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Equational Theory of Ordinals with Addition and Left Multiplication by ω

Christian Choffrut

IRIF, Université Paris Cité

Abstract. We show that the equational theory of the structure $\langle \omega^\omega : (x, y) \mapsto x + y, x \mapsto \omega x \rangle$ is finitely axiomatizable and give a simple axiom schema when the domain is the set of transfinite ordinals.

1. Introduction

Consider $A_n = \{a_i \mid 1 \leq i \leq n\}$, $n \leq \infty$ and let W_n denote the set of labeled ordinal words obtained from the empty word and each letter a by closing under concatenation $\cdot$ ($(u, v) \to u \cdot v$) and ω-*iteration* ($u^\omega = u \cdot u \cdots$), example $((ab)c)^\omega bc \in W_3$. Let $\mathcal{W}_n = \langle W_n : 1, \cdot, ^\omega \rangle$ denote the resulting algebra. It satisfies the following infinite (but reasonably simple) set Σ of axioms

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{1}$$

$$(x \cdot y)^\omega = x \cdot (y \cdot x)^\omega \tag{2}$$

$$(x^p)^\omega = x^\omega \quad p \geq 1 \tag{3}$$

$$x \cdot 1 = x \tag{4}$$

$$1 \cdot x = x \tag{5}$$

It is known that W_n is isomorphic to the free algebra with n generators in the variety generated by the equalities and that for $n > 1$, its equational theory is axiomatized by the system Σ, cf. [1]. The algebra $\mathcal{W}_1$ satisfies identities that do not hold in $\mathcal{W}_n$ with $n \geq 2$, for example $xyyx = yxyx$. Actually $\mathcal{W}_1$ is isomorphic to the set of ordinals less than ω^ω with the sum and right multiplication by ω ($x \to x\omega$).

In [2] it is shown that for the structure $\langle \omega^\omega : 0, 1, (x,y) \mapsto x+y, x \mapsto x\omega\rangle$ (in particular all constants less than ω^ω are allowed) equality between two terms is polynomial. Furthermore the more general algebra of all transfinite ordinals satisfies the same equalities.

Other different collections of operations on linearly ordered labeled words have been studied in the literature. By adding the ω^{opp}-iteration ($u^{\omega^{\text{opp}}} = \cdots u \cdot u$) we get the set W'_n and the corresponding algebra $\mathcal{W}'_n = \langle W'_n : 1, \cdot, ^\omega, ^{\omega^*}\rangle$. It is shown that for all $n \geq 1$ $\mathcal{W}'_n$ is isomorphic to the free n-generated algebra in the variety defined by a system of equations which is an axiomatization of the equational theory [3, Theorem 3.18]. In [4] a further operation is added, the *shuffle* operator, resulting in the set W''_n and the corresponding algebra $\mathcal{W}''_n = \langle W'_n : 1, \cdot, ^\omega, ^{\omega^*}, \eta\rangle$. Here again for all $n \geq 1$ $\mathcal{W}''_n$ is isomorphic to the free n-generated algebra in the variety defined by a system of equations which is an axiomatization of the equational theory

Summarising the situation if Eq($\mathcal{A}$) denotes the system of equations satisfied by an algebra $\mathcal{A}$, we have Eq($\mathcal{W}'_n$) =Eq($\mathcal{W}'_1$) and Eq($\mathcal{W}''_n$) =Eq($\mathcal{W}''_1$) because $\mathcal{W}'_n$ and $\mathcal{W}''_n$ are embeddable in $\mathcal{W}'_1$ and $\mathcal{W}''_1$ respectively. Yet Eq($\mathcal{W}_n$) is strictly included in Eq($\mathcal{W}_1$) if $n > 1$ and an axiom schema is missing for $\mathcal{W}_1$. A semantic approach was considered in [2] Indeed, the right and left handsides of an identity containing n variables may be viewed as mappings of $(\omega^\omega)^n$ into ω^ω. It is proved that an identity holds in $\mathcal{W}_1$ if and only if the two functions associated with the two handsides coincide over some so-called "test sets" such as all the powers ω^n with $n < \omega$ along with 0, for example. Refining this result allows one to exhibit a polynomial time algorithm which determines whether or not two expressions with the same set of variables define the same mapping, thus hold in $\mathcal{W}_1$. It is also proved that over ω^ω and the transfinite ordinals the equational theories are the same.

