On Completely Edge-Independent Spanning Trees in Locally Twisted Cubes

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Abstract. A network can contain numerous spanning trees. If two spanning trees T_i, T_j do not share any common edges, T_i and T_j are said to be *pairwisely edge-disjoint*. For spanning trees $T_1, T_2, ..., T_m$, if every two of them are pairwisely edge-disjoint, they are called *completely edge-independent spanning trees* (CEISTs for short). CEISTs can facilitate many network functionalities, and constructing CEISTs as maximally allowed as possible in a given network is a worthy undertaking. In this paper, we establish the maximal number of CEISTs in the *locally twisted cube* network, and propose an algorithm to construct $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n, the n-dimensional locally twisted cube. The proposed algorithm has been actually implemented, and we present the outputs. Network broadcasting in the LTQ_n was simulated using $\lfloor \frac{n}{2} \rfloor$ CEISTs, and the performance compared with broadcasting using a single tree.

Keywords: Broadcasting; Edge-disjoint; CEISTs; Locally twisted cubes; Spanning trees; Tree embedding.

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1. Introduction

One important indicator for a network's robustness is its ability to effectively embed useful structures, such as paths, rings, trees, etc. that satisfy certain constraints in the network. In recent years, the question of finding *completely edge-independent spanning trees* (CEISTs) in interconnection networks has particularly interested researchers in the field. The completely edge-independent spanning trees in a network can be an instrumental facilitator of many network functions, such as reliable communication, fault-tolerant broadcasting, secure messages distribution, etc [1, 2, 3, 4]. If we can determine the maximum number of CEISTs to optimize the utilization of the edges in a given network, we can achieve the maximum channel utilization, and reduce the communication delay. If there are k CEISTs in a network, then the network can tolerate, in the worst-case scenario, as many as $(k-1) \times (n-1)$ tree-edges to be faulty, and still achieve broadcasting.

In the study of computer networks, a network is almost exclusively modeled by a *graph* $G = (V, E)$ as in graph theory, with V being the set of vertices (nodes in network terms), and E being the set of edges (links/lines/channels between nodes). In this paper, these terms—graphs and networks, vertices and nodes, edges and links/lines/channels—will be used interchangeably.

There are several different categories of independent spanning trees, and they are broadly classified into *edge-independent spanning trees* (EISTs), *node-independent spanning trees* (NISTs), *completely independent spanning trees* (CISTs), and *edge-disjoint spanning trees* (EDSTs), etc. (see [4] for all these definitions). In general, EISTs is a set of spanning trees rooted at u in G such that there are no common internal edges between u and any other node among the paths in these spanning trees [4], and the difference between EISTs and EDSTs is that in EISTs, all spanning trees share one particular node as their common root, while all spanning trees in EDSTs are rootless. For that reason, CEISTs as EDSTs have been interchangeably used in the literature.

However, we noted that the definition of EDSTs in [21] is actually our definition of EISTs. Furthermore, Lin et al. proved that Hsieh and Tu's spanning trees are indeed node-independent spanning trees [27], that is, the spanning trees they constructed are rooted trees. Therefore, the spanning trees dealt with in our paper are different than those in [21]. Following the nomenclature of CIST (where "completely" means "rootless"), in the rest of this paper we employ the term CEISTs instead of EDSTs to avoid further confusion.

The *hypercube*, denoted by Q_n, is a classical interconnection network that has many desirable properties, such as low diameter, high connectivity, symmetry, etc. [5, 6, 7]. The CEISTs in Q_n have been extensively studied. Barden et al. proposed a method for obtaining the maximum number of CEISTs in Q_n, and proved that there exist $\lfloor \frac{n}{2} \rfloor$ CEISTs in Q_n [8]. For the *Cartesian product network* $G \times F$, Ku et al. gave two methods to embed CEISTs [9]. The first method constructed $n_1 + n_2$ CEISTs in $G \times F$ with certain assumptions, while the second one with no assumptions constructed $n_1 + n_2 - 1$ CEISTs in $G \times F$, where n_1 (resp. n_2) is the number of CEISTs in G (resp. F). In 2021, Zhou et al. proposed the maximum number of CEISTs and edge-disjoint spanning c-forests of equiarboreal graphs [10]. In 2022, Wang et al. proposed an algorithm that constructed completely independent spanning trees in the line graph of graph G based on G's CEISTs [15]. Furthermore, CEISTs have also been studied for some special networks [11, 12, 13, 14].

Fan et al. introduced the notion of *bijective connection networks* (BC networks for short) [16, 28], which include many well-known networks, such as hypercubes, locally twisted cubes, crossed cubes, etc., and other less known networks. Based on BC networks, *conditional BC networks* were defined by Cheng et al. in [17]. It has been shown that hypercubes, crossed cubes, locally twisted cubes, and Möbius cubes all belong to conditional BC networks. As of today, CEISTs in hypercubes [8] and crossed cubes [18] have been obtained, while CEISTs in locally twisted cubes and Möbius cubes [24] still remain unsolved.

As a prominent variant of the hypercube, the *locally twisted cube* LTQ_n, proposed by Yang et al. [19], has advantageous properties including low diameter and low fault-diameter. Hsieh et al. proved that LTQ_n is $(2n-5)$-Hamiltonian (for $n \geq 3$) if each vertex in LTQ_n is associated with at least two fault edges [20]. In this paper, we study how to embed the maximal number of CEISTs in the LTQ_n. Our work in this paper can be outlined as follows:

1) We propose a recursive algorithm, named *CEISTs_LTQ*, that constructs $\lfloor \frac{n}{2} \rfloor$ CEISTs $T_1, T_2, ...,$ $T_{\lfloor \frac{n}{2} \rfloor}$ in the LTQ_n;

2) We prove theoretically the correctness of *CEISTs_LTQ*, and determine its time complexity as $O(n \cdot 2^n)$, where n is the dimension of LTQ;

3) To solidly validate the algorithm, *CEISTs_LTQ* has been actually implemented, and we present the running outputs;

4) We have simulated the broadcasting in LTQ_n using *CEISTs*, and the outcomes are presented and discussed.

It is worth pointing out that the CEISTs-construction methods developed earlier for the hypercube and the crossed cube [8, 18] cannot be directly applied to *LTQ*. The link connection between subcubes in *LTQ* is more complicated than in the hypercube/crossed cube, calling for different, new approaches in the CEISTs-construction process.

