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Constructing Disjoint Steiner Trees in Sierpiński Graphs*

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Abstract. Let G be a graph and $S \subseteq V(G)$ with $|S| \geq 2$. Then the trees $T_1, T_2, \cdots, T_\ell$ in G connecting S are *internally disjoint Steiner trees* (or S-Steiner trees) if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers $1 \leq i, j \leq \ell$. Similarly, if we only have the condition $E(T_i) \cap E(T_j) = \emptyset$ but without the condition $V(T_i) \cap V(T_j) = S$, then they are *edge-disjoint Steiner trees* S-Steiner trees. The *generalized k-connectivity*, denoted by $\kappa_k(G)$, of a graph G, is defined as $\kappa_k(G) = \min\{\kappa_G(S) | S \subseteq V(G) \text{ and } |S| = k\}$, where $\kappa_G(S)$ is the maximum number of internally disjoint S-Steiner trees. The *generalized k-edge-connectivity* $\lambda_k(G)$ of G is defined as $\lambda_k(G) = \min\{\lambda_G(S) \,|\, S \subseteq V(G)\ and\ |S| = k\}$, where

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$\lambda_G(S)$ is the maximum number of edge-disjoint Steiner trees connecting S in G. These concepts are generalizations of the concepts of connectivity and edge-connectivity, and they can be used as measures of vulnerability of networks. It is, in general, difficult to compute these generalized connectivities. However, there are precise results for some special classes of graphs. In this paper, we obtain the exact value of $\lambda_k(S(n,\ell))$ for $3 \leq k \leq \ell^n$, and the exact value of $\kappa_k(S(n,\ell))$ for $3 \leq k \leq \ell$, where $S(n,\ell)$ is the Sierpiński graphs with order ℓ^n. As a direct consequence, these graphs provide additional interesting examples when $\lambda_k(S(n,\ell)) = \kappa_k(S(n,\ell))$. We also study the some network properties of Sierpiński graphs.

Keywords: Steiner Tree; Generalized Connectivity; Sierpiński Graph.
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1. Introduction

All graphs considered in this paper are undirected, finite and simple. We refer the readers to [1] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. The *neighborhood set* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \,|\, uv \in E(G)\}$. The *degree* of a vertex v in G is denoted by $d(v) = |N_G(v)|$. Denote by $\delta(G)$ ($\Delta(G)$) the minimum degree (maximum degree) of the graph G. For a vertex subset $S \subseteq V(G)$, the subgraph induced by S in G is denoted by $G[S]$ and similarly $G[V \setminus S]$ for $G \setminus S$ or $G - S$. Especially, $G - v$ is $G[V \setminus \{v\}]$. Let $\overline{G}$ be the complement of G. For a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_t\}$ of $V(G)$, let $G/\mathcal{P}$ be the graph obtained from G by deleting $\bigcup_{i\in[t]} E(G[V_i])$ and then identifying each V_i, respectively. For any positive integers n, we always use the convenient notation $[n]$ to denote the set $\{1, 2, \cdots, n\}$.

1.1. Generalized (edge-)connectivity

Connectivity and edge-connectivity are two of the most basic concepts of graph-theoretic measures. Such concepts can be generalized, see, for example, [16]. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Note that when $|S| = 2$ a minimal S-Steiner tree is just a path connecting the two vertices of S.

Let G be a graph and $S \subseteq V(G)$ with $|S| \geq 2$. Then the trees $T_1, T_2, \cdots, T_\ell$ in G are *internally disjoint S-trees* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers i, j, $1 \leq i, j \leq \ell$. Similarly, if we only have the condition $E(T_i) \cap E(T_j) = \emptyset$ but without the condition $V(T_i) \cap V(T_j) = S$, then they are *edge-disjoint S-trees* (Note that while we do not have the condition $V(T_i) \cap V(T_j) = S$, it is still true that $S \subseteq V(T_i) \cap V(T_j)$ as T_i and T_j are S-trees.) The *generalized k-connectivity*, denoted by $\kappa_k(G)$, of a graph G, is defined as $\kappa_k(G) = \min\{\kappa_G(S) | S \subseteq V(G)$ and $|S| = k\}$, where $\kappa_G(S)$ is the maximum number of internally disjoint S-trees. The *generalized local edge-connectivity* $\lambda_G(S)$ is the maximum number of edge-disjoint S-trees in G. The *generalized k-edge-connectivity* $\lambda_k(G)$ of G is defined as $\lambda_k(G) = \min\{\lambda_G(S) \,|\, S \subseteq V(G)\ and\ |S| = k\}$. Since internally disjoint S-trees are edge-disjoint but not vice versa, it follows from the definitions that $\kappa_k(G) \leq \lambda_k(G)$. There are many results on generalized (edge-)connectivity; see the book [15] by Li and Mao.

For a graph G and two distinct vertices x, y of G, the local connectivity $p_G(x, y)$ of x and y is defined as the maximum number of pairwise internally disjoint paths between x and y, and the local edge-connectivity $\lambda_G(x, y)$ is defined as the maximum number of pairwise edge-disjoint paths between x and y. The connectivity of G is defined as $\kappa(G) = \min\{p_G(x, y) \mid x, y \in V(G),\ x \neq y\}$, and the edge-connectivity of G is defined as $\lambda(G) = \min\{\lambda_G(x, y) \mid x, y \in V(G),\ x \neq y\}$. It is clear that when $|S| = 2$, $\lambda_2(G)$ is just the standard edge-connectivity $\lambda(G)$ of G, $\kappa_2(G) = \kappa(G)$, that is, the standard connectivity of G. Thus $\kappa_k(G)$ and $\lambda_k(G)$ are the generalized connectivity of G and the generalized edge-connectivity of G, respectively.