The algebra investigated here differs from $\mathcal{W}_1$ in that it considers the left (and not the right) multiplication by ω, i.e., we consider the signature $\langle (x,y) \mapsto x+y, x \mapsto \omega x\rangle$. We show that it is finitely axiomatizable in ω^ω and that the more general algebra over the same signature but with universe the transfinite ordinals satisfies a different set of equations of which we give a simple axiom schema.

2. Ordinals

We recall the elementary properties needed to understand this paper and refer to the numerous standard handbooks such as [5, 6] for a more thorough exposition of the theory. In particular each nonzero ordinal α is uniquely represented by its so-called Cantor normal form.

$$\omega^{\alpha_n} a_n + \cdots + \omega^{\alpha_0} a_0$$

$0 < a_0, \ldots a_n < \omega$ and $\alpha_n > \cdots > \alpha_0$ (a strictly decreasing sequence of ordinals). The *degree* of α denoted $\partial(\alpha)$ is α_n, its *valuation* denoted $\nu(\alpha)$ is α_0 and its *length* $|\alpha|$ is the sum $a_0 + \cdots + a_n$. The length of 0 is 0.

The sum $\alpha+\beta$ of two nonnull ordinals α and β of Cantor normal forms $\alpha=\sum_{i=p}^{i=0}\omega^{\alpha_i}a_i$ and $\beta=\sum_{j=q}^{j=0}\omega^{\beta_j}b_j$ is the ordinal with the following Cantor normal form

$$\left(\sum_{i=p}^{i=\ell+1}\omega^{\alpha_i}a_i\right)+\omega^{\beta_1}(a_\ell+b_q)+\left(\sum_{j=q-1}^{j=0}\omega^{\beta_j}b_j\right) \text{ if } \beta_q=\alpha_\ell \text{ for some } \ell$$

$$\left(\sum_{i=p}^{i=\ell}\omega^{\alpha_i}a_i\right)+\left(\sum_{j=q}^{j=0}\omega^{\beta_j}b_j\right) \quad \begin{array}{l}\text{if } \beta_q<\alpha_\ell \text{ for some } \ell\\ \text{and either } \ell=0\\ \text{or } \alpha_{\ell-1}<\beta_q\end{array}$$

$$\left(\sum_{j=q}^{j=0}\omega^{\beta_j}b_j\right) \text{ if } \beta_q>\alpha_p$$

In particular,

$$\begin{aligned}\partial(\alpha+\beta)&=\max(\partial(\alpha),\partial(\beta))\\ \nu(\alpha+\beta)&=\text{ if } \beta>0 \text{ then } \nu(\beta) \text{ else } \nu(\alpha)\end{aligned} \tag{6}$$

Addition is associative but not commutative: $\omega=1+\omega\neq\omega+1$.

If $\alpha+\beta+\gamma=\alpha+\gamma$ we say that β *does not count* which occurs exactly when $\partial(\beta)<\partial(\gamma)$.

Remark 1. $\partial(\beta_1),\partial(\beta_2)<\nu(\alpha_1)=\nu(\alpha_2)$ *and* $\alpha_1+\beta_1=\alpha_2+\beta_2$ *implies* $\beta_1=\beta_2$

We recall that the left multiplication by ω is distributive over the sum

$$\omega(\omega^{\alpha_1}a_1+\cdots+\omega^{\alpha_n}a_n)=\omega^{1+\alpha_1}a_1+\cdots+\omega^{1+\alpha_n}a_n \tag{7}$$

and that $\omega 0=0$.

Every $\alpha\geq\omega^\omega$ has a unique decomposition $\alpha=\alpha_1+\alpha_2$ with $\nu(\alpha_1)\geq\omega^\omega$ and $\partial(\alpha_2)<\omega^\omega$. In this case because of expression (7) we have $\omega\alpha_1=\alpha_1$ and thus

$$\omega(\alpha_1+\alpha_2)=\alpha_1+\omega\alpha_2 \tag{8}$$

3. Equational axiomatization for $\langle\omega^\omega:(x,y)\mapsto x+y,x\mapsto\omega x\rangle$

We consider the signature with one binary operation $+$ and one unary operation ω which we interpret in the structures $\mathcal{S}=\langle\omega^\omega:(x,y)\mapsto x+y,x\mapsto\omega x\rangle$ and $\mathcal{O}=\langle\text{Ord}:(x,y)\mapsto x+y,x\mapsto\omega x\rangle$ whose universes are respectively the set of all ordinals less than ω^ω and the set of all transfinite ordinals.