The rest of this paper proceeds as follows. Section 2 provides the preliminaries. Section 3 includes the paper's main work—describing the algorithm for constructing $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n, proving its correctness, and analyzing its time complexity. Section 3 also discusses the technical similarity/difference between *CEISTs_LTQ* and existing CEISTs algorithms for hypercubes and crossed cubes. Section 4 presents the simulation experiments to verify the validity of the algorithm, and evaluates its performance in efficient broadcasting. Section 5 concludes the paper.

2. Preliminaries

2.1. Terminology and notation

A network can be abstracted as a graph $G(V(G), E(G))$, where $V(G)$ denotes the vertex set and $E(G)$ denotes the edge set, representing the servers and the links between them respectively. As pointed out earlier, graphs and networks are interchangeably used throughout this paper.

T is said to be a spanning tree of graph G, if (1) T contains all the vertices of G; (2) T has exactly $|V(G)| - 1$ edges from $E(G)$. A path started from u and ended at v is denoted (u, v)-path. Given two (u, v)-paths P and Q started at u and ended with v, P and Q are *edge-disjoint* if they share no common edges. Let $T_1, T_2, ..., T_m$ be spanning trees in graph G, if for every pair of them do not contain the common edges, then $T_1, T_2, ..., T_m$ are called completely edge-independent spanning trees (CEISTs for short) in G.

A binary string u with length n can be written as $u_{n-1}u_{n-2}...u_iu_{i-1}...u_0$,where $u_i \in \{0, 1\}$ and $0 \leq i \leq n-1$. The complement of u_i will be denoted by $\overline{u_i}$ ($\overline{0} = 1$ and $\overline{1} = 0$). $|u|$ is the decimal value of u. A path P from $v^{(1)}$ to $v^{(n)}$ can be denoted as $v^{(1)}$-$v^{(2)}$-$\cdots$-$v^{(n)}$. $|u| - |v|$ means the decimal value of vertex u minus the decimal value of vertex v.

2.2. Locally twisted cubes

We adopt the definition of LTQ_n as follows.

Definition 1 [19]. For integer $n \geq 2$. The n-dimensional locally twisted cube, denoted by LTQ_n, is defined recursively as follows.

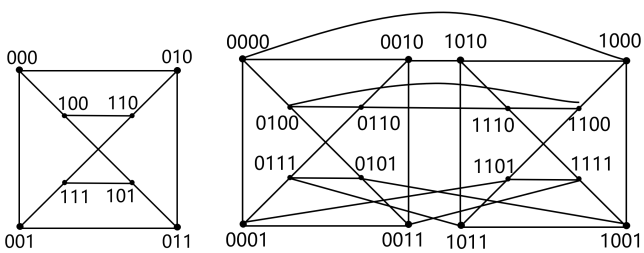
(1) LTQ_2 is a graph consisting of four vertices labeled with $00, 01, 10$, and 11, respectively, connected by four edges $(00, 01), (00, 10), (01, 11)$, and $(10, 11)$.

(2) For integer $n \geq 3$, LTQ_n is built from two disjoint copies of LTQ_{n-1} according to the following steps. Let LTQ^0_{n-1} (respectively, LTQ^1_{n-1}) denote the graph obtained by prefixing the label of each vertex in one copy of LTQ_{n-1} with 0 (respectively, 1). Each vertex $u = 0(u_{n-2}u_{n-3}...u_0)$ in LTQ^0_{n-1} is connected with the vertex $1(u_{n-2} \oplus u_0)u_{n-3}...u_0$ in LTQ^1_{n-1} by an edge, where"$\oplus$"represents the modulo 2 addition.

The locally twisted cube LTQ_n can also be equivalently defined in the following non-recursive fashion.

Definition 2 [19]. For integer $n \geq 2$, the n-dimensional locally twisted cube, LTQ_n, is a graph with $\{0, 1\}^n$ as the vertex set. Two vertices $u = u_{n-1}u_{n-2}...u_0$ and $v = v_{n-1}v_{n-2}...v_0$ in LTQ_n are adjacent if and only if either (1) For some $2 \leq i \leq n-1$, $u_i = \overline{v_i}$ and $u_{i-1} = v_{i-1} \oplus u_0$, and $u_j = v_j$ for all the remaining bits or, (2) For some $i \in \{0, 1\}$, $u_i = \overline{v_i}$, and $u_j = v_j$ for all the remaining bits.

Fig. 1 shows the examples of LTQ_3 and LTQ_4.



(a) (b)

Figure 1. (a) LTQ_3; (b) LTQ_4.

It follows that LTQ_n is an n-regular graph, and any two adjacent vertices in LTQ_n differ by at most two consecutive bits. Moreover, the following lemma holds.

Lemma 1. [21] Let $u = u_{n-1}u_{n-2}...u_0$ and $v = v_{n-1}v_{n-2}...v_0$ be two adjacent vertices in $LTQ_n(n \geq 2)$ with $u > v$. Then, the following statements hold.

(1) If $|u|$ is even, then $|u| - |v| = 2^i$ for some $0 \leq i \leq n-1$.

(2) If $|u|$ is odd, then either $|u| - |v| = 2^i$ for some $i \in \{0, 1\}$ or $|u| - |v| = 2^i - \left[(-1)^{u_i - 1} \times 2^{i-1}\right]$ for some $i \geq 2$.

When two adjacent vertices u and v have a leftmost differing bit at position d, that is $u_d \neq v_d$ and $u_{n-1}u_{n-2}...u_{d+1} = v_{n-1}v_{n-2}...v_{d+1}$. Furthermore, for $W \subseteq V(LTQ_n), N_d(W) = \{N_d(w)|w \in W\}$ denotes the set of the d-neighbors of all vertices in W. When two vertices u and v are adjacent, and both u's decimal value and v's decimal value are even (odd), we say (u, v) is even (odd) edge.

3. Completely edge-independent spanning trees in locally twisted cubes

In this section, we propose an algorithm to construct $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n, where $n \geq 2$, prove its correctness, and analyze its time complexity. In what follows, we represent each vertex in the trees by its decimal value.

3.1. Construction Algorithm of CEISTs for locally twisted cubes

A CEIST can be constructed easily in LTQ_2 and an example is presented in Fig. 2.

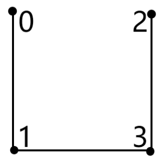


Figure 2. CEIST T_1 in LTQ_2.