As it is well-known, for any graph G, we have polynomial-time algorithms to find the classical connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$. Given two fixed positive integers k and ℓ, for any graph G the problem of deciding whether $\lambda_k(G) \geq \ell$ can be solved by a polynomial-time algorithm. If k $(k \geq 3)$ is a fixed integer and ℓ $(\ell \geq 2)$ is an arbitrary positive integer, the problem of deciding whether $\kappa(S) \geq \ell$ is NP-complete. For any fixed integer $\ell \geq 3$, given a graph G and a subset S of $V(G)$, deciding whether there are ℓ internally disjoint Steiner trees connecting S, namely deciding whether $\kappa(S) \geq \ell$, is NP-complete. For more details on the computational complexity of generalized connectivity and generalized edge-connectivity, we refer to the book [15].

In addition to being a natural combinatorial measure, generalized k-connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one wants to "connect" a pair of vertices of G "minimally", then a path is used to "connect" them. More generally, if one wants to "connect" a set S of vertices of G, with $|S| \geq 3$, "minimally", then it is desirable to use a tree to "connect" them. Such trees are precisely S-trees, which are also used in computer communication networks (see [8]) and optical wireless communication networks (see [6]).

From a theoretical perspective, generalized edge-connectivity is related to Nash-Williams-Tutte theorem and Menger theorem; see [15]. The generalized edge-connectivity has applications in $VLSI$ circuit design. In this application, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. Another application, which is our primary focus, arises in the Internet Domain. Imagine that a given graph G represents a network. We arbitrarily choose k vertices as nodes. Suppose one of the nodes in G is a *broadcaster*, and all other nodes are either *users* or *routers* (also called *switches*). The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via a tree connecting all the users and the broadcaster. So, in essence we need to find the maximum number Steiner trees connecting all the users and the broadcaster, namely, we want to get $\lambda(S)$, where S is the set of the k nodes. Clearly, it is a Steiner tree packing problem. Furthermore, if we want to know whether for any k nodes the network G has above properties, then we need to compute $\lambda_k(G) = \min\{\lambda(S)\}$ in order to prescribe the reliability and the security of the network. For more details, we refer to the book [15].

1.2. Sierpiński graphs

In 1997, Klavžar and Milutinović introduced the concept of Sierpiński graph $S(n, \ell)$ in [11]. We denote n-tuples V^n by the set

$$V^n = \{\langle u_0 u_1 \cdots u_{n-1}\rangle \mid u_i \in \{0, 1, \ldots, \ell - 1\} \text{ and } i \in \{0, 1, \ldots, n-1\}\}.$$

A *word* u of size n are denoted by $\langle u_0u_1, \cdots, u_{n-1}\rangle$ in which $u_i \in \{0, \dots, \ell-1\}$. The concatenation of two words $u = \langle u_0u_1 \cdots u_{n-1}\rangle$ and $v = \langle v_0v_1 \cdots v_{n-1}\rangle$ is denoted by uv.

Definition 1. *The Sierpiński graph $S(n, \ell)$ is defined as below, for $n \geq 1$ and $\ell \geq 3$, the vertex set of $S(n, \ell)$ consists of all n-tuples of integers $0, 1, \cdots, \ell - 1$. That is, $V(S(n, \ell)) = V^n$, where $uv = (u_0u_1 \cdots u_{n-1}, v_0v_1 \cdots v_{n-1})$ is an edge of $E(S(n, \ell))$ if and only if there exists $d \in \{0, 1, \dots, \ell-1\}$ such that:* (1) $u_j = v_j$, *if* $j < d$; (2) $u_d \neq v_d$; (3) $u_j = v_d$ *and* $v_j = u_d$, *if* $j > d$.

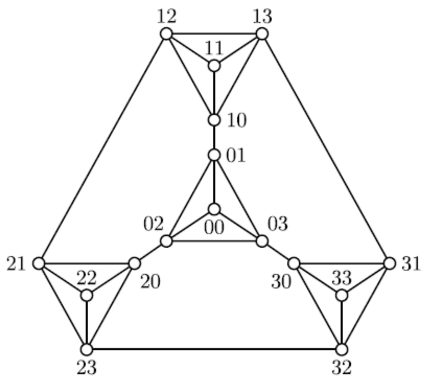

Figure 1: $S(2, 4)$

Sierpiński graph $S(2, 4)$ is shown in Figure 1. Note that $S(n, \ell)$ can be constructed recursively as follows: $S(1, \ell)$ is isomorphic to K_ℓ, which vertex set is 1-tuples set $\{0, \dots, \ell - 1\}$. To construct $S(n, \ell)$ for $n > 1$, we take copies of ℓ times $S(n - 1, \ell)$ and add the letter i on the top of the vertices in i-th copy of $S(n - 1, \ell)$, denoted by $S^i(n, \ell)$. Note that there is exactly one edge (*bridge edge*) between $S^i(n, \ell)$ and $S^j(n, \ell)$, $i \neq j$, namely the edge between vertices $\langle ij \cdots j\rangle$ and $\langle ji \cdots i\rangle$.