Let X be a fixed infinite countable set of elements called variables. The family of *terms* is inductively defined: variables and 0 are terms, if E and F are terms then $E+F$ is a term and if E is a term then ωE is a term. We will avoid unnecessary parentheses by adopting the usual conventions. We write $\Sigma_{i=1}^{i=k}E_i$ in place of $((\ldots(E_1+E_2)+\cdots)+E_{k-1})+E_k$ and ω^kE in place of $(\omega(\ldots((\omega E\ldots))))$. In case $k=0$ we let ω^0E be E.

We write $E(x_1, \ldots, x_n)$ when we need to make explicit the variables on which the term is inductively constructed. When working in ω^ω we will also consider $E(x_1, \ldots, x_n)$ as a function from $(\omega^\omega)^n$ into ω^ω in the natural way. Two terms E and F over the same set of variables are *equivalent* and we write $E \equiv F$ if they are equal as functions from $(\omega^\omega)^n$ into ω^ω. If ϕ is an assignment of the variables in ω^ω we write $\phi(E)$ to mean $E(\phi(x_1), \ldots, \phi(x_n))$. The notion of being equivalent in $\mathcal{O}$ is defined similarly.

An *identity* is a pair $(E(x_1, \ldots, x_n), F(x_1, \ldots, x_n))$ which we write

$$E(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)$$

in order to comply with the tradition, not to be confused with the equality as well-formed sequences of symbols over the vocabulary consisting of the variables and $0, +, \omega$. An identity is *satisfied* in the structure if E and F define the same functions.

A set Σ of identities called *axioms* is an *axiomatization* of the equational theory of $\mathcal{S}$ if all pairs of equivalent terms E, F are derivable from the axioms by the rules of equational logic and we write $E \underset{\mathtt{Ax}}{\equiv} F$.

Definition 2. *We consider the following set Σ of identities.*

$$x + (y + z) = (x + y) + z \tag{9}$$
$$\omega(x + y) = \omega x + \omega y \tag{10}$$
$$x + y + \omega x = y + \omega x \tag{11}$$
$$x + y + z + x + t + y = y + x + z + x + t + y \tag{12}$$
$$x + 0 = x \tag{13}$$
$$0 + x = x \tag{14}$$
$$\omega 0 = 0 \tag{15}$$

Let us verify that identity (12) holds in $\mathcal{S}$. Set $\mu = \max(\partial x, \partial y, \partial z, \partial t)$. If $\partial x, \partial y < \mu$ then both handsides reduce to $z+x+t+y$ because the leftmost occurrences of x and y do not count. If $\partial x = \mu$ and $\partial y < \mu$ both handsides reduce to $x + z + x + t + y$ and if $\partial x < \mu$ and $\partial y = \mu$ both handsides reduce to $y+z+x+t+y$. It remains the case where $\partial x = \partial y = \partial(z+x+t+y)$ say $x = \omega^\mu a+\alpha$, $y = \omega^\mu b + \beta$ and $z + x + t + y = \omega^\mu c + \gamma$ with $\partial\alpha, \partial\beta, \partial\gamma < \mu$. Then both handsides are equal to $\omega^\mu(a + b + c) + \gamma$.

The other identities are consequences of the definition of the sum.

Observe the consequence of (11)

$$\omega^p x + y + \omega^q x = y + \omega^q x \quad p < q \tag{16}$$

Indeed,

$$\omega x^p + y + \omega^q x = \omega x^p + y + \omega^{p+1} x \cdots + \omega^q x = y + \omega^{p+1} x \cdots + \omega^q x = y + \omega^q x$$

3.1. Elementary properties

Definition 3. *Given a variable x, an x-*monomial *is a term of the form $\omega^e x$ where e is a nonnegative integer and ω^e its* coefficient*. We simply speak of monomial when there is no need to specify which variable.*

Lemma 4. *Via the axioms* (10) *and* (11)*, every term in $\mathcal{S}$ is equivalent to a term of the form*

$$\sum_{i=n}^{1} \omega^{e_i} x_i, \quad e_i \geq 0 \tag{17}$$

with $i > j$ and $x_i = x_j$ implies $e_i \geq e_j$.