For $n \geq 3$ and n is odd, LTQ_n consists of two subcubes LTQ^0_{n-1} and LTQ^1_{n-1}, denoted by A and B. If there are $\lfloor \frac{n}{2} \rfloor$ CEISTs in A, we can construct $\lfloor \frac{n}{2} \rfloor$ CEISTs that are one-to-one isomorphic with $\lfloor \frac{n}{2} \rfloor$ trees of A in B accordingly. Then, for every two isomorphic trees in LTQ_n, we expect to connect them through specific edges to obtain $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n. Based on the above discussion, we propose a function (see algorithm *Odd_CEISTs*) to obtain $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n.

`Algorithm` *Odd_CEISTs*

`Input:` γ_A $(n-1)$, `and` P_A : v^1_A-v^2_A-$\cdots$-$v^{\lceil \frac{n}{2} \rceil}_A$.

(P_A `is constructed recursively by algorithm` *CEISTs_LTQ*).

`Output:` γ (n).

Begin

Step 1. Construct T_1^B, T_2^B, ..., $T_{\lfloor \frac{n}{2} \rfloor}^B$ and the path P_B : v_B^1-v_B^2-$\cdots$-$v_B^{\lceil \frac{n}{2} \rceil}$.

for $i = 1$ to $\lfloor \frac{n}{2} \rfloor$ do
  Construct tree T_i^B by adding 2^{n-1} to each vertex in T_i^A.
end for
for $i = 1$ to $\lceil \frac{n}{2} \rceil$ do
  $|v_B^i| = |v_A^i| + 2^{n-1}$.
end for

Step 2. Construct T_1, T_2, ..., and $T_{\lfloor \frac{n}{2} \rfloor}$ as follows.

for $i = 1$ to $\lfloor \frac{n}{2} \rfloor$ do
  $V(T_i) = V(LTQ_n)$.
  $E(T_i) = E(T_i^A) \cup E(T_i^B) \cup \{(v_A^i, v_B^i)\}$.
end for
return $\gamma(n)$ = $\{T_1, T_2, \ldots, T_{\lfloor \frac{n}{2} \rfloor}\}$.

end

Example 1. By algorithm *Odd_CEISTs*, Fig. 3 demonstrates the construction of CEIST T_1 in LTQ_3, and Fig. 4 demonstrates the construction of two CEISTs T_1 and T_2 in LTQ_5. We take the LTQ_5 as an example. Firstly, the two CEISTs $\gamma_A(4) = \{T_1^A, T_2^A\}$ and the path P_A : 0-2-10 in LTQ_4 as the input of the algorithm are presented in Fig. 5. Next, according to the step 1 of the algorithm, T_1^B, T_2^B and P_B : 16-18-26 are obtained. Then, according to the step 2 of the algorithm, T_1 is constructed by connecting T_1^A and T_1^B through edge (0, 16). T_2 is constructed by connecting T_2^A and T_2^B through edge (2, 18).

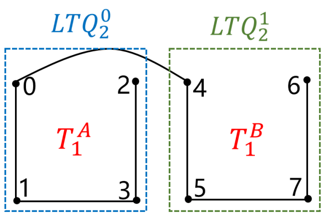
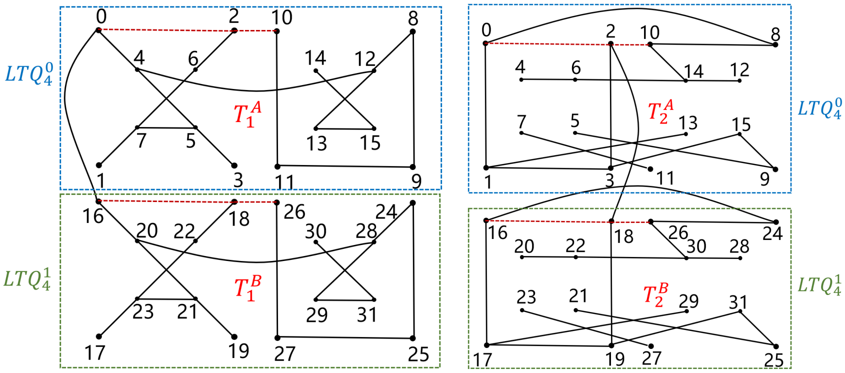


Figure 3. CEIST T_1 in LTQ_3.

For $n \geq 4$ and n is even, LTQ_n consists of four subcubes LTQ_{n-2}^{00}, LTQ_{n-2}^{10}, LTQ_{n-2}^{11} and LTQ_{n-2}^{01}, denoted by A, B, C and D, which have the common prefix 00, 10, 11 and 01, respectively. If there are $\frac{n-2}{2}$ CEISTs in A, we can construct $\frac{n-2}{2}$ CEISTs that are one-to-one isomorphic with $\frac{n-2}{2}$ trees of A in B accordingly, and so are C and D. Then, for every four isomorphic trees in LTQ_n, we expect to connect them through specific edges to obtain $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n, and after the construction of $\lfloor \frac{n}{2} \rfloor$ CEISTs, we also expect to get $\frac{n}{2}$ unused edges to form exactly path. Based on the above discussion, we propose a function (see algorithm *Even_CEISTs*) to obtain $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n.



(a) T_1 (b) T_2

Figure 4. Two CEISTs T_1 and T_2 in LTQ_5.

Algorithm *Even_CEISTs*

Input: γ_A $(n-2)$ and P_A : v_A^1-v_A^2-$\cdots$-$v_A^{\frac{n}{2}}$.

(P_A is constructed recursively by algorithm *CEISTs_LTQ*).

Output: γ (n), and P : v^1-v^2-$\cdots$-$v^{\frac{n}{2}+1}$.

Begin

Step 1. For $1 \leq i \leq \frac{n-2}{2}$, construct T_i^B, T_i^C, T_i^D, P_B, P_C, P_D.

  for $i = 1$ to $\frac{n-2}{2}$ do

    Construct T_i^B, T_i^C, T_i^D by adding 2^{n-1}, $3*2^{n-2}$, 2^{n-2} to each vertex in T_i^A, respectively.

  end for

  Construct paths P_B, P_C, P_D by adding 2^{n-1}, $3*2^{n-2}$, 2^{n-2} to each vertex in P_A, respectively.

Step 2. Construct T_i, for $1 \leq i \leq \frac{n}{2}-2$.

