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Elimination Distance to Bounded Degree on Planar Graphs Preprint

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Abstract. We study the graph parameter *elimination distance to bounded degree*, which was introduced by Bulian and Dawar in their study of the parameterized complexity of the graph isomorphism problem. We prove that the problem is fixed-parameter tractable on planar graphs, that is, there exists an algorithm that given a planar graph G and integers d, k decides in time $f(k, d) \cdot n^c$ for a computable function f and constant c whether the elimination distance of G to the class of degree d graphs is at most k.

1. Introduction

Structural graph theory offers a wealth of parameters that measure the complexity of graphs or graph classes. Among the most prominent parameters are *treedepth* and *treewidth*, which intuitively measure the resemblance of graphs with stars and trees, respectively. Other commonly studied structurally restricted graph classes are the class of *planar graphs*, classes that exclude a fixed graph as a *minor* or *topological minor*, classes of *bounded expansion* and *nowhere dense* classes.

Once we have gained a good understanding of a graph class $\mathscr{C}$, it is natural to study classes whose members are *close* to graphs in $\mathscr{C}$. One of the simplest measures of distance to a graph class $\mathscr{C}$ is the number of vertices or edges that one must delete (or add) to a graph G to obtain a graph from $\mathscr{C}$. Guo et al. [1] formalized this concept under the name *distance from triviality*. For example, the size of a vertex cover is the distance to the class of edgeless graphs and the size of a feedback vertex set is the distance to the class of acyclic graphs. More generally, for a graph G a vertex set X is called a

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c-treewidth modulator if the treewidth of $G-X$ is at most c, hence, the size of a c-treewidth modulator corresponds to the distance to the class of graphs of treewidth at most c. This concept was introduced and studied by Gajarský et al. in [2].

The *elimination distance* to a class $\mathscr{C}$ of graphs measures the number of recursive deletions of vertices needed for a graph G to become a member of $\mathscr{C}$. More precisely, a graph G has elimination distance 0 to $\mathscr{C}$ if $G \in \mathscr{C}$, and otherwise elimination distance $k+1$, if in every connected component of G we can delete a vertex such that the resulting graph has elimination distance k to $\mathscr{C}$. Elimination distance was introduced by Bulian and Dawar [3] in their study of the parameterized complexity of the graph isomorphism problem. Elimination distance naturally generalizes the concept of treedepth, which corresponds to the elimination distance to the class $\mathscr{C}_0$ of edgeless graphs.

Elimination distance has very nice algorithmic applications. On the one hand, small elimination distance to a class $\mathscr{C}$ on which efficient algorithms for certain problems are known to exist, may allow to lift the applicability of these algorithms to a larger class of graphs. For example, Bulian and Dawar [3] showed that the graph isomorphism problem is fixed-parameter tractable when parameterized by the elimination distance to the class $\mathscr{C}_d$ of graphs with maximum degree bounded by d, for any fixed integer d. Recently, Hols *et al.* [4] proved the existence of polynomial kernels for the vertex cover problem parameterized by the size of a deletion set to graphs of bounded elimination distance to different classes of graphs. The related concept of recursive backdoors has recently been studied in the context of efficient SAT solving [5, 6].

On the other hand, it is an interesting algorithmic question by itself to determine the elimination distance of a given graph G to a class $\mathscr{C}$ of graphs. It is well known (see e.g. [7, 8, 9]) that computing treedepth, i.e. elimination distance to $\mathscr{C}_0$, is fixed-parameter tractable. More precisely, we can decide in time $f(k) \cdot n$ whether an n-vertex graph G has treedepth at most k. Bulian and Dawar proved in [10] that computing the elimination distance to any minor-closed class $\mathscr{C}$ is fixed-parameter tractable when parameterized by the elimination distance. They also raised the question whether computing the elimination distance to the class $\mathscr{C}_d$ of graphs with maximum degree at most d is fixed-parameter tractable when parameterized by the elimination distance and d. Note that this question is not answered by their result for minor-closed classes, since $\mathscr{C}_d$ is not closed under taking minors.