The vertices $(\underbrace{i, i, \cdots, i}_{n})$, $i \in \{0, 1, \dots, \ell - 1\}$ are the *extreme vertices* of $S(n, \ell)$. Note that an extreme vertex u of $S(n, \ell)$ has degree $d(u) = \ell - 1$. For $i \in \{0, \dots, \ell - 1\}$ and $n \geq 2$, let $S^i(n - 1, \ell)$ denote the subgraph of $S(n, \ell)$ induced by the vertices of the form $\{\langle iu_1 \cdots u_{n-1}\rangle \,|\, 0 \leq u_i \leq \ell - 1\}$. The vertex set $V(S(n, \ell))$ can be partitioned into ℓ parts $V(S^0(n - 1, \ell)), V(S^1(n - 1, \ell)), \dots, V(S^{\ell-1}(n - 1, \ell))$. For each $0 \leq i \leq \ell - 1$, $S^i(n - 1, \ell)$ is isomorphic to $S(n - 1, \ell)$. Note that $V(S(n, \ell)) = V(S^0(n - 1, \ell)) \cup \cdots \cup V(S^{\ell-1}(n - 1, \ell))$ and $S(n, \ell)$ is the graph obtained from $S^0(n - 1, \ell), \dots, S^{\ell-1}(n - 1, \ell)$ by adding exactly one edge (*bridge edge*) between $S^i(n - 1, \ell)$ and $S^j(n - 1, \ell)$, $i \neq j$, and the bridge edge joins $\langle ij \cdots j\rangle$ and $\langle ji \cdots i\rangle$ (notices that $\langle ij \cdots j\rangle$ and $\langle ji \cdots i\rangle$ are extreme vertices of $S^i(n - 1, \ell)$ and $S^j(n - 1, \ell)$, respectively, if we regard $S^i(n - 1, \ell)$ and $S^j(n - 1, \ell)$ as two copies of $S(n - 1, \ell)$).

Sierpiński graphs generalize Hanoi graphs which can be viewed as "discrete" finite versions of a Sierpiński gasket [23, 10]. Xue considered the Hamiltionicity and path t-coloring of Sierpiński-like graphs in [25]; furthermore, they proved that $Val(S(n, k)) = Val(S[n, k]) = \lfloor k/2 \rfloor$, where $Val(S(n, k))$ is the linear arboricity of Sierpiński graphs. We remark that although Sierpiński graphs are not regular, they are “almost” regular as the extreme vertices have degrees one less than the degrees of non-extreme vertices.

1.3. Preliminaries and our results

Chartrand et al. [4] and Li et al. [16] obtained the exact value of $\kappa_k(K_n)$.

Theorem 1.1. ([16, 4])
For every two integers n and k with $2 \le k \le n$,

$$\kappa_k(K_n) = \lambda_k(K_n) = n - \lceil k/2 \rceil.$$

The following result is on the Hamiltonian decomposition of Sierpiński graphs.

Theorem 1.2. [25] (1) *For even $\ell \ge 2$, $S(n,\ell)$ can be decomposed into edge disjoint union of $\frac{\ell}{2}$ Hamiltonian paths of which the end vertices are extreme vertices.*

(2) *For odd $\ell \ge 3$, there exist $\frac{\ell-1}{2}$ edge-disjoint Hamiltonian paths whose two end vertices are extreme vertices in $S(n,\ell)$.*

In fact, Theorem 1.2 is used in the proof of Theorem 1.5, which give an lower bound for the generalized k-edge connectivity of Sierpiński graphs $S(n,\ell)$. We require the following result.

Theorem 1.3. ([5, 26])
Suppose that G is a complete graph with $V(G) = \{v_0, \cdots, v_{N-1}\}$. If $N = 2n$, then G can be decomposed into n Hamiltonian paths

$$\{i \in \{1,2,\ldots,n\} : \ L_i = v_{0+i}v_{1+i}v_{2n-1+i}v_{2+i}v_{2n-2+i}\cdots v_{n+1+i}v_{n+1}\},$$

where the subscripts take modulo $2n$. If $N = 2n+1$, then G can be decomposed into n Hamiltonian paths

$$\{i \in \{1,2,\ldots,n\} : \ L_i = \ v_{0+i}v_{1+i}v_{2n+i}v_{2+i}v_{2n-1+i}\cdots v_{n+i}v_{n+1+i}\}$$

and a matching $M = \{v_{n-i}v_{n+1} : \ i \in \{1,2,\ldots,n\}\}$, where the subscripts take modulo $2n+1$.

The following result is derived from Theorem 1.3, and we will use it later.

Corollary 1.4. *Let s be an integer with $s \le \frac{N}{2}$. Suppose that G is the complete graph with $V(G) = \{v_1, \cdots, v_N\}$ and $\mathcal{S} = \{\,\{v_{i_1}, v_{i_2}\} : i \in \{1,2,\ldots,s\}\}$ is a collection of pairwise disjoint 2-subsets of $V(G)$. Then there are s edge-disjoint Hamiltonian paths $L_1, \cdots, L_s$ such that v_{i_1}, v_{i_2} are endpoints of L_i.*

Our main result is as follows.

Theorem 1.5. (i) *For $3 \le k \le \ell$, we have*

$$\kappa_k(S(n,\ell)) = \lambda_k(S(n,\ell)) = \ell - \lceil k/2 \rceil.$$

(ii) *For $\ell + 1 \le k \le \ell^n$, we have*

$$\kappa_k(S(n,\ell)) \le \lfloor \ell/2 \rfloor \text{ and } \lambda_k(S(n,\ell)) = \lfloor \ell/2 \rfloor.$$

2. Proof of Theorem 1.5

This section is organized as follows. We first introduce some notations, operations and auxiliary graphs that are used in the proof. Then we give a transitional theorem (Theorem 2.1), which is the main tool for the proof of Theorem 1.5. Theorem 2.1 will be proved by constructing an algorithm, and the proof takes up almost entire Section 2 (the proof is divided into two parts: the construction of algorithm and the proof of its correctness). Finally, we complete the proof of Theorem 1.5 by using Theorem 2.1.