  $V(T_i) = V(LTQ_n)$,

  $E(T_i) = E(T_i^{\epsilon\in\{A,B,C,D\}}) \cup \{(v_A^i, v_B^i)\} \cup \{(v_B^i, v_C^i)\} \cup \{(v_C^i, v_D^i)\}$.

Step 3. Construct $T_{\frac{n}{2}-1}(i = \frac{n}{2}-1)$.

  $V(T_i) = V(LTQ_n)$,

  $E(T_i) = E(T_i^{\epsilon\in\{B,C,D\}}) \cup \{(v_B^i, v_C^i)\} \cup \{(v_C^i, v_D^i)\}$
  $\cup\{(v, N_{n-2}(v)) | v \in V(T_i^A)\}$.

Step 4. Construct $T_{\frac{n}{2}}(i = \frac{n}{2})$.

  $V(T_i) = V(LTQ_n)$,

  $E(T_i) = E(T_{i-1}^A) \cup \{(v, N_{n-1}(v)) | v \in V(T_{i-1}^A) \textit{ and } |v| \textit{ is even}\}$
  $\setminus \{(v_A^j, v_B^j) \mid j = 1 \textit{ to } i-2, \textit{ and } j = i\}$
  $\cup \{(v, N_{n-2}(v)) | v \in V(T_{i-1}^B) \textit{and } |v| \textit{ is even}\}$
  $\setminus \{(v_B^j, v_C^j) \mid j = 1 \textit{ to } i-1\}$

$\cup \ \{(v,\ N_{n-1}(v))|\ v\ \in\ V(T^C_{i-1})\ and\ |v|\ is\ even\}$
$\setminus \ \{(v^j_C, v^j_D)\ |\ j\ =\ 1\ to\ i-1\}$
$\cup \ \{(v,\ N_{n-1}(v))\ |\ v\ \in\ V(T^A_{i-1})\ and\ |v|\ is\ odd\}$
$\cup \ \{(v,\ N_{n-2}(v))\ |\ v\ \in\ V(T^C_{i-1})\ and\ |v|\ is\ odd\}$
$\cup \ \{(v,\ N_{n-1}(v))\ |\ v\ \in\ V(T^B_{i-1})\ and\ |v|\ is\ odd\}$
$\cup \ \{(v^j_r,\ v^{j+1}_r)\ |\ r\ \in\ \{B,\ C,\ D\},\ j\ =\ 1\ to\ i-1\}$.

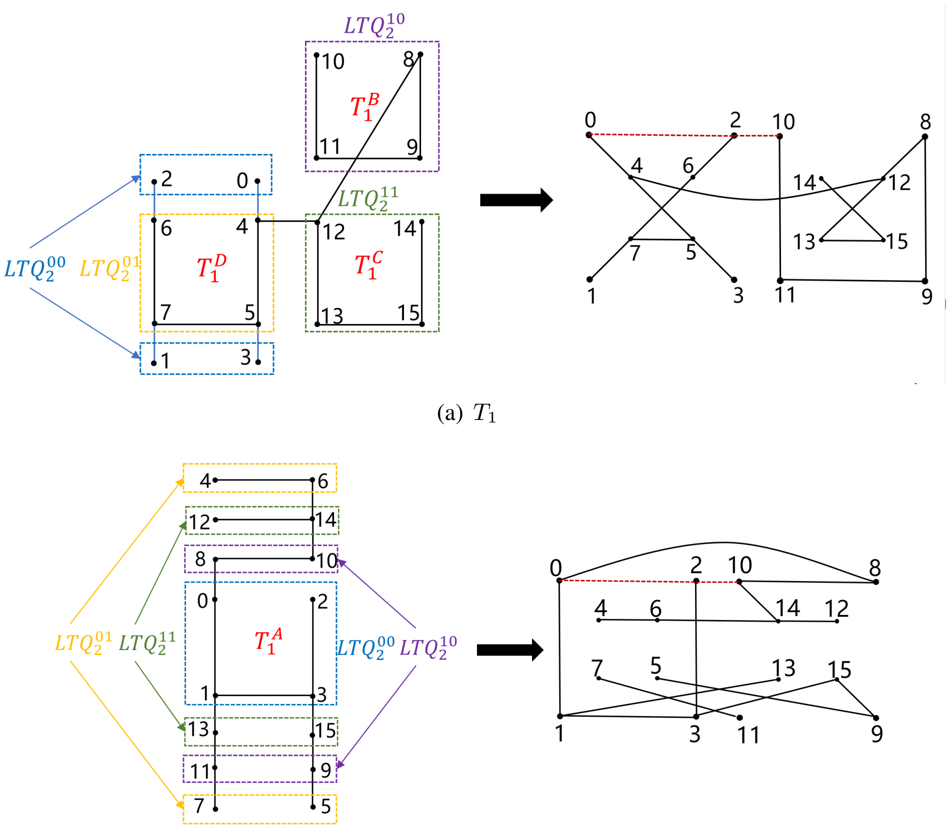
`Step 5. Construct the path` $P:\ v^1\text{-}v^2\text{-}\cdots\text{-}v^{\frac{n}{2}+1}$ `as follows.`

  11: $v^i\ =\ v^i_A$, for $1\ \leq\ i\ \leq\ \frac{n}{2}$.

  12: $v^{\frac{n}{2}+1}_A\ =\ v^{\frac{n}{2}}_A\ +\ 2^{n-1}$.

  13: return $\gamma\ (n)$ = $\{T_1,\ T_2, \ldots,\ T_{\frac{n}{2}}\}$, and P.

end



(a) T_1

(b) T_2

Figure 5. Two CEISTs T_1 and T_2 in LTQ_4.

Example 2. By algorithm *Even_CEISTs*, Fig. 5 demonstrates the construction of two CEISTs T_1 and T_2 in LTQ_4, and Fig. 6 demonstrates the construction of three CEISTs T_1, T_2 and T_3 in LTQ_6. We take LTQ_6 as an example, firstly, the two CEISTs $\gamma_A(4) = \{T_1^A, T_2^A\}$ and the path P_A : 0-2-10 in LTQ_4 as the input of the algorithm are presented in Fig. 5. Next, according to the step 1 of the algorithm *Even_CEISTs*, T_1^B, T_2^B, T_1^C, T_2^C, T_1^D, T_2^D and P_B : 32-34-42, P_C : 48-50-58, P_D : 16-18-26 are obtained. Then, according to the step 2 of the algorithm *Even_CEISTs*, T_1 is constructed by connecting T_1^A, T_1^B, T_1^C and T_1^D, through edges (0, 32), (32, 48) and (48, 16). T_2 is constructed by connecting T_2^B, T_2^C and T_2^D through edges (34, 50) and (50, 18), and all the edges between A and D. T_3 is constructed by the following steps: (1) T_2^A is contained in T_3. (2) First, connect all the even edges between A and B except (0, 32) and (10, 42), all the even edges between B and C except (32, 48) and (34, 50), and all the even edges between C and D except (16, 48) and (18, 50). (3) Then, connect all the odd edges between A and C, all the odd edges between C and B, and all the odd edges between B and D. (4) Connect (32, 34), (34, 42), (48,50), (50, 58), (16, 18), (18, 26). Finally, the path P is 0-2-10-42.