Proof:
Trivial by induction and identity (16). □

A term as in (17) is said to be *flat*. Since a flat term can be considered as a sequence of monomials, the slightly abusive expression "suffix of a term" has to be understood in the natural way.

Example 1. *The flat term corresponding to $E = \omega(x + \omega(\omega x + y + \omega x) + y) + x + x$ is*

$$\omega^3 x + \omega^2 y + \omega y + \omega^3 x + \omega y + x + x$$

which can be identified to the sequence $\omega^3 x, \omega^2 y, \omega y, \omega^3 x, \omega y, x, x$.

Lemma 5. *If two flat terms are equivalent, for every $x \in X$ their subsums consisting of the subsequence of their x monomials are equal.*

Proof:
Put $y = 0$ for all $y \neq x$. It suffices to observe that two flat terms with a unique variable x are equivalent if and only if they are equal since for $x = 1$ they evaluate in the same ordinal 1. □

A *decomposition* $E = E_k + \cdots + E_j + \cdots + E_1$ of a nonzero flat term E as in (17) is defined by a subsequence $n = i_k > i_{k-1} \ldots > i_1 > i_0 = 0$ and by grouping successive monomials

$$E_j = \sum_{i=i_j}^{i_{j-1}+1} \omega^{e_i} x_i \tag{18}$$

We will use two types of decomposition in Lemma 6 and in Theorem 9.

Lemma 6. *Consider the decomposition of a flat term*

$$E = F + \omega^e x + G$$

with $e > 0$ where G contains no occurrence of $\omega^e x$. Let $F' + \omega^e x + G'$ be the decomposition of an equivalent flat term where G' contains no occurrence of $\omega^e x$. Then G and G' contain the same multiset of occurrences of y-monomials for all variables y.

Proof:
Observe that the statement makes sense because by Lemma 5 if E and F are equivalent they have the same different monomials. It suffices to prove that the set of y-monomials in G with maximal coefficient is an invariant of the equivalence class of E. The statement is clearly true for $y = x$ because we are dealing with flat terms, so we assume $y \neq x$. Without loss of generality we may assume that the term contains no z-monomials for z different from x and y. If there is no occurrence of an y-monomial we are done. Let ω^f, $f < e$ be the greatest coefficient of an x-monomial in G if such an occurrence exist and ω^g the greatest coefficient of a y-monomial in G. Set $H = F + \omega^e x$ and let a, b be two integers such that $a+f = b+g$. Then $\nu(H(\omega^a, \omega^b)) = e+a$ and $G(\omega^a, \omega^b) = \omega^{a+f} \cdot n+\alpha$ where $\partial(\alpha) < a + f$ and n is the number of occurrences of $\omega^f x$ plus the number of occurrences of $\omega^g y$. If G contains no x-monomial, then $\nu(H(\omega^g, 1)) = e + g$ and $G(\omega^g, 1) = \omega^g \cdot n + \alpha$ where $\partial(\alpha) < g$ and n is the number of occurrences of $\omega^g y$. □

Definition 7. *With the notations of* (17) *the* new monomial decomposition *(NMD) of E is the sum $E = E_n + E_{n-1} + \cdots + E_1$ where $E_i = E'_i + \omega^{e_i} x_i$ for some E'_i such that $\omega^{e_i} x_i$ does not occur in $E_{i-1} + \cdots + E_1$ (by convention $E_0 = 0$).*

Observe that n is the number of different monomials in E. The idea is to record the moment when a new monomial appears in a scan from right to left.

Lemma 8. *Let E and F be two flat terms and their NMD*

$$\begin{aligned} E &= E'_n + \omega^{e_n} x_n + \cdots + E'_1 + \omega^{e_1} x_1 \\ F &= F'_m + \omega^{f_m} y_m + \cdots + F'_1 + \omega^{f_1} y_1 \end{aligned} \tag{19}$$

If $E \equiv F$ then $n = m$, $\omega^{e_i} x_i = \omega^{f_i} y_i$ for $i = 1, \ldots, n$ and for $i = 1, \ldots n$ E'_i and F'_i differ by a permutation of their monomials.