2.1. Notations, operations and auxiliary graphs

For $1 \leq s < n$, by the definition of the Sierpiński graph $S(n, \ell)$, we have that $G = S(n, \ell)$ consists of ℓ^{n-s} Sierpiński graphs $S(s, \ell)$, and we call each such $S(s, \ell)$ an *s-atom* of G. Note that $\mathcal{P} = \{V(H) : \ H \text{ is an } s\text{-atom of } G\}$ is a partition of $V(G)$. We use G_s to denote the graph $G/\mathcal{P}$. It is obvious that G_s is a simple graph and $G_s = S(n-s, \ell)$. Note that G_n is a single vertex and $G_0 = G$. For a vertex u of G_s, we use $A_{s,u}$ to denote the s-atom of G which is contracted into u. See Figures 2, 3 and 4 for illustrations.

Notation	**Corresponding interpretation**
s-atom	A copy of $S(s, \ell)$ in $G = S(n, \ell)$, where $0 \leq s \leq n$.
G_s	The graph obtained from G by contracting all ℓ^{n-s} s-atoms in G.
$A_{u,s}$	The s-atom in G which is contracted into u, where $u \in V(G_s)$.
H^u	G_{s-1} can be obtained form G_s by expending each $u \in V(G_s)$ to a complete graph $H^u = K_\ell$.
u^e, v^e	If $e = uv$ is an edge of G_s, then e is also an edge of G_{s-1}. let u_e, v_e denote endpoints of e in G_{s-1} such that $u_e \in H^u$ and $v_e \in H^v$.
U	The set of labelled vertices in G ($\lvert U \rvert = k$).
U^s	The set of labelled vertices in G_s.
W^u	The set of labelled vertices in H^u.

Table 1: Notations for and their meanings.

Note that G_{s-1} is the graph obtained from G_s by replacing each vertex u with a complete graph K_ℓ. We denote by H^u the complete graph replacing u (see Figures 3 and 4 for illustrations). For an edge $e = uv$ of G_s, e is also an edge of G_{s-1}. We use u_e and v_e to denote the ends of e in G_{s-1}, respectively, such that $u_e \in V(H^u)$ and $v_e \in V(H^v)$. (See Figure 5 for an illustration.)

Fix a subset U of $V(G)$ arbitrary such that $\lvert U \rvert = k$, where $3 \leq k \leq \ell$. For the graph G_s, a vertex u of G_s is *labelled* if $V(A_{s,u}) \cap U \neq \emptyset$, and u is *unlabelled* otherwise. We use U^s to denote the set of labelled vertices of G_s. For a labelled vertex u of G_s, let $W^u = V(H^u) \cap U^{s-1}$, that is, the set of labelled vertices in H^u. See Figure 2 as an example, if $U = \{u_{000}, u_{001}, u_{013}, u_{122}\}$, then

$U^1 = \{u_{00}, u_{01}, u_{12}\}$ and $W^{u_0} = \{u_{00}, u_{01}\}$ in Figure 3. For ease of reading, above notations and their meanings are summarized in Table 1.

We construct an out-branching $\overrightarrow{T}$ with vertex set $V(\overrightarrow{T}) = \bigcup_{i\in\{0,1,\ldots,n\}}\{x :\ x \in V(G_i)\}$ and arc set $A(\overrightarrow{T}) = \{(x,y) :\ y \in V(H^x)\}$. The root of $\overrightarrow{T}$ is denoted by v_{root} (it is clear that $V(G_n) = \{v_{root}\}$). For two different vertices x, y of $V(\overrightarrow{T})$, if there is a direct path from x to y, then we say that $x \prec y$, and denote the directed path by $x\overrightarrow{T}y$. The following result is straightforward.

Fact 1. *If $x_i \in V(\overrightarrow{T})$ for $i \in [p]$ and $x_1 \prec x_2 \prec \ldots \prec x_p$, then $\sum_{i\in[p]} |W^{x_i}| \leq k + p - 1$.*

2.2. A transitional theorem and its proof

In order to prove our main result (Theorem 1.5), we first prove the following transitional theorem by constructing $\ell - \lceil k/2\rceil$ internally disjoint U-Steiner trees in G.

Theorem 2.1. *Let $G = S(n,\ell)$ and $U \subseteq V(G)$ with $|U| = k$ (U is defined before), and let $c = \ell - \lceil k/2\rceil$. For $0 \leq s \leq n$, G_s has c internally disjoint U^s-Steiner trees $T_1, \cdots, T_c$ such that the following statement holds.*

($\star$) For each $u \in U^s$ (say $u \in W^v$ and hence v is a labelled vertex of G_{s+1} if $s \neq n$) and $i \in [c]$, $d_{T_i}(u) \leq 2$.

The rest part of this subsection is the proof of Theorem 2.1.

Proof of Theorem 2.1 Suppose that s' is the minimum integer such that $G_{s'}$ has exactly one labelled vertex (note that s' exists since G_n consists of a single labelled vertex). This means that $G_{s'-1}$ has at least two labelled vertices. Let v be the labelled vertex in $G_{s'}$. Then $U \subseteq A_{s',v}$. Thus, in order to find c internally disjoint U-Steiner trees of G satisfying ($\star$), we only need to find c internally disjoint U-Steiner trees in $A_{s',v}$ satisfying ($\star$). Hence, without loss of generality, we can assume that G is a graph with

$$|W^{v_{root}}| = |U^{n-1}| \geq 2. \tag{1}$$

Therefore, $|U^i| \geq 2$ for each $i \leq n - 1$.