Then we synthesize the two algorithms and propose an integrated algorithm, named algorithm *CEISTs_*LTQ, to generate $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n, where $n \geq 2$.

Algorithm *CEISTs_*LTQ
Input: Integer n , with $n \geq 2$.
Output: $\gamma\ (n)$, for $n \geq 2$, and a path P: v^1-v^2-$\cdots$-$v^{\frac{n}{2}+1}$ when n is even.
Begin
if n = 2 then
    $V(T_1) = \{0, 1, 2, 3\}$, $E(T_1) = \{(0,1), (1,3), (3,2)\}$,
    $v^1 = 0$, $v^2 = 2$.
if $n \geq 3$ and n is odd then
    $\gamma\ (n-1)$, $P_A \leftarrow$ *CEISTs_*$LTQ(n-1)$,
    $\gamma\ (n) \leftarrow$ *Odd_CEISTs* $(\gamma\ (n-1), P_A)$.
if $n \geq 4$ and n is even then
    $\gamma\ (n-2)$, $P_A \leftarrow$ *CEISTs_*$LTQ(n-2)$,
    $\gamma\ (n-1)$, $P \leftarrow$ *Even_CEISTs*$(\gamma\ (n-2), P_A)$.
return $\gamma\ (n)$, and path P for n is even.
end

3.2. Correctness of *CEISTs_*LTQ

To verify the correctness of the CEISTs obtained by algorithm *CEISTs_*LTQ, we present the following theorems.

Theorem 3.1. For $n \geq 3$ and n is odd, $T_1, T_2, ..., T_{\lfloor \frac{n}{2} \rfloor}$ constructed by algorithm *CEISTs_*LTQ are $\lfloor \frac{n}{2} \rfloor$ CEISTs in LTQ_n.

Proof:
By the step 1 of algorithm *Odd_CEISTs*, T_i^B is constructed by adding 2^{n-1} to each vertex in T_i^A, we have $E(T_i^B) \cap E(T_j^B) = \emptyset$, for any $1 \leq i<j \leq \lfloor \frac{n}{2} \rfloor$, and P_B is constructed by adding 2^{n-1} to each

vertex in P_A, we have $V(P_A) \cap V(P_B) = \emptyset$. Since $|v_A^i|$ is even, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, according to Lemma 1, v_A^i and v_B^i are two adjacent vertices. Thus $(v_A^i, v_B^i) \neq (v_A^j, v_B^j)$, for any $1 \leq i<j \leq \lfloor \frac{n}{2} \rfloor$, $\{E(T_i^A) \cup E(T_i^B) \cup \{(v_A^i, v_B^i)\}\} \cap \{E(T_j^A) \cup E(T_j^B) \cup \{(v_A^j, v_B^j)\}\} = \emptyset$, for any $1 \leq i<j \leq \lfloor \frac{n}{2} \rfloor$. Therefore, $E(T_i) = \{E(T_i^A) \cup E(T_i^B) \cup \{(v_A^i, v_B^i)\}\}, E(T_i) \cap E(T_j) = \emptyset$, for any $1 \leq i<j \leq \lfloor \frac{n}{2} \rfloor$, there exist $\lfloor \frac{n}{2} \rfloor$ CEISTs $T_1, T_2, ..., T_{\lfloor \frac{n}{2} \rfloor}$ in LTQ_n, where $n \geq 3$ and n is odd. □

Theorem 3.2. For $n \geq 4$ and n is even, $T_1, T_2, ..., T_{\frac{n}{2}}$ constructed by algorithm *CEISTs_LTQ* are $\frac{n}{2}$ CEISTs in LTQ_n.

Proof:
By the step 1 of algorithm *Even_CEISTs*, trees T_i^B, T_i^C and T_i^D are constructed by adding 2^{n-1}, $3 * 2^{n-1}$ and 2^{n-2} to each vertex in T_i^A, respectively, we have $\{E(T_i^r) \cap E(T_j^r) = \emptyset | r \in \{B, C, D\}, 1 \leq i<j \leq \frac{n}{2} - 2\}$, and P_B, P_C, P_D are constructed by adding $2^{n-1}, 3 * 2^{n-1}, 2^{n-2}$ to each vertex in P_A, respectively, we have $V(P_A) \cap V(P_B) \cap V(P_C) \cap V(P_D) = \emptyset$. Since $|v_A^i|$ is even, for $i \in \{1, 2, ..., \frac{n}{2}\}$, according to Lemma 1, v_A^i and v_B^i are two adjacent vertices, v_B^i and v_C^i are two adjacent vertices, v_C^i and v_D^i are two adjacent vertices, and v_A^i and v_D^i are two adjacent vertices. We have the following cases:

Case 1. Construct T_i, for $1 \leq i \leq \frac{n}{2} - 2$. By the step 2 of algorithm *Even_CEISTs*, we can know $E(T_i) = E(T_i^A) \cup E(T_i^B) \cup E(T_i^C) \cup E(T_i^D) \cup \{(v_A^i, v_B^i), (v_B^i, v_C^i), (v_C^i, v_D^i)\}$. Therefore, $E(T_i) \cap E(T_j) = \emptyset$, for any $1 \leq i<j \leq \frac{n}{2} - 2$.

Case 2. Construct $T_{\frac{n}{2}-1}$. From the topology of LTQ_n, We know both even vertices and odd vertices are adjacent only between A and D, or between B and C, the simple topology of LTQ_n is presented in Fig. 7. Thus, we choose $T_{\frac{n}{2}-1}^B, T_{\frac{n}{2}-1}^C$ and $T_{\frac{n}{2}-1}^D$ as the infrastructure, and connect them by $(v_B^{\frac{n}{2}-1}, v_C^{\frac{n}{2}-1}), (v_C^{\frac{n}{2}-1}, v_D^{\frac{n}{2}-1})$. Then, we connect all the vertices between A and D to obtain tree $T_{\frac{n}{2}-1}$. Obviously, $E(T_i) \cap E(T_j) = \emptyset$, for any $1 \leq i<j \leq \frac{n}{2} - 1$.