For $k, d \in \mathbb{N}$, we denote by $\mathscr{C}_{k,d}$ the class of all graphs that have elimination distance at most k to $\mathscr{C}_d$. It is easy to see that for every fixed k and d we can formulate the property that a graph is in $\mathscr{C}_{k,d}$ by a sentence in monadic second-order logic (MSO). By the famous theorem of Courcelle [11] we can test every MSO-property φ in time $f(|\varphi|, t) \cdot n$ on every n-vertex graph of treewidth t for some computable function f. Hence, we can decide for every n-vertex graph G of treewidth t whether $G \in \mathscr{C}_{k,d}$ in time $f(k, d, t) \cdot n$ for some computable function f. However, for $d \geq 3$ already the class $\mathscr{C}_d$ has unbounded treewidth, and so the same holds for $\mathscr{C}_{k,d}$ for all values of k. Thus, Courcelle's Theorem cannot be directly applied to derive fixed-parameter tractability of the problem in full generality.

On the other hand, it is easy to see that the graphs in $\mathscr{C}_{k,d}$ exclude the complete graph K_{k+d+2} as a topological minor, and hence, for every fixed k and d, the class $\mathscr{C}_{k,d}$ in particular has bounded expansion and is nowhere dense. We can efficiently test first-order (FO) properties on bounded expansion and nowhere dense classes [12, 13], however, first-order logic is too weak to express the elimination distance problem. This follows from the fact that first-order logic is too weak to express even connectivity of a graph or to define connected components.

While we are unable to resolve the question of Bulian and Dawar in full generality, in this work we initiate the quest of determining the parameterized complexity of elimination distance to bounded degree graphs for restricted classes of inputs. We prove that for every n-vertex graph G that excludes K_5 as a minor (in particular for every planar graph) we can test whether $G \in \mathscr{C}_{k,d}$ in time $f(k,d) \cdot n^c$ for a computable function f and constant c. Hence, the problem is fixed-parameter tractable with parameters k and d when restricted to K_5-minor-free graphs.

Theorem 1.1. (Main result)
There is an algorithm that for a K_5-minor-free input graph G with n vertices and integers k, d, decides in time $f(k,d) \cdot n^c$ whether G belongs to $\mathscr{C}_{k,d}$, where f is a computable function and c is a constant.

In case G is K_5- and $K_{3,3}$-minor-free (that is, G is planar), the running time is $f(k,d) \cdot n^3$ for a computable function f.

Observe that the result is not implied by the result of Bulian and Dawar for minor-closed classes, as the K_5-minor-free subclass of $\mathscr{C}_d$ is not minor-closed. It is natural to consider as a next step classes that exclude some fixed graph as a minor or as a topological minor, and finally to resolve the problem in full generality.

To solve the problem on K_5-minor-free graphs we combine multiple techniques from parameterized complexity theory and structural graph theory. First, we use the fact that the property of having elimination distance at most k to $\mathscr{C}_d$ for fixed k and d is MSO definable, and hence efficiently solvable by Courcelle's Theorem on graphs of bounded treewidth. If the input graph G has small treewidth, we can hence solve the instance by Courcelle's Theorem. If G has large treewidth, we conclude that it contains a large grid minor [14]. We then have two cases.

First, if in sufficiently many branch sets there are vertices of degree exceeding d, we conclude that the graph does not belong to $\mathscr{C}_{k,d}$. This is because the grid minor will not be sufficiently scattered by k elimination rounds.

Second, if only few branch sets contain vertices of degree exceeding d, we can find a branch set that is "far" from any problematic vertices. The vertices of such branch set are called *irrelevant*, that is, vertices whose deletion does not change containment in $\mathscr{C}_{k,d}$. By iteratively removing irrelevant vertices we arrive either in the first case, or at an instance of small treewidth. In both cases, we can conclude. The irrelevant vertex technique was introduced in [15] and is by now a standard technique in parameterized algorithms, see [16] for a survey.

