3, \ldots, k\}\} \cup \{u_j^{p-1} u_j^p \,|\, 2 \leq p \leq \ell-1, j \in \{1, 2, \ldots, i-3, i+3, \ldots, k\}\}$. Without loss of generality, let $u_1^1 \in M$. Then $\cup_{i\in\{1,4,\ldots,k-2\}} E(P_i) \subseteq EM(u_1^1)$. Since $EM(u_i^1) = \{u_i^1 u_{i+1}^1, u_i^1 u_{i-1}^1, cu_i^1\} \cup \{cu_j^1 \,|\, j \in \{1, 2, \ldots, i-3, i+3, \ldots, k\}\} \cup \{u_j^{p-1} u_j^p \,|\, 2 \leq p \leq \ell-1, j \in \{1, 2, \ldots, i-3, i+3, \ldots, k\}\}$ and $k \geq 9$, it follows that $E(P_2) \subseteq EM(u_{k-3}^1)$, $E(P_3) \subseteq EM(u_{k-3}^1)$, $E(P_2) \subseteq EM(u_{k-2}^1)$, $E(P_3) \subseteq EM(u_{k-2}^1)$, $E(P_k) \subseteq EM(u_3^1)$ and $E(P_{k-1}) \subseteq EM(u_3^1)$

For any edge $e \in E(P_i)$, where $i \in \{2, 3, k, k-1\}$, if k is even, then $e \in EM(u_3^1) \cup EM(u_{k-3}^1)$; if k is odd, then $e \in EM(u_3^1) \cup EM(u_{k-2}^1)$. Therefore, $e \in \cup_{x\in M} EM(x)$ for any $e \in E(P_i)$, where $1 \leq i \leq k$, and so $\operatorname{dem}(G) \leq \sum_{i=1}^{a_k} \lceil k/(4i-2) \rceil + 2(\ell - a_k)$. Thus, $\operatorname{dem}(G) = \sum_{i=1}^{a_k} \lceil k/(4i-2) \rceil + 2(\ell - a_k)$. For $\ell \leq a_k$, it is similar to the case that $\ell \geq a_k + 1$, as desired. □

Theorem 4.5. For any positive integer $k \geq 9$, there exists a connected graph G such that such that

$$\operatorname{dem}(G \setminus v) - \operatorname{dem}(G) = \lfloor k/2 \rfloor - \lceil k/6 \rceil,$$

where $v \in V(G)$.

Proof:
Let $G = C(2, k)$, where $\ell = 2$ and $k \geq 5$. Note that $G \setminus v_0 = C_k \square K_2$. From Theorem 3.4, we have $\operatorname{dem}(G \setminus v_0) = \operatorname{dem}(C_k \square K_2) = k$. From Theorem 4.3, we have $\operatorname{dem}(C(2, k)) = \sum_{i=1}^{2} \lceil k/(4i-2) \rceil = \lceil k/2 \rceil + \lceil k/6 \rceil$, and hence $\operatorname{dem}(G - v) - \operatorname{dem}(G) = k - \lceil k/2 \rceil - \lceil k/6 \rceil = \lfloor k/2 \rfloor - \lceil k/6 \rceil$, as desired. □

Let $G = C(\ell, k)$ and $H = C_k \square P_\ell$. From Theorems 4.3 and 3.4, if $\ell \gg k$, then $\operatorname{dem}(G)/\operatorname{dem}(H) \approx 1$. From Theorems 4.3 and 3.4, if $k = 402$ and $\ell = 100$, then $\operatorname{dem}(G)/\operatorname{dem}(H) \approx 0.561453$.

Corollary 4.6. There exist two connected graphs H and G such that

$$\frac{\operatorname{dem}(G)}{\operatorname{dem}(H)} \approx 0.561453,$$

where H is an induced subgraph of G.