Proof:
Clearly $n = m$ because E and F have the same different monomials by Lemma 5. The last claim is a consequence of Lemma 6. □

Theorem 9. *Σ is an axiomatization of $\mathcal{S}$.*

Proof:
Consider the new monomial decompositions (19). By Definition 7 for all successive monomials in E'_i (resp. in F'_i) there exists an occurrence of these monomials in $\omega^{e_i} x_i + \cdots + E'_1 + \omega^{e_1} x_1$ resp. in $\omega^{e_i} x_i + \cdots + F'_1 + \omega^{e_1} x_1$. Since each permutation is a product of transpositions, we get $E \underset{\mathtt{Ax}}{\equiv} F$ by repetitive applications of axiom (12). □

4. Equational axiomatization for $\langle \mathcal{O} : (x, y) \mapsto x + y, x \mapsto \omega x \rangle$

Property (12) is valid in $\mathcal{O}$, by interpreting in the proof of Section 3, the exponent μ as an element of $\mathcal{O}$. Beyond ω^ω axiom (11) no longer holds. Indeed since $\omega\omega^\omega = \omega^{1+\omega} = \omega^\omega$.

$$\omega^\omega \cdot 2 = \omega^\omega + \omega\omega^\omega \neq \omega\omega^\omega = \omega^\omega \tag{20}$$

We consider the system Σ' consisting of the axioms of Σ except axiom (11) along with the following new axioms.

$$\omega^p x + y + \omega^q x = x + y + \omega^q x \quad 0 < p < q \tag{21}$$

$$x + \omega^r y + t + \omega^p x + u + \omega^q y = \omega^r y + x + t + \omega^p x + u + \omega^q y \quad p > 0 \text{ or } q = r, \tag{22}$$

$$x + \omega^r y + t + \omega^q y + u + \omega^p x = \omega^r y + x + t + \omega^q y + u + \omega^p x \quad p > 0 \text{ or } q = r \tag{23}$$

Lemma 10. *Identities* (21), (22) *hold in the structure* $\langle \mathcal{O} : (x, y) \mapsto x + y, x \mapsto \omega x \rangle$.

Proof:
Indeed, (21) holds in ω^ω since by identity (11) both hand sides reduce to $y + \omega^q x$. If $x \geq \omega^\omega$ we decompose $x = x_1 + x_2$ with $\nu x_1 \geq \omega$ and $\partial x_2 < \omega$ as in expression (8). Because $\omega^p \cdot x_1 = x_1$ we get $\omega^p \cdot x = x_1 + \omega^p \cdot x_2$ and thus

$$\begin{array}{cll} & \omega^p x + y + \omega^q x & \\ = & x_1 + \omega^p \cdot x_2 + y + \omega^q \cdot x & (\text{decomposition of } x) \\ = & x_1 + y + \omega^q \cdot x & (\partial \omega^p x_2 < \partial \omega^q x) \\ = & x_1 + x_2 + y + \omega^q \cdot x & (\partial x_2 < \partial \omega^q x) \\ = & x + y + \omega^q \cdot x & (\text{recomposition of } x) \end{array}$$

Concerning (22) and (23), if $\partial(x), \partial(y) \geq \omega$ in both cases the two hand sides reduce

$$x + y + t + x + u + y = y + x + t + x + u + y$$

which holds because of identity (12). Consider (22). If $p = 0$, thus $q = r$, then this is axiom (12). If $p > 0$ and $x < \omega^\omega$ the leftmost x does not count and if $x \geq \omega^\omega$ then $\omega^r y$ does not count. Identity (23) is proved similarly. □

Because of identity (21) every term is equivalent to a term of the form

$$E = \sum_{k=m}^{1} \omega^{e_k} x_k, \quad e_k \geq 0 \tag{24}$$

where $i > j$, $x_i = x_j$ and $e_i, e_j > 0$ implies $e_i \geq e_j$ which we call *pseudo flat*. An occurrence of a variable x (i.e., a monomial with coefficient 1) to the left of some monomial $\omega^e x$ with $e > 0$ is called *hidden*.