Recent advances. We first presented Theorem 1.1 in a conference paper [17]. In this journal version, the presentation of the proof has been vastly improved and simplified. While the main result is the same, some observations enabled us to remove unnecessary complicated notion and to simplify the proof.

Since the first publication of our results much further work has been done on algorithms computing elimination distances. Most notably, Agrawal et al. [18] establishing fixed-parameter tractability of the problem on general graphs. In fact, they showed that their approach yields an efficient algorithm to decide the elimination distance to any graph class that can be described by a finite set of forbidden induced subgraphs. In a second work it was shown that there exists a FPT algorithm for computing elimination distance to classes characterized by the exclusion of a family of finite graphs as topological minors [19]. In another line of research, Agrawal and Ramanujan initiated the study of computing elimination distance to the class of cluster graphs [20].

Very recently, Diner et al. introduced in [21] the notion of block elimination distance, which replaces the connectivity property in Bulian and Dawar's classical notion of elimination distance with biconnectivity, while Fomin, Golovach and Thilikos further generalized several of these prior results by considering elimination distances to graph classes expressible by a restricted first-order logic formulas [22]. The results are also implied by the very recent algorithmic meta theorem for separator logic [23, 24].

2. Preliminaries

A *graph* G consists of a set of vertices $V(G)$ and a set of edges $E(G)$. We assume that graphs are finite, simple and undirected, and we write $\{u, v\}$ for an edge between the vertices u and v. For a set of vertices $S \subseteq V(G)$, we denote the subgraph of G induced by the vertices $V(G) \setminus S$ by $G - S$. If $S = \{a\}$, we write $G - a$. The *degree* of a vertex v is the number of edges e such that $v \in e$. The maximum degree of a graph is the largest degree of its vertices.

A *partial order* on a set V is a binary relation $\leq$ on V that is reflexive, anti-symmetric and transitive. A set $W \subseteq V$ is a *chain* if it is totally ordered by $\leq$. If $\leq$ is a partial order on V, and for every element $v \in V$ the set $V_{\leq v} := \{u \in V \mid u \leq v\}$ is a chain, then $\leq$ is a *tree order*. Note that the covering relation of a tree order is not necessarily a tree, but may be a forest. An *elimination order* on a graph G is a tree order $\leq$ on $V(G)$ such that for every edge $\{u, v\} \in E(G)$ we have either $u \leq v$ or $v \leq u$. The *depth* of a vertex v in a tree order $\leq$ is the size the set $V_{<v} := \{u \in V \mid u < v\}$. The *depth* of a tree order $\leq$ is maximal depth among all vertices.

The *treedepth* of a graph G is defined recursively as follows.

$$\mathrm{td}(G) = \begin{cases} 0 & \text{if } G \text{ is edgeless,} \\ 1 + \min\{\mathrm{td}(G - v) \mid v \in V(G)\} & \text{if } G \text{ is connected and not edgeless,} \\ \max\{\mathrm{td}(H) \mid H \text{ connected component of } G\} & \text{otherwise.} \end{cases}$$

A graph G has treedepth at most k if and only if there exists an elimination order on G of depth at most k. If the longest path in G has length k, then its treedepth is bounded by k and an elimination order of at most this depth can be found in linear time by a depth-first-search.

Elimination distance to a class $\mathscr{C}$ naturally generalizes the concept of treedepth. Let $\mathscr{C}$ be a class of graphs. The *elimination distance* of G to $\mathscr{C}$ is defined recursively as

$$\mathrm{ed}_{\mathscr{C}}(G) = \begin{cases} 0 & \text{if } G \in \mathscr{C}, \\ 1 + \min\{\mathrm{ed}_{\mathscr{C}}(G - v) \mid v \in V(G)\} & \text{if } G \notin \mathscr{C} \text{ and } G \text{ is connected,} \\ \max\{\mathrm{ed}_{\mathscr{C}}(H) \mid H \text{ connected component of } G\} & \text{otherwise.} \end{cases}$$

We denote by $\mathscr{C}_d$ the class of all graphs of maximum degree at most d and by $\mathscr{C}_{k,d}$ the class of all graphs with elimination distance at most k to $\mathscr{C}_d$. Note for instance that $\mathrm{td}(G) = k$ if and only if $G \in \mathscr{C}_{k,0}$. We write $\mathrm{ed}_d(G)$ for $\mathrm{ed}_{\mathscr{C}_d}(G)$.