Proof of Theorem 1.17: Let $G = \widehat{K}_{2k+2}$. From Proposition 1.16, there exists a vertex $u \in V(G)$ such that $\operatorname{dem}(G) - \operatorname{dem}(G \setminus u) = k$. Note that $G \setminus u = P_{2k+2}$ is a connected graph. In addition, let $H = C(2, t)$, $k = \lfloor t/2 \rfloor - \lceil t/6 \rceil$ and $v \in V(H)$. From Theorem 4.5, we have $\operatorname{dem}(C(2, t) \setminus v) - \operatorname{dem}(C(2, t)) = k$, where $C(2, t) \setminus v = C_t \square K_2$ is a connected graph.

In fact, $G \setminus v$ is a subgraph of G. From Theorem 1.17, for any positive integer $k \geq 3$, there exists a graph G such that $\operatorname{dem}(G \setminus v) - \operatorname{dem}(G) \geq k$. □

Let $G = C(2,t)$ and $H_1 = C(2,t) \setminus v$, where $t \geq 8$. From Theorem 1.17, $\mathrm{dem}(H_1) \geq \mathrm{dem}(G)$. Note that G is not tree, it follows from Theorem 1.6 that $\mathrm{dem}(G) \geq 2$. Let H_2 be a tree satisfying $H_2 \sqsubseteq G$. From Theorem 1.6, we have $\mathrm{dem}(H_2) = 1$, and hence $\mathrm{dem}(H_2) \leq \mathrm{dem}(G)$, and so Corollary 4.7 holds.

Corollary 4.7. There exists a connected graph G and two non-spanning subgraphs $H_1, H_2 \sqsubseteq G$ such that $\mathrm{dem}(H_1) \geq \mathrm{dem}(G)$ and $\mathrm{dem}(H_2) \leq \mathrm{dem}(G)$.

Proof of Proposition 1.18: For any graph G with order n and $G \setminus v$ with at least one edge, we have $\mathrm{dem}(G) \leq n-1$ and $\mathrm{dem}(G \setminus v) \geq 1$, and hence $\mathrm{dem}(G) - \mathrm{dem}(G \setminus v) \leq n-2$. Furthermore, let $G = K_3$, then $\mathrm{dem}(G) - \mathrm{dem}(G \setminus v) = n-2$, and hence the upper bound is sharp. Conversely, since $\mathrm{dem}(G) - \mathrm{dem}(G \setminus v) = n-2$, it follows that $\mathrm{dem}(G) = n-1$ and $\mathrm{dem}(G \setminus v) = 1$. From Theorem 1.6, $G \setminus v$ is a tree. Suppose that $|V(G)| \geq 4$. Since $\mathrm{dem}(G) = n-1$, it follows from Theorem 1.6 that $G = K_n$, and hence $G \setminus v = K_{n-1}$ which contradicts to the fact that $G \setminus v$ is a tree. Suppose that $|V(G)| \leq 3$. Since $\mathrm{dem}(G) = n-1$, it follows that $G = K_n$, where $n \leq 3$. If $G = K_2$, then $G \setminus v = K_1$, which contradicts to the fact that $G \setminus v$ contains at least one edge. Therefore, $G = K_3$, as desired. □

Next, we consider the subgraph H of G. If H is a proper subgraph of G satisfying $\mathrm{dem}(H) \leq \mathrm{dem}(G)$, then what is the relation between H and G? A natural question is what is the maximum number of edges we can delete from G without changing the number of distance-edge monitoring? We give a partial answer as follows.

Recall that the base graph G_b is a subgraph of G with $\mathrm{dem}(G) = \mathrm{dem}(G_b)$. Therefore, we can give a lower bound for the edge set E such that $\mathrm{dem}(G) = \mathrm{dem}(G - E)$.

Observation 4.1. Let G be a connected graph, and let $E_1 = E(G) - E(G_b)$. For $E \subseteq E(G)$, if $\mathrm{dem}(G) = \mathrm{dem}(G - E)$ and $G - E$ is a connected graph with order at least 2, then $|E| \geq |E_1|$.