Example 2. *In*

$$z + \omega x + \underline{x} + \underline{y} + \omega y + \omega x + y + \omega x + y + x$$

the unique hidden occurrences are the leftmost underlined occurrences of x and y.

The definition 7 of new monomial decomposition extends naturally to the present structure.

Example 3. *(Example 2 continued) With the notations of definition 7 we have*

$$(z) + (\omega x + x + y + \omega y) + (\omega x + y + \omega x) + (y) + (x)$$

Indeed, when reading the expression from right to left we define five groups of terms and start a group whenever we find a monomial that never occurred before (the rightmost monomial of each group). For example the third group (from the right) starts with the new monomial ωx and contains no new monomial; it ends just before a new monomial (ωy) appears. Observe that the hidden occurrences are not deleted.

The following is obtained by simple adaptation of Lemma 8.

Lemma 11. *Consider the new monomial decomposition of two equivalent pseudo-flat terms E and F. Then*

$$E = E'_n + \omega^{e_n} x_n + \cdots + E'_1 + \omega^{e_1} x_1$$
$$F = F'_n + \omega^{e_n} x_n + \cdots + F'_1 + \omega^{e_1} x_1$$

where for all $i = 1, \ldots, n$ E'_i and F'_i contain the same occurrences of nonhidden monomials.

Now instead of splitting accordingly to a "new monomial" we split according to a "new variable" (have in mind a scan from right to left).

Definition 12. *With the notations of* (24) *the* new variable decomposition *(NVD) of a pseudo-flat E is the sum $E = E_n + E_{n-1} + \cdots + E_1$ where for each $i = 1, \ldots, n$ $E_i = E'_i + \omega^{e_i} x_i$ such that no x_i-monomial appears in $E_{i-1} + \cdots + E_1$ with the convention $E_0 = 0$.*

No hidden occurrence can be the right-most monomial of some E_i and that the number of subterms in the decomposition equals the number of variables in the term.

Example 4. *(Example 2 continued). The new variable decomposition is a sum of three subterms.*

$$(z) + (\omega x + x + y + \omega y + \omega x + y + \omega x + y) + (x)$$

Observe the difference with the new monomial decomposition. Here the expression is decomposed in three groups. From right to left: the x-monomials, then the y-monomials with possibly some x-monomials and finally the z-monomials and possibly some y- or x-monomials. here

Lemma 13. *Let $E(x_1, \ldots, x_n)$ and $F(x_1, \ldots, x_n)$ be two equivalent pseudo-flat terms and consider their new variable decompositions $E = E_n + E_{n-1} + \cdots + E_1$ and $F = F_m + F_{n-1} + \cdots + F_1$. Then $n = m$ and for all $i \leq j$ the number of hidden occurrences of each x_i in E_j is equal to the number of hidden occurrences x_i in F_j.*

Proof:
Equality $n = m$ is obvious because it is the number of variables of the two equivalent terms E and F. By possibly renaming the variables we may assume that for $i = 1, \ldots, n$ the rightmost monomial in E_i is $\omega^{e_i} x_i$ for some $e_i \geq 0$. By definition a hidden occurrence of x_i can only belong to the set of monomials in $E_n + \cdots + E_i$. We set for all $i \leq j \leq n + 1$

- $H_{i,j}$ the number of hidden occurrences of x_i in $E_{j-1} + \cdots + E_i$
- $N_{i,j}$ the number occurrences of nonhidden x_i-monomials in $E_{j-1} + \cdots + E_i$

Consider the assignment $\phi(x_i) = 1$, $\phi(x_j) = \omega^a$ and $\phi(x_\ell) = 0$ for $\ell \neq i, j$. For $a < \omega$ greater than the exponents of the coefficients of all x_i-monomials we have $\phi(E) = \alpha + \beta$ where $\nu(\alpha) \geq a > \partial(\beta)$. Then $N_{i,j} = |\beta|$. Now consider the assignment $\phi(x_i) = \omega^\omega$, $\phi(x_j) = \omega^{\omega+1}$ and $\phi(x_\ell) = 0$ for $\ell \neq i, j$. Then we have $\phi(E) = \omega^{\omega+1}c + \omega^\omega d, c, d < \omega$ and $H_{i,j} = d - N_{i,j}$. Then the number of hidden occurrences of x_i in E_j is equal to $H_{i,j+1} - H_{i,j}$ □