Definition 2.1. (Definition 4.2 of [3])
A tree order $\leq$ on G is an *elimination order to degree* d if for every $v \in V(G)$ the set $S_v := \{u \in G \mid \{u, v\} \in E(G), u \not\leq v$ and $v \not\leq u\}$ satisfies either:

1. $S_v = \emptyset$ or
2. v is $\leq$-maximal, $|S_v| \leq d$, and for all $u \in S_v$, we have $\{w \mid w < u\} = \{w \mid w < v\}$.

A more general notion of elimination order to a class $\mathscr{C}$ was given in the dissertation thesis of Bulian [25], which is however not needed in this generality for our purpose.

Proposition 2.2. (Proposition 4.3 in [3])
A graph G satisfies $\mathrm{ed}_d(G) \leq k$ if, and only if, there exists an elimination order to degree d of depth k for G.

The following lemma is easily proved by induction on k.

Lemma 2.3. For every graph G and elimination order $\leq$ to degree d for G, we can compute in quadratic time an elimination order $\preceq$ to degree d for G, with depth not larger than the depth of $\leq$, and with the additional property that for every $v \in V(G)$, if C, C' are distinct connected components of $G - V_{\preceq v}$ (or of G), then the vertices of C and C' are incomparable with respect to $\preceq$.

Let G be a graph. A graph H with vertex set $\{v_1, \ldots, v_n\}$ is a *minor* of G, written $H \preceq G$, if there are connected and pairwise vertex disjoint subgraphs $H_1, \ldots, H_n \subseteq G$ such that if $\{v_i, v_j\} \in E(H)$, then there are $w_i \in V(H_i)$ and $w_j \in V(H_j)$ such that $\{w_i, w_j\} \in E(G)$. We call the subgraph H_i the *branch set* of the vertex v_i in G. If all H_i have radius at most r, then we say that H is a *depth-r minor* of G, in symbols $H \preceq_r G$. If G is a graph and $\mathcal{H} = \{H_1, \ldots, H_n\}$ is a set of pairwise vertex disjoint subgraphs of G, then the graph H with vertex set $\{v_1, \ldots, v_n\}$ and edges $\{v_i, v_j\} \in E(H)$ if and only if there is an edge between a vertex of H_i and a vertex of H_j in G, the *minor induced by* $\mathcal{H}$. If $\bigcup_{1\leq i\leq n} V(H_i) = V(G)$, we call $\mathcal{H}$ a *minor model of* H that *subsumes all vertices* of G.

We denote by K_t the complete graph on t vertices. We denote by $G_{m\times n}$ the grid with m rows and n columns, that is, the graph with vertex set $\{v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and edges $\{v_{i,j}, v_{i',j'}\}$ for $|i - i'| + |j - j'| = 1$.

For our purpose we do not have to define the notion of treewidth formally. It is sufficient to note that if a graph G contains an $n \times n$ grid as a minor, then G has treewidth at least $n + 1$ and vice versa, that large treewidth forces a large grid minor, as stated in the next theorem.

Theorem 2.1. (Excluded Grid Theorem)
There exists a function g such that for every integer $t \geq 1$, every graph of treewidth at least $g(t)$ contains the $t \times t$ grid as a minor. Furthermore, such a grid minor can be computed in polynomial time.

The theorem was first proved by Robertson and Seymour in [14]. Improved bounds and corresponding efficient algorithms were subsequently obtained. We refer to the work of Chuzhoy and Tan [26] for the currently best known bounds on the function g and further pointers to the literature concerning efficient algorithms. For planar graphs we employ the following much better bounds.

Theorem 2.2. (Planar Excluded Grid Theorem, see Theorem 7.23 of [27])
There exists an $O(n^2)$ algorithm that, for a given n-vertex planar graph G and integer t either outputs a tree decomposition of G of width at most $5t$, or construct a minor model of the $t \times t$-grid in G.