Proof of Theorem 1.19: Let $E \subseteq E(G)$ satisfying $\mathrm{dem}(G) = \mathrm{dem}(G - E)$ and $M = \{u, v\}$ be a DEM set of G with $|M| = \mathrm{dem}(G) = 2$. From Theorem 1.8, we have $|EM(u)| \leq n-1$ and $|EM(v)| \leq n-1$. If $uv \in E(G)$, then $e(G) \leq 2(n-1)-1$. Since $\mathrm{dem}(G) = \mathrm{dem}(G - E) = 2$, it follows from Theorem 1.6 that $G - E$ must contain a cycle, and hence $|E| \leq 2(n-1)-1-3 = 2n-6$.

Suppose that $uv \notin E(G)$. If $|EM(u) \cap EM(u)| \geq 1$, then $|E| \leq 2n-6$, which is similar to the case that $uv \in E(G)$. If $EM(u) \cap EM(v) = \emptyset$, it follows from Theorem 1.8 that $e(G) \leq 2(n-1)$. Since $\mathrm{dem}(G - E) = 2$, then it follows from Theorem 1.6 that $G - E$ must contain a cycle, and hence $|E| \leq 2(n-1) - 3 = 2n-5$. Furthermore, we give the following claim.

Claim 4.8. $|E| \leq 2n-6$.

Proof:
Assume, to the contrary, that $|E| = 2n-5$. Since $\mathrm{dem}(G - E) = 2$, it follows from Theorem 1.6 that $G - E = C_3$. Without loss of generality, let $V(G - E) = \{v_1, v_2, v_3\}$. In addition, from Theorem 1.8, the subgraph induced by the edge set $EM(u)$ and $EM(v)$ are the spanning trees of G. If $u, v \in \{v_1, v_2, v_3\}$, then $uv \in E(G)$, a contradiction. Thus, $u \notin \{v_1, v_2, v_3\}$ or $v \notin \{v_1, v_2, v_3\}$. Without loss generality, suppose that $u \notin \{v_1, v_2, v_3\}$.

If $d_G(u) = 1$, then $|N(u)| = 1$. Let $N(u) = \{w\}$. Since the subgraph induced by the edge set $EM(u)$ and $EM(v)$ are the spanning trees of G, it follows that $uw \in EM(u) \cap EM(v)$, which contradicts to the fact that $EM(u) \cap EM(v) = \emptyset$. Therefore, $d_G(u) \geq 2$. Since the subgraph induced by the edge set $EM(v)$ is a spanning tree of G, it follows that there exists a vertex $u_1 \in N(u)$ such that $uu_1 \in EM(v)$. From Theorem 1.4, we have $uu_1 \in EM(u)$, and hence $uu_1 \in EM(u) \cap EM(v)$, which contradicts to the fact that $EM(u) \cap EM(u) = \emptyset$. □

From Claim 4.8, we have $|E| \leq 2n - 6$. Furthermore, let $G = (n-2)K_1 \vee K_2$ with vertex set $V(G) = \{v_i \,|\, 1 \leq i \leq n\}$ and edge set $E(G) = \{v_1v_2\} \cup \{v_1v_i, v_2v_i | 3 \leq i \leq n\}$. Then, $\operatorname{dem}(G) = 2$. Let $E = \{v_1v_i, v_2v_i \,|\, 4 \leq i \leq n\} \subseteq E(G)$. From Observation 3.1, $\operatorname{dem}(G - E) = \operatorname{dem}(K_3)$ $+(n-1)\operatorname{dem}(K_1) = 2$, and hence there exists an edge set E_1 such that $\operatorname{dem}(G - E) = 2$ and $|E| = 2n - 6$, as desired. □

In the end of this section, we give the proof of Theorem 1.22 as follows.