Case 3. Construct $T_{\frac{n}{2}}$. By Case 1 and Case 2, we know the unused edges are $E(T_{\frac{n}{2}-1}^A)$, even edges between A and B except for $(v_A^i, v_B^i)(1 \leq i \leq \frac{n}{2} - 2)$ and $(v_A^{\frac{n}{2}}, v_B^{\frac{n}{2}})$, even edges between B and C except for $(v_B^i, v_C^i)(1 \leq i \leq \frac{n}{2} - 1)$, even edges between C and D except for $(v_C^i, v_D^i)(1 \leq i \leq \frac{n}{2} - 1)$, all the odd edges in LTQ_n, and (v_r^j, v_r^{j+1}), for $r \in \{B, C, D\}$, and $j = 1$ to $\frac{n}{2} - 1$. Thus, we choose $T_{\frac{n}{2}-1}^A$ as the infrastructure, (1) Connect all the even edges between A and B except for (v_A^i, v_B^i) and $(v_A^{\frac{n}{2}}, v_B^{\frac{n}{2}})$, for $1 \leq i \leq \frac{n}{2} - 2$. Then, we connect $v_B^i(1 \leq i \leq \frac{n}{2} - 2)$ and $v_B^{\frac{n}{2}}$ to the tree by $v_B^{\frac{n}{2}-1}$, that is we connect $(v_B^i, v_B^{i+1})(1 \leq i \leq \frac{n}{2} - 1)$. (2) Connect all the even edges between B and C except for $(v_B^i, v_C^i)(1 \leq i \leq \frac{n}{2} - 1)$. Then we connect $v_C^i(1 \leq i \leq \frac{n}{2} - 1)$ to the tree by $v_C^{\frac{n}{2}}$, that is we connect $(v_C^i, v_C^{i+1})(1 \leq i \leq \frac{n}{2} - 1)$. (3) Connect all the even edges between C and D except for $(v_C^i, v_D^i)(1 \leq i \leq \frac{n}{2} - 1)$. Then we connect $v_D^i(1 \leq i \leq \frac{n}{2} - 1)$ to the tree by $v_D^{\frac{n}{2}}$, that is we connect $(v_D^i, v_D^{i+1})(1 \leq i \leq \frac{n}{2} - 1)$. (4) Connect all the odd edges in LTQ_n. Thus, For $n \geq 4$ and n is even, $T_1, T_2, ..., T_{\frac{n}{2}}$ constructed by algorithm *CEISTs_LTQ* are $\frac{n}{2}$ CEISTs in LTQ_n. □

Theorem 3.3. For $n \geq 2$ and n is even, after $\frac{n}{2}$ CEISTs are constructed, there still remain $\frac{n}{2}$ unused edges.

Proof:
Since LTQ_n has $n \cdot 2^{n-1}$ edges, and each spanning tree in LTQ_n has $2^n - 1$ edges, $\frac{n}{2}$ spanning trees have $(2^n - 1) \cdot \frac{n}{2} = n \cdot 2^{n-1} - \frac{n}{2}$ edges. Thus, after $\frac{n}{2}$ CEISTs are constructed, there still remain $\frac{n}{2}$ unused edges. In order to reduce the time complexity, we specify the $\frac{n}{2}$ unused edges are v_A^1-v_A^2-$\cdots - v_A^{\frac{n}{2}}$-$v_A^{\frac{n}{2}+1}$, where $|v_A^1| = 0, |v_A^{\frac{n}{2}+1}| = |v_A^{\frac{n}{2}}| + 2^{n-1}$.

In summary, for $n \geq 2$, $T_1, T_2, ..., T_{\frac{n}{2}}$ constructed by algorithm *CEISTs_LTQ* are $\frac{n}{2}$ CEISTs in LTQ_n. □

Theorem 3.4. For integer $n \geq 2$, algorithm *CEISTs_LTQ* obtains $\lfloor \frac{n}{2} \rfloor$ CEISTs in $O(n \cdot 2^n)$ time, where n is the dimension of LTQ_n.

Proof:
Let $T(n)$ denote the running time of the algorithm *CEISTs_LTQ*. The time complexity of algorithm 1 and algorithm 2 is $O(2^n)$, we have a recurrence equation that bounds:

$$T(n) = \begin{cases} 1, n = 2; \\ T(n-1) + O(2^n), n \text{ is odd}; \\ T(n-2) + O(2^n), n \text{ is even}. \end{cases}$$

Solving the recurrence equation yields that the time complexity is $O(n \cdot 2^n)$. □

3.3. CEISTs in LTQ vs. in hypercubes/crossed cubes

We make some remarks pertaining to CEISTs algorithms for hypercubes and crossed cubes. As has been noted, algorithms for embedding CEISTs in hypercubes and crossed cubes have been provided in [8] and [18], respectively. Although *CEISTs_LTQ* also uses a recursive scheme, it is not a straightforward, direct application of the existing methods in [8] or [18]. Due to these cubes' differences in topology, different techniques are used for the task.

Note that when n is odd, LTQ_n and LTQ_{n-1} have *the same number* (i.e. $\frac{n-1}{2}$) of CEISTs. That means we can just splice the CEISTs in the two LTQ_{n-1}s to build the $\frac{n-1}{2}$ CEISTs in LTQ_n. That is, we just need to choose splicing edges between the two LTQ_{n-1}s, and no additional CEISTs need to be found in the process. For this case, LTQ/hypercube/crossed cube's treatments are similar.

The complication arises when n is even. Now the LTQ_n's $\frac{n}{2}$ CEISTs are recursively constructed from four LTQ_{n-2}s, each of which contains $\frac{n-2}{2}$ CEISTs:

> **Step** 1: Since $n-2$ is even, four LTQ_{n-2}s can be spliced into one LTQ_n, which still contains $\frac{n-2}{2}$ CEISTs;
>
> **Step** 2: Choose unused edges inside and between the four LTQ_{n-2}s to build one more CEIST, so that we have $\frac{n-2}{2} + 1 = \frac{n}{2}$ CEISTs.