Theorem 14. *The system of identities Σ' is an axiomatization of the equational theory of the structure* $\langle \text{Ord} : (x,y) \mapsto x + y, x \mapsto \omega x \rangle$

Proof:
Let E and F be two equivalent pseudo-flat terms and let $E = E_n + E_{n-1} + \cdots + E_1$ and $F = F_n + F_{n-1} + \cdots + F_1$ be their NVD. We show that we have $E \underset{\mathtt{Ax}}{\equiv} \widehat{E}$ where $\widehat{E} = F_n + E_{n-1} + \cdots + E_1$. By Lemma 13 E_n and F_n have the same number of hidden occurrences for all variables. We now consider their NMD. By Lemma 11 we have

$$
\begin{aligned}
E_n &= E'_{n,s} + \omega^{e_{n,s}} x_{n,s} + \cdots + E'_{n,1} + \omega^{e_{n,1}} x_{n,1} \\
F_n &= F'_{n,s} + \omega^{e_{n,s}} x_{n,s} + \cdots + F'_{n,1} + \omega^{e_{n,1}} x_{n,1}
\end{aligned}
$$

where for $j = s, \ldots, 1$, $E'_{n,j}$ and $F'_{n,j}$ have the same set of occurrences of nonhidden monomials. By equations (22), (23) (with $r = q$) and (12), all occurrences of nonhidden monomials in $E'_{n,j}$ commute pairwise and with the hidden monomials. So we have with $R = E_{n-1} + \cdots + E_1$

$$
\begin{aligned}
& E'_{n,s} + \omega^{e_{n,s}} x_{n,s} + \cdots + E'_{n,1} + \omega^{e_{n,1}} x_{n,1} + R \\
\underset{\mathtt{Ax}}{\equiv}\; & E''_{n,s} + \omega^{e_{n,s}} x_{n,s} + \cdots + E''_{n,1} + \omega^{e_{n,1}} x_{n,1} + R
\end{aligned}
$$

where for $j = s, \ldots, 1$, $E''_{i,j}$ and $F'_{i,j}$ have the same sequence of nonhidden monomials. They may only differ in the number and positions of the hidden monomials. Since all hidden monomials commute by (12) with all monomials in $E''_{n,s} + \omega^{e_{n,s}} x_{n,s} + \cdots + E''_{n,1} + \omega^{e_{n,1}} x_{n,1} + R$ and since the number of hidden monomials in F_n and $E''_{n,s} + \omega^{e_{n,s}} x_{n,s} + \cdots + E''_{n,1} + \omega^{e_{n,1}} x_{n,1}$ are the same for all variables, we have

$$
F_n \underset{\mathtt{Ax}}{\equiv} E''_{n,s} + \omega^{e_{n,s}} x_{n,s} + \cdots + E''_{n,1} + \omega^{e_{n,1}} x_{n,1} + R
$$

Now $E \underset{\mathtt{Ax}}{\equiv} \widehat{E}$ implies $E \equiv \widehat{E}$ thus $\widehat{E} \equiv F$ and finally by cancelation $E_{n-1} + \cdots + E_1 \equiv F_{n-1} + \cdots + F_1$ which allows us to conclude by induction. □

5. Complexity

We show that the complexity of determining the equivalence of two terms E and F is linear in the size of the expressions as element of the algebras $\mathcal{S}$ and Ord respectively.

The flattening of a term can be achieved as follows. Construct in linear time the syntactic tree whose nodes are of arity 1 for the left multiplication by ω and of arity 2 for the addition. The leaves of the tree are labeled by the different variables occurring in the term. Then a depth-first search assigns to each variable its coefficients ω^i.