The second black-box we use is Courcelle's Theorem, stating that we can test MSO properties efficiently on graphs of bounded treewidth. We use standard notation from logic and refer to the literature for all undefined notation, see e.g. [28].

Theorem 2.3. (Courcelle's Theorem [11])
There exists a function f such that for every MSO-sentence φ and every n-vertex graph G of treewidth t we can test whether $G \models \varphi$ in time $f(|\varphi|, t) \cdot n$.

3. The proof

We first show that in a grid not many vertices can be affected by k recursive deletions. This is due to the following unbreakability property of grids.

A *separation* in a graph G is a pair of vertex subsets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$ and there is no edge with one endpoint in $A \setminus B$ and the other in $B \setminus A$. The *order* of the separation (A, B) is the size of its *separator* $A \cap B$.

For $q, k \in \mathbb{N}$, a graph G is (q, k)-*unbreakable* if for every separation (A, B) of G of order at most k, we have $|A| \leq q$ or $|B| \leq q$.

Lemma 3.1. For all integers k, m, the $m \times m$-grid $G_{m \times m}$ is (k^2, k)-unbreakable.

Proof:
Let $G = G_{m \times m}$ and let A, B be a separation of order at most k in G. We show that $|A| \leq k^2$ or $|B| \leq k^2$. The statement is trivially true if $m \leq k$ as then $|V(G)| \leq k^2$. Hence, assume that $m > k$, and let $\overline{z} := z_1, \dots, z_k = A \cap B$.

We call a row or a column of G *untouched* if it does not intersect $\overline{z}$ and *touched* if it does intersect $\overline{z}$. As there are at least $k + 1$ rows (resp. columns) there is at least one untouched row (resp. column). An untouched row (resp. column) must be contained in one part of the separation. Assume w.l.o.g. that A contains an untouched row. Then A contains all untouched columns and, by symmetry, all untouched rows. Therefore every element in B must be part of a touched row and a touched column, and, as there are at most k touched rows resp. columns, there are at most k^2 such elements. □

Observe that when a graph with at least $3q$ vertices is (q, k)-unbreakable, then removing any k vertices from G leaves all but at most q vertices in the same connected component. Hence, this component does not break in the recursive process, and in particular, we delete at most k vertices from that component. On the other hand, the small part of size at most q can possibly be completely eliminated in the recursive process.

Corollary 3.2. (of Lemma 3.1)
For all integers k, m with $k < m$, for all connected graphs G for which there exists a minor model $\mathcal{G}_{m,m} = \{H_{i,j} \mid 1 \leq i, j \leq m\}$ that subsumes all vertices of G, k rounds of recursive elimination on G can delete vertices in at most k^2 many different branch sets.

Definition 3.3. Let G be a K_5-minor-free graph and let $k, d \in \mathbb{N}$. Assume there exists a minor model $\mathcal{G}_{m,m} = \{H_{i,j} \mid 1 \leq i, j \leq m\}$ (for $m \geq 4k + 7$) that subsumes all vertices of G and that induces a supergraph of the grid $G_{m,m}$. We call the branch set $H_{i,j}$ *(k, d)-safe* if $2k + 3 \leq i, j \leq m - 2k - 3$ and if $H_{i',j'}$ contains no vertex of degree at least $d + 1$ (in G) for $|i - i'|, |j - j'| \leq 2k + 2$.

Lemma 3.4. Let G be a K_5-minor-free graph and let $k, d \in \mathbb{N}$. Assume there exists a minor model $\mathcal{G}_{m,m} = \{H_{i,j} \mid 1 \leq i, j \leq m\}$ (for $m \geq 4k + 7$) that subsumes all vertices of G and that induces a supergraph of the grid $G_{m,m}$. Assume $H_{i,j}$ is (k, d)-safe. Let $a \in V(H_{i,j})$, let $B \subseteq V(G) \setminus \{a\}$ with $|B| \leq k$ and let $x, y \in V(G) \setminus (B \cup \{a\})$ be of degree at least $d + 1$. Then x and y are connected in $G - B$ if and only if x and y are connected in $G - B - a$.