Proof of Theorem 1.22: Let $G = K_n$ with vertex set $\{v_i \,|\, 1 \leq i \leq n\}$ and edge set $\{v_iv_j \,|\, 1 \leq i < j \leq n\}$. Let T be a spanning tree in K_n. For any edge $uv \in E(T)$ and vertex $w \in ((N_G(u) \cup N_G(v)) \setminus \{u, v\}) \cap V(T)$, we have $d_G(w, u) = d_G(w, v) = 1$ and $d_{G-uv}(w, u) = d_{G-uv}(w, v) = 1$, and hence $uv \notin EM(w)$. From Proposition 3.1, any edge $uv \in E(T)$ is only monitored by u or v, and hence $\operatorname{dem}(K_n|_T) \geq \beta(T)$. From Theorem 1.9, $\operatorname{dem}(K_n|_T) \leq \beta(T)$, and hence $\operatorname{dem}(K_n|_T) = \beta(T)$. Since T is tree with order n, it follows that T is a bipartite graph. Without loss of generality, let $V(T) = U \cup V$ ($|U| \leq |V|$), which is a bipartite partition of $V(T)$. From the pigeonhole principle, we have $|U| \leq \lfloor \frac{n}{2} \rfloor$. For any $uv \in E(T)$, we have $\{u, v\} \cap U \neq \emptyset$, and hence $\beta(T) \leq \lfloor \frac{n}{2} \rfloor$. In addition, T contains at least one edge, and hence $\beta(T) \geq 1$, and so $1 \leq \beta(T) \leq \lfloor \frac{n}{2} \rfloor$.

Suppose that $T = S_n$ with vertex set $\{v_i \,|\, 1 \leq i \leq n\}$ and edge set $E(S_n) = \{v_1v_i \,|\, 2 \leq i \leq n\}$. Then $\{v_1\}$ is the vertex cover set of S_n, and hence $\beta(T) = 1$, and so the lower bound is sharp. Suppose that $T = P_n$ with vertex set $\{v_i \,|\, 1 \leq i \leq n\}$ and edge set $E(P_n) = \{v_iv_{i+1} \,|\, 1 \leq i \leq n-1\}$. Then, $\{v_i \,|\, i \equiv 0 \pmod 2, 1 \leq i \leq n\}$ is a minimum vertex cover set of P_n, and hence $\beta(T) = \lfloor \frac{n}{2} \rfloor$, and so the upper bound is sharp. □

Similar to Theorem 1.22, we have the following corollary.

Corollary 4.9. Let H be a subgraph of G with $|V(H)| = p$. Then,

$$1 \leq \operatorname{dem}(G|_H) \leq p - 1.$$

Furthermore, the bounds are sharp.

Theorem 4.10. If H is a connected induced subgraph of graph G, then

$$\operatorname{dem}(G) - \operatorname{dem}(G|_H) \leq |V(G)| - |V(H)|.$$

Moreover, if G and H are both complete graphs, then the bound is sharp.

Proof:
For any graph G and H, where H is an induced subgraph of G. Let $M_1 \subseteq V(H)$ be a restrict-DEM set of H in G with $|M_1| = \text{dem}(G|_H)$. Let $M = (V(G) - V(H)) \cup M_1$. We will prove that $\text{dem}(G) \leq |M|$. For any edge $uv \in E(G)$, if u or v in $V(G) - V(H)$, then it follows from Theorem 1.4 that e is monitored by the vertex in $V(G) \setminus V(H)$. For any edge $e \in E(H)$, since M_1 is a restrict-DEM set, it follows that e is monitored by the vertex in M_1, and hence M is a DEM set in G. Since $|M| = |M_1| + (|V(G)| - |V(H)|) = |V(G)| - |V(H)| + \text{dem}(G|_H)$, it follows that $\text{dem}(G) \leq |M| = \text{dem}(G|_H) + (|V(G)| - |V(H)|)$, and so $\text{dem}(G) - \text{dem}(G|_H) \leq (|V(G)| - |V(H)|)$. Furthermore, let $G = K_n$ and $H = K_m$ ($3 \leq m \leq n$). From Theorem 1.7, $\text{dem}(G) = n - 1$. For any $uv \in E(H)$ and $w \in (N(u) \cup N(v)) \setminus \{u, v\}$, we have $d_G(w, u) = d_G(w, v) = 1$ and $d_{G-uv}(w, u) = d_{G-uv}(w, v) = 1$, and hence $uv \notin EM(w)$. From Proposition 3.1, any edge $uv \in E(H)$ is only monitored by u or v, and so $\text{dem}(K_n|_{K_m}) \geq \beta(K_m) = m - 1$. From Theorem 1.9, $\text{dem}(K_n|_{K_m}) \leq \beta(K_m) = m - 1$, and hence $\text{dem}(K_n|_{K_m}) = m - 1$. Therefore, $\text{dem}(G) - \text{dem}(G|_H) = (n - 1) - (m - 1) = |V(G)| - |V(H)|$, as desired. □

5. Perturbation results for some known graphs

Firstly, we study the change of DEM numbers for some well-known graphs when any edge (or vertex) of the graph is deleted.