The proof is technical and is via induction. Note that since G_n is a single vertex and $G = G_0$, the induction is from G_n to G_0. The basic idea is to use the U^s-Steiner trees of G_s to construct appropriate U^{s-1}-Steiner trees of G_{s-1}. Since each vertex in G_s corresponds to a complete graph in G_{s-1}, it is not a straightforward process to extend U^s-Steiner trees of G_s to U^{s-1}-Steiner trees of G_{s-1}.

If $s = n$, then each T_i in G_n is the empty graph and the result holds. Thus, suppose $s \leq n - 1$. Hence, the labelled vertex v of G_{s+1} always exists. The following implies that the result holds for $s = n - 1$. Note that $U^{n-1} = W^{v_{root}}$.

Claim 1. *If $s = n - 1$, then we can construct c internally disjoint U^s-Steiner trees, say $T_1, \ldots, T_c$, such that for each $i \in [c]$ and $u \in U^s$, $d_{T_i}(u) \leq 2$. Moreover,*

(i) If $|U^s| \leq \lceil k/2\rceil$, then for each $i \in [c]$ and $u \in U^s$, $d_{T_i}(u) = 1$;

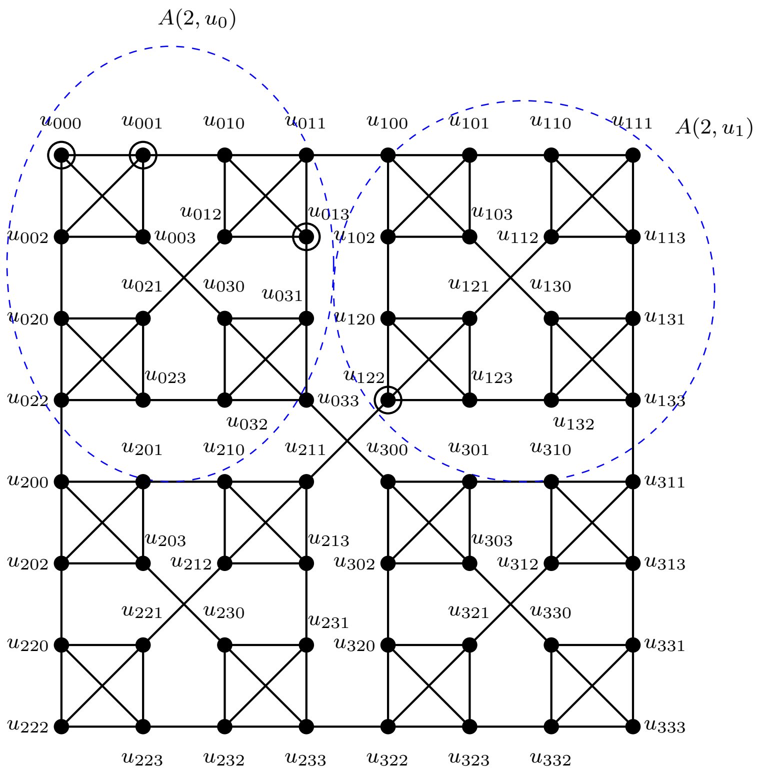

Figure 2: $G = G_0 = S(3, 4)$

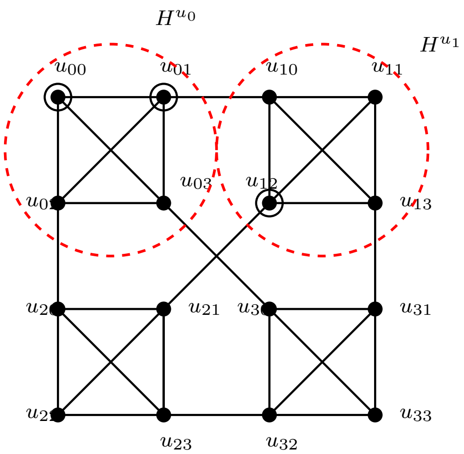

Figure 3: The graph G_1

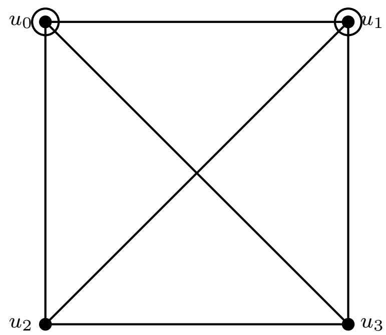

Figure 4: The graph G_2

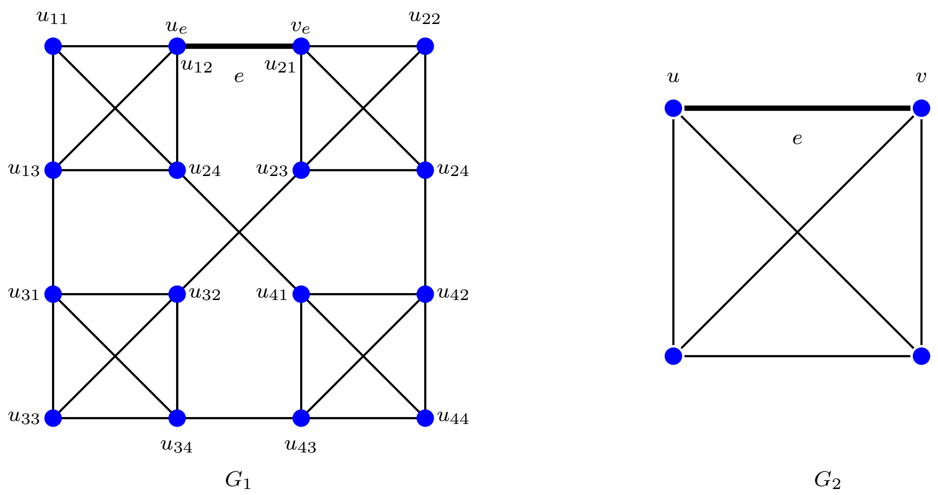

Figure 5: From graph G_1 to G_2 by u_ev_e contraction edge uv.