Proof:
Assume that x and y are connected in $G - B$ and let P be a path witnessing this. Assume that this path contains the vertex a.

We define sets X_ℓ for $0 \leq \ell \leq k + 1$ of branch sets as follows. Let X_0 be the set consisting only of $H_{i,j}$ and for $\ell \geq 1$ let X_ℓ be the set of all $H_{i',j'}$ with $2\ell - 1 \leq |i - i'|, |j - j'| \leq 2\ell$ that do not already belong to $X_{\ell-1}$. The sets X_ℓ are the borders (of thickness 2) of the $(4\ell + 1) \times (4\ell + 1)$-subgrid

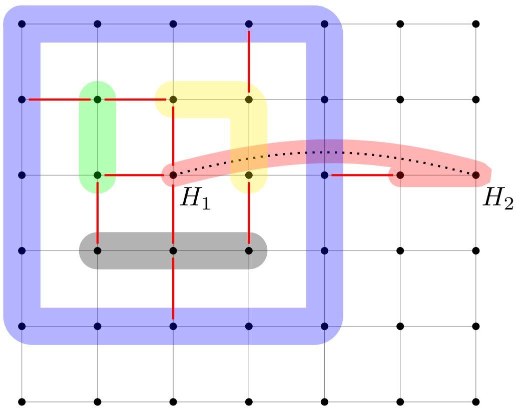

Figure 1. Construction of a K_5 minor as soon as a branch set (here H_1) connected to a branch set (here H_2) that is at distance more than 2 in the grid [27, Figure 7.10]. The blue part marks the "outer part" of the border of thickness 2, the "inner part" decomposes into a green, yellow and gray part.

around $H_{i,j}$. Observe that the vertices x and y do not belong to any of the X_ℓ, as $H_{i,j}$ is (k,d)-safe by assumption. For $0 \leq \ell \leq k+1$ let Y_ℓ be the subgraph of G induced by the vertices of X_ℓ.

We claim that for $1 \leq \ell \leq k+1$, the sets Y_ℓ are connected sets that separate a from x and analogously a from y. Clearly, the Y_ℓ are connected. Now observe that there is no edge between a vertex of $\bigcup_{0 \leq i \leq \ell-1} Y_i$ and a vertex of $G - \bigcup_{0 \leq i \leq \ell} Y_i$. The existence of such a connection would create a K_5 minor, see [27, Figure 7.10] or Figure 1. Hence, any path between a and x (or y) must pass through Y_ℓ.

As $|B| \leq k$, there is one Y_ℓ with $1 \leq \ell \leq k+1$ that does not intersect B. Let u be the first time that P visits Y_ℓ on its way from x to a and let v be the last time that P visits Y_ℓ on its way from a to y. As Y_ℓ is connected, we can reroute the subpath between u and v through Y_ℓ and thereby construct a path between x and y in $G - B - a$. □

Note that in the proof it is important that all vertices belong to some branch set. Otherwise, we could have a vertex x in G that does not belong to any branch set while being adjacent to $H_{i,j}$ and the whole argumentation would fail.

Corollary 3.5. Let G be a K_5-minor-free graph and let $k, d \in \mathbb{N}$. Assume there exists a minor model $\mathcal{G}_{m,m} = \{H_{i,j} \mid 1 \leq i,j \leq m\}$ (for $m \geq 4k+7$) that subsumes all vertices of G and that induces a supergraph of the grid $G_{m,m}$. Assume $H_{i,j}$ is (k,d)-safe. Then every vertex $a \in H_{i,j}$ is *irrelevant*, i.e., $G - a \in \mathscr{C}_{k,d}$ if and only if $G \in \mathscr{C}_{k,d}$.