5.1. Deleting one edge or vertex from some known graphs

Let $NV_i(G) = \{v \mid d_G(v) = i, v \in V(G)\}$ and $NE_{a,b}(G) = \{uv \mid uv \in E(G), d_G(u) = a, d_G(v) = b\}$. Note that $E(P_n) = NE_{1,2}(P_n) \cup NE_{2,2}(P_n)$. If $e \in NE_{1,2}(P_n)$, then $\text{dem}\,(P_n - e) = 1$, and hence $\text{dem}\,(P_n) - \text{dem}\,(P_n - e) = 0$. If $e \in NE_{2,2}(P_n)$, then $\text{dem}\,(P_n - e) = 2$, and hence $\text{dem}\,(P_n) - \text{dem}\,(P_n - e) = -1$.

Corollary 5.1. Let P_n be a path of order n, where $n \geq 2$. For any $e \in E(P_n)$, we have

$$\text{dem}(P_n - e) = \begin{cases} \text{dem}(P_n), & \text{if } e \in NE_{1,2}(P_n); \\ \text{dem}(P_n) + 1, & \text{if } e \in NE_{2,2}(P_n). \end{cases}$$

Foucaud et al. [10] obtained the DEM numbers of complete bipartite graph K_{ℓ_1,ℓ_2}.

Theorem 5.2. [10] Let ℓ_1 and ℓ_2 be two integers with $\ell \geq 1$ and $\ell_2 \geq 1$. Then

$$\text{dem}\,(K_{\ell_1,\ell_2}) = \min\{\ell_1, \ell_2\}.$$

The following corollary is immediate.

Corollary 5.3. Let $n \geq 3$ be an integer. Then,
(i) for any edge $e \in E(C_n)$, $\text{dem}(C_n - e) = \text{dem}(C_n) - 1 = 1$;
(ii) for any edge $e \in E(K_n)$, $\text{dem}(K_n - e) = \text{dem}(K_n) - 1 = n - 2$;
(iii) for any edge $e \in E(K_{n,n})$, $\text{dem}(K_{n,n} - e) = \text{dem}(K_{n,n}) = n$.

Proof:

(i) From Theorem 1.6, we have $\mathrm{dem}(C_n - e) = \mathrm{dem}(P_n) = 1$. Since $\mathrm{dem}(C_n) = 2$, it follows that $\mathrm{dem}(C_n - e) = \mathrm{dem}(C_n) - 1 = 1$.

(ii) Let $G = K_n$. From Theorem 1.7, we have $\mathrm{dem}(G) = n-1$ and $\mathrm{dem}(G-uv) \leq n-2$ for any edge $uv \in E(G)$. Then, we prove that $\mathrm{dem}(G-uv) \geq n-2$. Suppose that $\mathrm{dem}(G-uv) \leq n-3$. Let M be a DEM set of $G - uv$ with the minimum cardinality. For any edge $xy \in E(G - uv)$ and vertex $w \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}$, we have $d_G(w, x) = d_{G-xy}(w, x)$ and $d_G(w, y) = d_{G-xy}(w, y)$, and hence $xy \notin EM(w)$. From Proposition 3.1, any edge $xy \in E(G - uv)$ is only monitored by x and y, and hence $M \cap \{x, y\} \neq \emptyset$. If $u, v \in M$, then there exist two vertices $v_1, v_2 \in V(G) \setminus \{u, v\}$ such that $v_1, v_2 \notin M$, and hence $v_1v_2 \notin \cup_{x\in M} EM(x)$, which contradicts to the fact that $\cup_{x\in M} EM(x) = E(G - uv)$. Suppose that u or $v \notin M$. Without loss of generality, let $u \notin M$. Then there exists a vertex $v_1 \in V(G)\setminus\{u, v\}$ such that $v_1 \notin M$, and hence $uv_1 \notin \cup_{x\in M} EM(x)$, which contradicts to the fact that $\cup_{x\in M} EM(x) = E(G-uv)$. Therefore, $\mathrm{dem}(G-uv) \leq n-2$, and so $\mathrm{dem}(G-uv) = n-2$. This implies that $\mathrm{dem}(K_n - e) = \mathrm{dem}(K_n) - 1 = n - 2$.