A flat expression is a sum of monomials. Thus we may consider the finite set A whose elements a_{ij} are in one-to-one correspondence with the different monomials $\omega^{e_j} x_i$, $1 \leq i \leq n, 1 \leq j \leq m_i$ occurring in the term. Two terms can only be equivalent if they have the same multiset of monomials. We define $A_i = \{a_{i,j} : 1 \leq j \leq m_i\}$ for all $1 \leq i \leq n$ and thus $A = \bigcup_{i=1}^{n} A_i$. A flat expression can be identified with a sequence $u \in A^*$ satisfying the conditions

- for all i there exists $1 \leq j \leq m_i$ and v, w such that $u = va_{ij}w$ (by the definition of A)
- for all factorizations $u = va_{ij}wa_{ik}z$ it holds $j \geq k$. (see condition (17))

For all $v \in A^*$ we set $\mathrm{C}(v) = \{a_{ij} \in A : \exists w, z \ (v = wa_{ij}z)\}$. Henceforth by flat expression we mean a sequence on A subject to the above conditions. A flat expression has a unique factorization $u = u_n u_{n-1} \cdots u_1$, which we call *subalphabet factorization* (see the NMD) such that u_1 is the longest word such that $|C(u_1| = 1$ and for all $i > 1, u_i$ is the longest word such that $|C(u_i \cdots u_1) \setminus C(u_{i-1} \cdots u_1)| = 1$

The equivalence of two flat terms in $\mathcal{S}$ can be done by comparing in a single pass their subalphabet factorizations $u = u_n u_{n-1} \cdots u_1$ and $v_m v_{m-1} \cdots v_1$ and by checking whether or not $C(u_i) = C(v_i)$ for all $i \leq 1$. The equivalence of two pseudo-flat terms in Ord can be done by. additionally counting the hidden variables in the same pass.

The following shows that E and F are equivalent if and only if their restrictions to any pair of variables are equivalent (the restriction to $\{x, y\}$ consists in deleting all z-monomials for z different from x and y).

Proposition 15. *Two expressions E and F are equivalent if and only if their restrictions on any two pair of variables are equivalent.*

Proof:
We identify two expressions with two sequences $u, u' \in A^*$ as above. Consider their subalphabet factorizations $u = u_n u_{n-1} \cdots u_1$ and $u' = u'_n u'_{n-1} \cdots u'_1$ (they have the same length). For every $1 \leq i, j \leq n$ let $\pi_{i,j}$ be the projection of A^* onto $(A_i \cup A_j)^*$. The rightmost letter of u_i and u'_i are the same. This is due to the fact that for all pairs (i, j) we have $\pi_{i,j}(u) = \pi_{i,j}(u')$ showing that all rightmost occurrences of $a \in A$ in u and u' we have $u = vaw$ and $u' = v'aw'$ with $C(w) = C(w')$.

Now we prove that for all $a \in A_k$ the number of occurrences of a in u_i depends only on the projections $\pi_{i,\ell}$ for $1 \leq \ell \leq n$. Indeed, assume the rightmost letters of u_i and u_{i+1} are in A_r and A_s respectively. The number of occurrences of a in $u_n \cdots u_i$ is equal to the number of occurrences of a in $\pi_{k,s}(u_n u_{n-1} \cdots u_i)$ and the number of occurrences of a in $u_n \cdots u_{i+1}$ is equal to the number of occurrences of a in $\pi_{k,s}(u_n u_{n-1} \cdots u_{i+1})$. □

Corollary 16. *Equality $E = F$ is provable if and only if for all $1 \leq i < j \leq n$ equalities $E_{ij} = F_{ij}$ are provable.*

References

[1] S. L. Bloom and C. Choffrut, Long words: the theory of concatenation and ω-power, *Theoret. Comp. Sci.*, 259 (2001), 533–548.

[2] C. Choffrut and S. Grigorieff, Test sets for equality of terms in the additive structure of ordinals augmented with right multiplication by ω, *Algebra Universalis*, 81, 45, 2020.

[3] S. L. Bloom and Z. Ésik. Axiomatizing omega and omega-op powers of words. *RAIRO Theor. Informatics Appl.*, 38, 1, 3–17, 2004.

[4] S. L. Bloom and Z. Ésik. The equational theory of regular words. *Inf. Comput.*, 197, 1-2, 55–89, 2005.

[5] J.G. Rosenstein, *Linear ordering*, Academic Press, New-York, 1982

[6] W. Sierpiński, *Cardinal and Ordinal Numbers*, 2nd ed., PWN-Polish Scientific Publishers, 1965.