Proof:
Let $H := G - a$. We have to prove that $H \in \mathscr{C}_{k,d}$ implies $G \in \mathscr{C}_{k,d}$. Hence, assume $H \in \mathscr{C}_{k,d}$. Let $\leq_H$ be an elimination order to degree d of depth k for H. We also assume that $\leq_H$ satisfies the property of Lemma 2.3, that is, for every $v \in V(G)$, if C, C' are distinct connected components of $G - V_{\leq_H v}$, then the vertices of C and C' are incomparable with respect to $\leq_H$. Let A_0 be the connected component of G containing a. Note that A_0 may break into multiple connected components in $H = G - a$. We show below that at most one of these connected components contains a vertex of degree excedeeing d.

For $1 \leq i \leq k$, we define inductively:

- m_i as the unique $\leq_H$-minimal element of $A_{i-1} \setminus \{a\}$ (if it exists) such that there exists a vertex v with $m_i \leq_H v$ and of degree at least $d+1$ in $H[A_{i-1} \setminus \{a\}]$. If there is no such element m_i, the process stops.

  Let us prove that there is at most one candidate for m_i. Assume that there are incomparable m and m' satisfying these conditions. This means that there are vertices v and v' of degree at least $d+1$ in $A_{i-1} \setminus \{a\}$ (hence of degree at least $d+1$ in G) with $m \leq_H v$ and $m' \leq_H v'$. Note that we have $m \not\leq_H v'$ because $\leq_H$ is a tree order. By Lemma 2.3, we have that v, v' are both in $A_{i-1} \setminus \{a\}$, i.e. in the connected component of a in $G - \{m_1, \ldots, m_{i-1}\}$ (it follows also by induction that $\{m_1, \ldots, m_{i-1}\} = V_{\leq_H m_{i-1}}$, hence we may apply the lemma). Hence, v and v' are connected in $G - \{m_1, \ldots, m_{i-1}\}$. With Lemma 3.4, we also have that v and v' are connected in $G - \{a, m_1, \ldots, m_{i-1}\}$.

We take a witness path from v to v'. Since $m \leq_H v$ and $m \not\leq_H v'$, this path must contain two adjacent vertices w, w' with $m \leq_H w$ and $m \not\leq_H w'$. This contradicts the fact that $\leq_H$ is an elimination order satisfying the property of Lemma 2.3. Therefore, there is at most one possible such m_i.

- $T_i := \{v \in A_{i-1} \ : \ m_i \not\leq_H v\}$. These are the vertices that are disconnected from A_i in H, but not in G. As explained above, they all have degree at most d.
- A_i as the connected component of a in $G - \{m_1, \ldots, m_i\}$. Note that again, $A_i - \{a\}$ may be a union of connected components in $H - \{m_1, \ldots, m_i\}$.

The processes stops after at most k rounds. When the process stops, we have then defined m_i, T_i, and A_i up to $i = \omega$, with $\omega \leq k$ and every element in A_ω has degree at most d in $H[A_\omega]$.

We then define the new order $\leq_G$ as follows:

- for all x, y other than a and that are not in any of the T_i nor in A_ω, we have $x \leq_G y$ if and only if $x \leq_H y$,
- for all x in $A_\omega \cup \bigcup\limits_{i \leq \omega} T_i$, we have $m_i \leq_G x$ for all $i \leq \omega$. Note that a is in A_ω.

Note that all the elements in A_ω, and the T_i's, together with a are $\leq_G$-maximal.

We now prove that this new order is indeed an elimination order to degree d of depth k for G. We have that $\leq_G$ is a tree order and that it has depth at most k. Let us now take a vertex b and study S_b. Recall the definition of S_b from Definition 2.1. As we have two orders, we distinct S_b^G from S_b^H.

First, note that for $b = m_i$, we have $S_{m_i}^G = S_{m_i}^H = \emptyset$. So we don't have to check anything.

Then, assume that b is in $A_\omega \cup \bigcup\limits_{i \leq \omega} T_i$. Then b is both $\leq_G$ and $\leq_H$-maximal.