(iii) Let $G = K_{n,n}$ with vertex set $V(G) = \{u_i \,|\, 1 \leq i \leq n\} \cup \{v_i \,|\, 1 \leq i \leq n\}$ and edge set $E(G) = \{u_iv_j \,|\, 1 \leq i, j \leq n\}$. From Theorem 5.2, we have $\mathrm{dem}(K_{n,n}) = n$. Without loss of generality, let $G_1 = G - u_1v_1$. Firstly, we prove that $\mathrm{dem}(G_1) \geq n$. Suppose that M is a DEM set of G_1 with $|M| = n - 1$. Then, there exists two vertices $u_p, v_q \notin M$ with $u_pv_q \in E(G_1)$. For any $u \in M \cap \{u_i \,|\, 1 \leq i \leq n\}$, we have $d_{G_1-u_pv_q}(u, u_p) = d_{G_1}(u, u_p) = 2$ and $d_{G_1-u_pv_q}(u, v_q) = d_{G_1}(u, v_q) = 1$, and hence $u_pv_q \notin EM(u)$. Similarly, $u_pv_q \notin EM(v)$ for any $v \in M \cap \{v_i \,|\, 1 \leq i \leq n\}$, and hence $u_pv_q \notin \cup_{x\in M} EM(x)$, which contradicts to the fact that $\cup_{x\in M} EM(x) = E(G_1)$. Therefore, $\mathrm{dem}(G_1) \geq n$. Then, we prove that $\mathrm{dem}(G_1) \leq n$. Let $M = \{u_i \,|\, 1 \leq i \leq n\}$. Then M is a vertex cover set of G_1. From Theorem 1.9, $\mathrm{dem}(G_1) \leq |M| = n$, and hence $\mathrm{dem}(G_1) = n$. Therefore, $\mathrm{dem}(K_{n,n} - e) = \mathrm{dem}(K_{n,n}) = n$. □

There exist three graphs G_1, G_2, G_3 and $v \in V(G)$ such that $\mathrm{dem}(G_1) > \mathrm{dem}(G_1-v)$, $\mathrm{dem}(G_2) = \mathrm{dem}(G_2 - v)$, $\mathrm{dem}(G_3) < \mathrm{dem}(G_3 - v)$, respectively.

The following corollary is immediate.

Corollary 5.4. Let $n \geq 3$ be an integer. For any $v \in v(G)$, we have

(i) $\mathrm{dem}(C_n \setminus v) = \mathrm{dem}(C_n) - 1 = 1$;
(ii) $\mathrm{dem}(K_n \setminus v) = \mathrm{dem}(K_n) - 1 = n - 2$;
(iii) $\mathrm{dem}(K_{n,n} \setminus v) = \mathrm{dem}(K_{n,n}) - 1 = n - 1$.

Proposition 5.5. Let P_n be a path with vertex set $\{v_i \,|\, 1 \leq i \leq n\}$ and edge set $\{v_iv_{i+1} \,|\, 1 \leq i \leq n - 1\}$, where $n \geq 5$. For any $v \in V(P_n)$, we have

$$\mathrm{dem}(P_n \setminus v) = \begin{cases} \mathrm{dem}(P_n), & \text{if } v \in \{v_1, v_2, v_{n-1}, v_n\}; \\ \mathrm{dem}(P_n) + 1, & \text{if } v \in \{v_i \,|\, 3 \leq i \leq n - 2\}. \end{cases}$$