It is in Step 2 above that the method for LTQ is more complex than in hypercubes/crossed cubes. Due to the less-regular, "locally twisted" connection between LTQ subcubes, the edge selection techniques for hypercubes/crossed cubes fail to work. A more restricted selection procedure is carried out by *CEISTs_LTQ*.

As a matter of fact, the edge selection procedure for *CEISTs*_LTQ will also work for both hypercubes and crossed cubes. However, it would introduce restrictions that are unnecessary when selecting CEISTs edges in hypercubes and crossed cubes.

4. Implementation and simulation

To attest algorithm *CEISTs*_LTQ's validity, it was actually implemented using the programming language Python. The program's main methods exactly follow the steps outlined in *CEISTs*_LTQ. Fig. 8 illustrates the algorithm output, CEISTs in an odd LTQ_7, while Fig. 9 illustrates the CEISTs in an even LTQ_8.

CEISTs can be used for efficient broadcasting, reliable broadcasting and secure distribution of information. Then, we can divide a message into several data packets, encrypt every packet, and transmit them through multiple CEISTs, finally we decrypt and merge them when we receive all the packets to achieve efficient broadcasting and secure distribution of information.

We simulate the scenario where a message of size less than $1M$ is broadcast from a source node to all nodes in LTQ_n, using multiple CEISTs. The most common Ethernet frame length can carry about 1500 bytes of the message (excluding the initial preamble, frame delimiter, and the frame check sequence at the end) [23]. Thus, the message can be divided into $1M/1500\ bytes = 700$ (data packets). We compare $\lfloor \frac{n}{2} \rfloor$ CEISTs as transmission channels with a single spanning tree as transmission channel for data broadcasting. We employ a round-robin strategy to call $\lfloor \frac{n}{2} \rfloor$ channels for packet transmission to balance the load of all transmission channels. That is the first $\lfloor \frac{n}{2} \rfloor$ packets are transmitted by the $\lfloor \frac{n}{2} \rfloor$ channels in sequence, the $\lfloor \frac{n}{2} \rfloor + 1$ packet is transmitted by the first channel, the $\lfloor \frac{n}{2} \rfloor + 2$ packet is transmitted by the second channel, and so on.

We use the following two metrics to evaluate broadcasting efficiency, one is the *maximum broadcasting latency* (MBL for short), and the other is the *average broadcasting latency* (ABL for short). For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, let $mt(k)$ be the delivery the whole message time between the farthest two vertices in the kth tree. There are $s = 2^{2n-1} - 2^{n-1}$ pairs of vertices in LTQ_n. For $1 \leq i \leq s$, let $arbt(i)$ be the maximum distance between the ith pairs of vertices in $\lfloor \frac{n}{2} \rfloor$ CEISTs, and let $t(i)$ be the distance between the ith pairs of vertices in one spanning tree.

Suppose a message is divided into x data packets. Firstly, the MBL of broadcasting latency using $\lfloor \frac{n}{2} \rfloor$ CEISTs is: $MBL = \lceil \frac{x}{\lfloor \frac{n}{2} \rfloor} \rceil \cdot \max\limits_{1 \leq k \leq \lfloor \frac{n}{2} \rfloor} \{mt(k)\}$, and the MBL of broadcasting latency using a single spanning tree is: $MBL = x \cdot \min\limits_{1 \leq k \leq \lfloor \frac{n}{2} \rfloor} \{mt(k)\}$. Secondly, the ABL of broadcasting latency using $\lfloor \frac{n}{2} \rfloor$ CEISTs is: $ABL = \frac{\sum\limits_{i=1}^{s} arbt(i)}{s}$, and the ABL of broadcasting latency using a single spanning tree is: $ABL = \frac{\sum\limits_{i=1}^{s} t(i)}{s}$. All the experimental results showing ABL and MBL are depicted in Fig. 10. To demonstrate the efficiency of broadcasting in multiple CEISTs, we choose the spanning tree with the minimal $mt(k)$ in $\lfloor \frac{n}{2} \rfloor$ CEISTs to calculate the MBL of the single spanning tree, and choose the spanning tree with the minimal ABL in $\lfloor \frac{n}{2} \rfloor$ CEISTs to calculate the ABL of the single spanning tree.

For both ABL and MBL, the performance of broadcasting latency using $\lfloor \frac{n}{2} \rfloor$ CEISTs is better than that using a single spanning tree. As the network size grows, in the odd dimension and even dimension, the ABL and MBL of $\lfloor \frac{n}{2} \rfloor$ CEISTs will rise, respectively, but the growth rate is very slow compared with the single spanning tree.

5. Conclusion

We proposed, and proved correctness for, an $O(n \cdot 2^n)$ algorithm, named *CEISTs*_LTQ, to construct $\lfloor \frac{n}{2} \rfloor$ CEISTs in the *locally twisted cube* network LTQ_n, where $n \geq 2$ is the dimension. The number of CEISTs constructed by our algorithm is optimal. Experiments were conducted to verify the validity of our algorithm, and to simulate broadcasting LTQ_n using the CEISTs. It is worth pointing out that our proposed algorithm for LTQ_n can also be used to construct CEISTs in hypercube Q_n and crossed cube CQ_n, which makes it a more general algorithm.

Many directions can be pursued for continuing our work. Among them, for example, CEIST-embedding in the presence of faulty (therefore missing) nodes can be explored [25, 26]. Since $\lfloor \frac{n}{2} \rfloor$ CEISTs have already been constructed for Q_n, CQ_n, and now for locally twisted cube LTQ_n, a rather reasonable conjecture would be that all hypercube variants, or even all bijective connection networks, have $\lfloor \frac{n}{2} \rfloor$ CEISTs.