(ii) *otherwise, for each* $u \in U^s$, *there are at most* $|U^s| - \lceil k/2 \rceil$ *internally disjoint* U^s*-Steiner trees* T_i *such that* $d_{T_i}(u) = 2$.

Proof:
Note that $G_{n-1} = S(1, \ell) = K_\ell$.

Suppose $|U^{n-1}| \leq \lceil k/2 \rceil$. Then $|V(G_{n-1}) - U^{n-1}| \geq \ell - \lceil k/2 \rceil = c$ and we can choose c stars of $\{x \vee U^{n-1} : x \in V(G_{n-1}) - U^{n-1}\}$ as internally disjoint U^{n-1}-Steiner trees $T_1, \dots, T_c$. It is easy to verify that $d_{T_i}(u) = 1$ for $i \in [c]$ and $u \in U^{n-1}$.

Suppose $|U^{n-1}| \geq \lceil k/2 \rceil$. Since $|U^{n-1}| \leq k$, it follows that $0 \leq |U^{n-1}| - \lceil k/2 \rceil \leq \lfloor |U^{n-1}|/2 \rfloor$. By Corollary 1.4, there are $|U^{n-1}| - \lceil k/2 \rceil$ edge-disjoint Hamiltonian paths in $G_{n-1}[U^{n-1}]$. So, we can choose c internally disjoint U^{n-1}-Steiner trees consisting of $|U^{n-1}| - \lceil k/2 \rceil$ edge-disjoint Hamiltonian paths in $G_{n-1}[U^{n-1}]$ and $\ell - |U^{n-1}|$ stars $\{x \vee U^{n-1} : x \in V(G_{n-1}) - U^{n-1}\}$. It is easy to verify that $d_{T_i}(u) \leq 2$ for $i \in [c]$ and $u \in U^{n-1}$, and there are at most $|U^{n-1}| - \lceil k/2 \rceil$ internally disjoint U^{n-1}-Steiner trees T_i such that $d_{T_i}(u) = 2$. □

Our proof is a recursive process that the c internally disjoint U^s-Steiner trees of G_s are constructed by using the c internally disjoint U^{s+1}-Steiner trees of G_{s+1}. We will find some ways to construct the c internally disjoint U^s-Steiner trees such that $(\star)$ holds. In the finial step, the c internally disjoint U-Steiner trees in $G_0 = G$ will be obtained. We have proved that the result holds for $s \in \{n, n-1\}$. Now, suppose that we have constructed c internally disjoint U^{s+1}-Steiner trees, say $\mathcal{F} = \{F_1, \dots, F_c\}$, and the trees satisfy $(\star)$, where $s \in \{0, \dots, n-1\}$. We need to construct c internally disjoint U^s-Steiner trees $\mathcal{T} = \{T_1, \dots, T_c\}$ of G_s satisfying $(\star)$.

Recall that each edge of G_{s+1} is also an edge of G_s. In addition, let $E_{u,i}$ denote the set of edges in F_i incident with the labelled vertex u in G_s and let $V_{u,i} = \{u_e : e \in E_{u,i}\}$ (recall that the edge

$e = uw$ in G_{s+1} is denoted by $e = u_e w_e$ in G_s, where $u_e \in V(H^u)$ and $w_e \in V(H^w)$). Note that $E_{u,i}$ and $V_{u,i}$ are the subsets of $E(G_s)$ and $V(G_s)$, respectively.

Our aim is to construct internally disjoint U^s-Steiner trees $T_1, \ldots, T_c$ that is obtained from U^{s+1}-Steiner trees $F_1, \ldots, F_c$. So, we need to "blow" up each vertex w of G_{s+1} into H^w and find internally disjoint W^w-Steiner trees $F_1^w, \ldots, F_c^w$ of H^w (F_i^w may be the empty graph if w is a unlabelled vertex) such that

$$\mathcal{T} = \left\{ T_i = F_i \cup \bigcup_{w \in V(G_{s+1})} F_i^w : i \in [c] \right\}.$$

Each F_i^w can be constructed as follows.

- If w is a labelled vertex of G_{s+1}, then w is in each U^{s+1}-Steiner tree of $\mathcal{F}$. We choose c internally disjoint W^w-Steiner trees $F_1^w, \ldots, F_c^w$ of H^w such that $V_{w,i} \subseteq V(F_i^w)$ for each $i \in [c]$.

- If w is an unlabelled vertex, then w is in at most one U^{s+1}-Steiner tree of $\mathcal{F}$. For each $i \in [c]$, if $w \in V(F_i)$, then let F_i^w be a spanning tree of H^w; if $w \notin V(F_i)$, then let F_i^w be the empty graph.

For $i \in [c]$, let $E_i = E(F_i) \cup \bigcup_{w \in V(G_{s+1})} E(F_i^w)$ and let $T_i = G_s[E_i]$. It is obvious that $T_1, \ldots, T_c$ are internally disjoint U^s-Steiner trees. We need to ensure that each labelled vertex of G^s satisfies ($\star$).