Additionally, $S_b^G = S_b^H$ if b is not adjacent to a, and $S_b^G = S_b^H \cup \{a\}$ otherwise. Note that as a is (k, d)-safe, it has no neighbor of degree $d + 1$, hence, if b is adjacent to a it has degree at most $d - 1$ in H, and $|S_b^H| \leq d - 1$ so $|S_b^G| \leq d$. For the last point of Definition 2.1, note that for any $v \in S_b^G$, v also belongs to $A_\omega \cup \bigcup\limits_{i \leq \omega} T_i$, we then have $\{w \ : \ w <_G v\} = \{w \ : \ w <_G b\} = (m_i)_{i<\omega}$.

Finally, we look at the case where b is not in A_ω, is not m_i nor in T_i for any $i \leq \omega$. In this case, we have that b cannot have neighbors in A_ω nor any of the T_i for $i \leq \omega$. To see this, assume that there is a $v \in T_i$ neighbor to b. This implies that b is in A_{i-1}, as it is connected to v, the latter being in A_{i-1}, which is the connected component of a in $G - \{m_1, \ldots, m_{i-1}\}$. As $b \notin T_i$, we have $m_i \leq_H b$ and $m_i \not\leq_H v$ which contradict that $\leq_H$ is an elimination order. This contradiction also holds if b has a neighbor in A_ω as this would imply that $b \in A_\omega$.

Therefore, in this final case, $S_b^G = S_b^H$. This also holds for any neighbor of b. Hence for any vertex v in S_b^G we have that:
$\{w \ : \ w <_G b\} = \{w \ : \ w <_H b\} = \{w \ : \ w <_H v\} = \{w \ : \ w <_G v\}$.

To conclude, we have that $\leq_G$ is indeed an elimination order to degree d of depth k for G. This ends the proof that a is irrelevant. $\square$

We can now prove our main result, Theorem 1.1.

Proof:
Let k, d be two integers and G be a connected K_5-minor free graph. We set $h(k) := (4k + 7) \cdot k$. Let $g(\cdot)$ be the function from Theorem 2.1.

First run the algorithm of Theorem 2.1 with parameter $h(k)$ (or, if G is planar, use the algorithm of Theorem 2.2).

Case 1. The algorithm outputs a tree-decomposition of width $g(h(k))$, or of width $5h(k)$ if G is planar. We then use Courcelle's Theorem (Theorem 2.3) to decide whether $G \in \mathscr{C}_{k,d}$.

Case 2. The algorithm outputs a minor model of the $h(k) \times h(k)$-grid in G. Furthermore we make sure that the set of branch sets subsumes all vertices. We then distinguish two cases depending on the number of branch sets that contain a node of degree at least $d + 1$.

Case 2.1 If more than k^2 branch sets contain a node of degree at least $d+1$, then using Corollary 3.2, we conclude that $G \notin \mathscr{C}_{k,d}$.

Case 2.2. There are at most k^2 branch sets containing a vertex of degree at least $d + 1$, and therefore, at most $k^2 \cdot (4k + 7)^2$ branch sets that are at distance at most $2k + 3$ to a vertex of degree at least $d + 1$. As $h(k)$ is large enough, there is a (k, d)-safe branch set which, by Corollary 3.5, implies the existence of an irrelevant vertex. We iteratively eliminate irrelevant vertices until we are in one of Case 1 or Case 2.1. □

We now do a quick complexity analysis of the algorithm. The algorithm of Theorem 2.1 runs in time $n^{O(1)}$. Then performing either Courcelle's Theorem, finding k^2 branch set with vertices of degree $d + 1$, or finding an irrelevant vertex can be done in linear time, i.e. $f(k, d) \cdot n$ for some computable function f. In the worst case, we might end up performing Case 2.2 up to $|G|$ many times before concluding via Case 1 or Case 2.1. Therefore, the overall complexity is $f(k, d) \cdot n^c$ for a computable function f and constant c.

In the slightly more restrictive case where G is planar, the algorithm of Theorem 2.2 runs in time $O(n^2)$. This improves the complexity of the overall algorithm yielding a total running time of $f(k, d) \cdot n^3$ for a computable function f.

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