Acknowledgment

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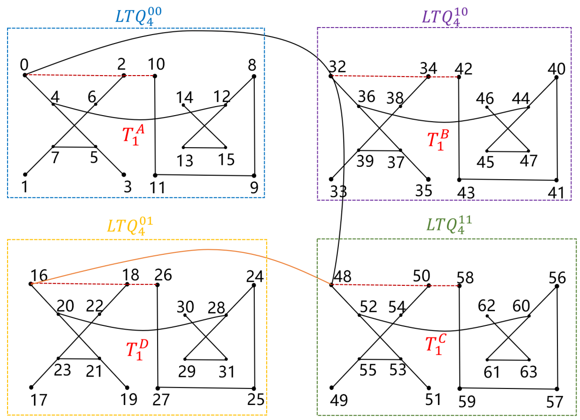
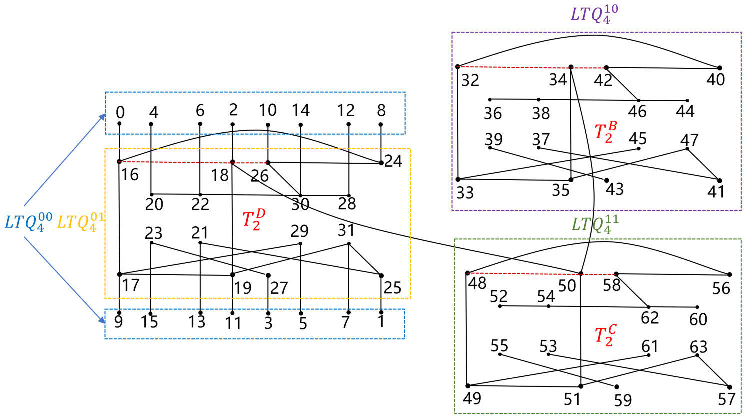
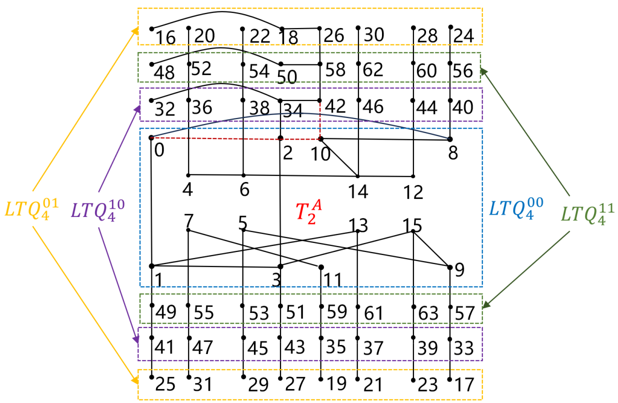


(a) T_1



(b) T_2



(c) T_3

Figure 6. Three CEISTs T_1, T_2 and T_3 in LTQ_6.

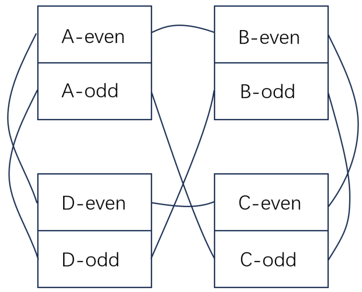
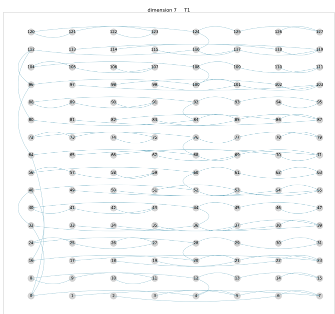
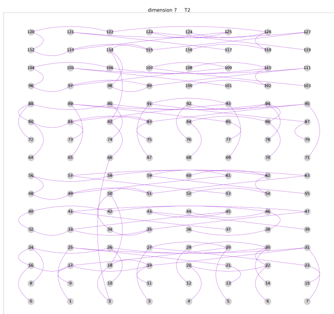


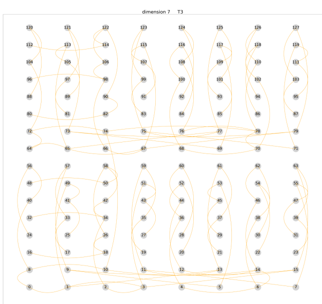
Figure 7. The topology of LTQ_n.



(a) T_1

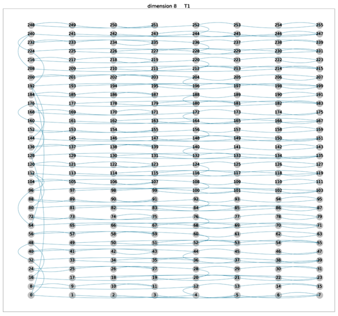


(b) T_2

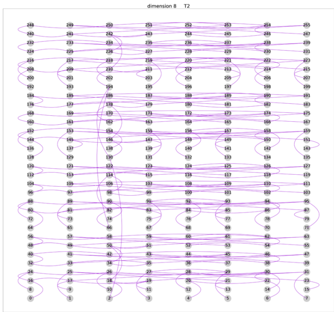


(c) T_3

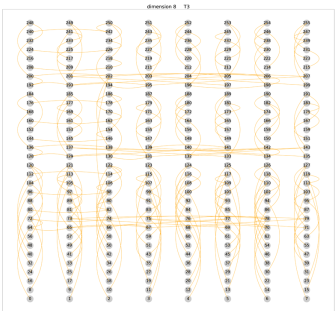
Figure 8. Three CEISTs in LTQ_7.



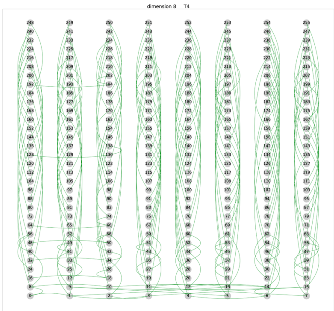
(a) T_1



(b) T_2



(c) T_3



(d) T_4

Figure 9. Four CEISTs in LTQ_8.

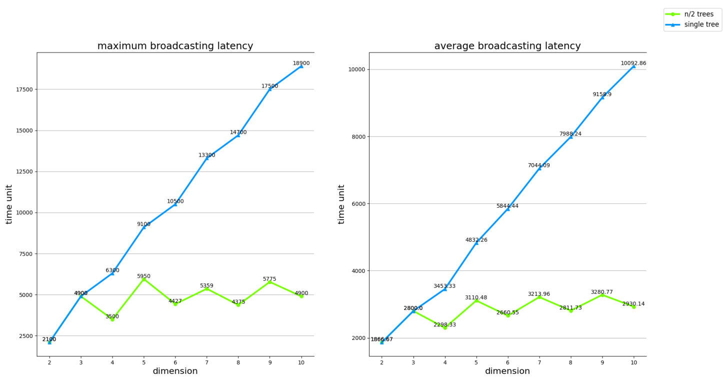


Figure 10. The comparisons of ABL and MBL between $\lfloor \frac{n}{2} \rfloor$ CEISTs and a single spanning tree.