In fact, without loss of generality, we only need to choose an arbitrary labelled vertex $u \in G_{s+1}$ and prove that each vertex of W^u satisfies ($\star$). This is because F_i^w is clear when w is an unlabelled vertex (in the case F_i^w is either a spanning tree of H^w or the empty graph). By Eq. (1) and Fact 1, $|W^u| \leq k - 1$ since $u \neq v_{root}$.

Since $d_{F_i}(u) \leq 2$ for each $i \in [c]$ and $u \in U^s$, it follows that $|E_{u,i}| = |V_{u,i}| \leq 2$. Therefore, we can divide $F_1, \ldots, F_c$ into the following five types on u:

Type 1: $|V_{u,i}| = 1$ and $V_{u,i} \subseteq W^u$.
Type 2: $|V_{u,i}| = 1$ and $V_{u,i} \nsubseteq W^u$.
Type 3: $|V_{u,i}| = 2$ and $V_{u,i} \subseteq W^u$.
Type 4: $|V_{u,i}| = 2$ and $V_{u,i} \cap W^u = \emptyset$.
Type 5: $|V_{u,i}| = 2$ and $|V_{u,i} \cap W^u| = 1$.

If F_i is a graph of Type j ($1 \leq j \leq 5$) on u, then we also call F_i^u a *W^u-Steiner tree of Type j*. Suppose there are $n_j(u)$ trees F_i that belong to Type j on u for $j \in [5]$. Then

$$\sum_{i=1}^{5} n_j(u) = c. \tag{2}$$

It is obvious that $n_j(v_{root}) = 0$ for each $j \in [5]$. Let

$$R(u) = V(H^u) - W^u - \bigcup_{i \in [c]} V_{u,i}.$$

Note that $R(u)$ is the set of unlabelled vertices in G_s, which will be used to construct the W^u-Steiner trees in H^u. It is clear that

$$|R(u)| = \ell - |W^u| - [n_2(u) + 2n_4(u) + n_5(u)]. \quad (3)$$

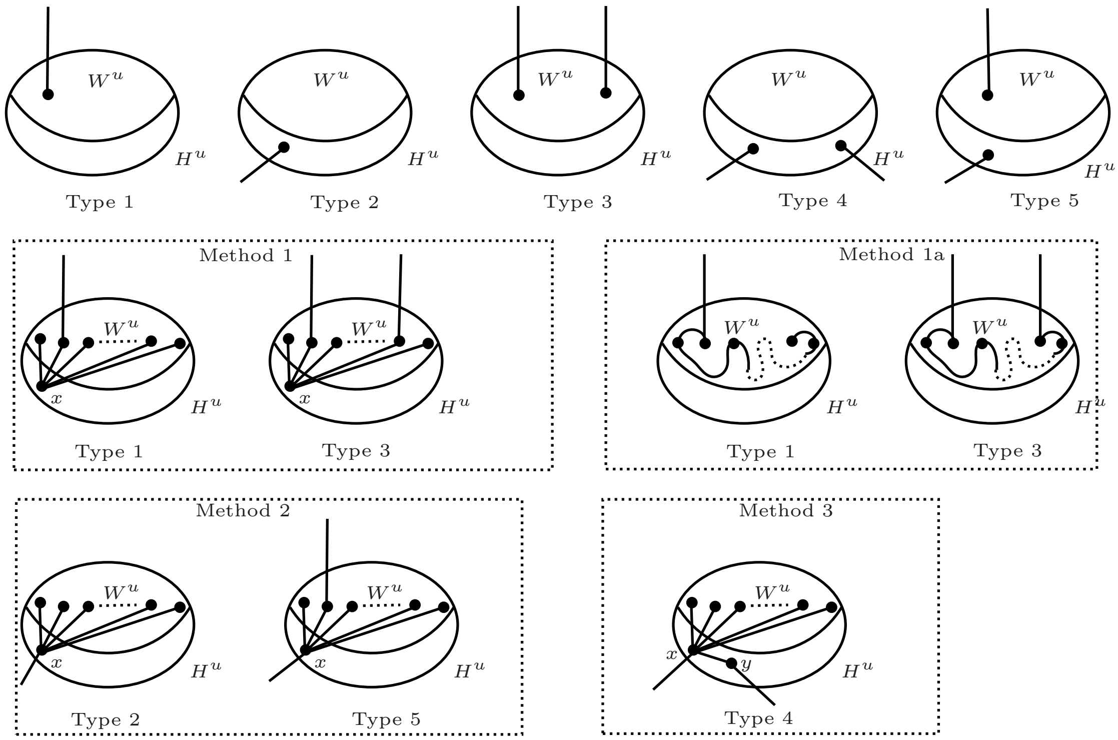

Figure 6: Five types and methods 1, 1a, 2 and 3.

We have the following five methods to choose $F_1^u, \ldots, F_c^u$ for the labelled vertex $u \in G_{s+1}$ (see Figure 6).

Method 1 If F_i is a graph of Type 1 and $|W^u| \geq 2$, or F_i is a graph of Type 3, then let $F_i^u = x \vee W^u$, where $x \in R(u)$.

Method 1a If F_i is a graph of Type 1 and $|W^u| \geq 2$, then let F_i^u be a Hamiltonian path of $H^u[W^u]$ such that the vertex in $V_{u,i}$ is an endpoint of this Hamiltonian path; if F_i is a graph of Type 3, then let F_i^u be a Hamiltonian path of $H^u[W^u]$ such that the endpoints of F_i are two vertices in $V_{u,i}$.