Proof:
From Theorem 1.6, we have $\text{dem}\,(P_n) = 1$. If $v \in \{v_1, v_n\}$, then $\text{dem}\,(P_n \setminus v) = 1$. If $v \in \{v_2, v_{n-1}\}$, then $\text{dem}\,(P_n \setminus v) = \text{dem}\,(P_{n-1}) + \text{dem}\,(K_1) = 1$, where $V(K_1) = \{v_1\}$ or $V(K_1) = \{v_n\}$, and hence $\text{dem}\,(P_n) - \text{dem}\,(P_n \setminus v) = 0$. If $v \in \{v_i \,|\, 3 \leq i \leq n-2\}$, then $\text{dem}\,(P_n - v) = 2$, and hence $\text{dem}\,(P_n) - \text{dem}\,(P_n \setminus v) = -1$. □

Proposition 5.5 shows that there exists a graph G and $v \in V(G)$ such that $\text{dem}(G) = \text{dem}(G \setminus v)$ or $\text{dem}(G) < \text{dem}(G \setminus v)$.

5.2. Whether the DEM set is still applicable?

Foucaud et al. proved in [10] that the problem DEM SET is NP-complete. For any graph G, a natural question is whether the original DEM set can monitor all edges if some edges or vertices in G are delated. We design the Algorithm 1 and the time complexity is polynomial.

WHETHER THE DEM SET IS STILL APPLICABLE?
Instance: A graph $G = (V, E)$, an edge $e \in E(G)$ and a DEM set M of G.
Question: Whether M is still a DEM set for the graph $G - e$?

Given a graph G, a DEM set M, and an edge $e \in E(G)$, our goal is to determine whether the original DEM set M is still valid in the resulting graph $G - e$. The algorithm is shown in Algorithm 1.

Algorithm 1 The algorithm for determining M is or not monitor set for $G - e$

Input: A graph G, $M \subseteq V(G)$ and $e \in E(G)$;
Output: $E(G-e) \subseteq \cup_{x \in M} EM(x)$ is TRUE or FALSE;
$M_1 \leftarrow E(G-e)$
for each vertex $v \in M$ **do**
    $M_1 \leftarrow M_1 - EM(v)$
end for
if $M_1 = \emptyset$ **then return** $E(G-e) \subseteq \cup_{x \in M} EM(x)$ is TRUE;
else return $E(G-e) \subseteq \cup_{x \in M} EM(x)$ is FALSE;
end if

The algorithm of how to compute the edge set $EM(x)$ from G is polynomial by the breadth-first spanning tree algorithm. Hence the time complexity of Algorithm 1 is polynomial.

6. Conclusion

In this paper, we studied the effect of deleting edges and vertices in a graph G on the DEM number. We obtained that $\text{dem}(G-e) - \text{dem}(G) \leq 2$ for any graph G and $e \in E(G)$. Furthermore, the bound is sharp. In addition, we can find a graph H and $v \in V(H)$ such that $\text{dem}(H \setminus v) - \text{dem}(H)$ can be arbitrarily large. This fact gives an answer to the monotonicity of the DEM number. This means that there exist two graphs H and G with $H \sqsubseteq G$ such that $\mathrm{dem}(H) \geq \mathrm{dem}(G)$.

It is interesting to consider the following problems for future work.

(1) Characterize the graphs $\mathrm{dem}(H) \geq \mathrm{dem}(G)$ if $H \sqsubseteq G$.

(2) For a graph G and $E \subseteq E(G)$, what is the maximum value of $|E|$ such that $\mathrm{dem}(G) = \mathrm{dem}(G - E)$?

(3) For any $\epsilon > 0$, whether the ratio $\frac{\mathrm{dem}(G)}{\mathrm{dem}(H)} \leq \epsilon$ holds, where H is an induced subgraph of G.

In addition, it would be interesting to study distance-edge monitoring sets in further standard graph classes, including circulant graphs, graph products, or line graphs. In addition, characterizing the graphs with $\mathrm{dem}(G) = n - 2$ would be of interest, as well as clarifying further the relation of the parameter $\mathrm{dem}(G)$ to other standard graph parameters, such as arboricity, vertex cover number and feedback edge set number.

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