Method 2 If F_i is a graph of Type 2 or Type 5, say x is the only vertex of $V_{u,i}$ with $x \notin W^u$, then let $F_i^u = x \vee W^u$.

Method 3 If F_i is a graph of Type 4, say $V_{u,i} = \{x, y\}$, then let $F_i^u = xy \cup (x \vee W^u)$.

Method 4 If F_i is a graph of Type 1 and $|W^u| = 1$, then let F_i^u be the empty graph.

It is worth noting that the method is deterministic if F_i^u is a graph of Types 2, 4 and 5, or F_i^u is a graph of Type 1 and $|W^u| = 1$. So we firstly construct these F_i^us by using Methods 2, 3 and 4, respectively, and then construct F_i^us of Type 1 with $|W^u| \geq 2$ and construct F_i^us of Type 3 by using Method 1 or Method 1a.

The following is an algorithm constructing c internally disjoint U-Steiner trees of G.

Algorithm 1: The construction of c internally disjoint U-Steiner trees of G

Input: U, c and $G = S(n, \ell)$ with $|W^{root}| \geq 2$

Output: c internally disjoint U-Steiner trees $T'_1, T'_2, \ldots, T'_c$ of G

```
T'_1, T'_2, ..., T'_c are empty graphs;
s = n //(* in initial step, G^n is a single vertex*);
for s ≥ 1 do
    for each vertex u of G_s do
        if u is an unlabelled vertex then
            for 1 ≤ i ≤ c do
                if u is a vertex of F_i then
                    F_i^u is a spanning tree of H^u;
                else
                    F_i^u is is the empty graph;
                end
                T'_i = T'_i ∪ F_i^u;
                i = i + 1;
            end
        end
        if u is a labelled vertex then
            μ = |R(u)|;
            for 1 ≤ i ≤ c do
                if T'_i is of Type 2 or Type 5 then
                    construct F_i^u by using Method 2;
                end
                if T'_i is of Type 4 then
                    construct F_i^u by using Method 3;
                end
                if T'_i is of Type 1 and |W^u| = 1 then
                    construct F_i^u by using Method 4;
                end
                if either T'_i is of Type 1 and |W^u| ≥ 2, or T'_i is of Type 3 then
                    if μ > 0 then
                        construct F_i^u by using Method 1;
                        μ = μ − 1;
                    end
                    if μ ≤ 0 then
                        construct F_i^u by using Method 1a;
                    end
                end
                T'_i = T'_i ∪ F_i^u;
                i = i + 1;
            end
        end
    end
    s = s − 1;
end
```

Algorithm 1 is an algorithm for finding c internally disjoint U-Steiner trees of G. For each step of s, the algorithm will construct c internally disjoint U^{s-1}-steiner trees $T'_1, T'_2, \ldots, T'_c$. For convenience, we denote each T'_i by T^s_i after the s step of outer "for", that is, $T^s_1, T^s_2, \ldots, T^s_c$ are c internally disjoint U^s-Steiner trees in G^s generated in Algorithm 1. If Algorithm 1 is correct, then Lines 16–40 indicate that $d_{T'_i}(x) \leq 2$ for any $x \in W^u$, and hence ($\star$) always holds.

We now check the correctness of the algorithm. Since F^u_i is deterministic when u is an unlabelled vertex (Lines 5–15 of Algorithm 1), and F^u_i is deterministic if u is a labelled vertex and either F^u_i is a graph of Types 2, 4 and 5, or F^u_i is a graph of Type 1 and $|W^u| = 1$ (Lines 19–27 of Algorithm 1), we only need to talk about the labelled vertex u with $|W^u| \geq 2$ and check the correctness of Lines 28–36 of Algorithm 1. That is, to ensure that there exist $n_1(u)$ F^u_is of Type 1 and $n_3(u)$ F^u_is of Type 3. If F^u_i is constructed by Method 1, then F^u_i is a star with center in $R(u)$ (say the center of the star is a_i); if F^u_i is constructed by Method 1a, then F^u_i is a Hamiltonian path of H^u with endpoints in $V_{u,i}$. Since a_is are pairwise differently and are contained in $R(u)$, and H^u has at most $\lfloor |W^u|/2 \rfloor$ Hamiltonian paths to afford by Corollary 1.4, we only need to ensure that $|R(u)| + \lfloor |W^u|/2 \rfloor \geq n_1(u) + n_3(u)$. Thus, we only need to prove the following result.

Lemma 2.1. *For each labelled vertex $u \in V(G_s)$ with $|W^u| \geq 2$, Ineq.*

$$n_1(u) + n_3(u) - |R(u)| \leq \lfloor |W^u|/2 \rfloor \tag{4}$$

holds.

Proof:
By Eqs. (2) and (3),

$$\begin{aligned} n_1(u) + n_3(u) - |R(u)| &= n_1(u) + n_3(u) - [\ell - |W^u| - n_2(u) - 2n_4(u) - n_5(u)] \\ &= [n_1(u) + n_2(u) + n_3(u) + n_4(u) + n_5(u)] + n_4(u) + |W^u| - \ell \\ &= c + n_4(u) + |W^u| - \ell. \end{aligned}$$

Since $c = \ell - \lceil k/2 \rceil$, it follows that

$$n_1(u) + n_3(u) - |R(u)| = n_4(u) + |W^u| - \lceil k/2 \rceil. \tag{5}$$

Before the proof of Lemma 2.1, we give a series of claims as preliminaries.

Claim 2. *Suppose that $a \in V(v_{root}\overrightarrow{T}u)$ is a labelled vertex with $a \in U^\iota$, where $1 \leq \iota \leq n$. If $|W^a| = 1$ (say $W^a = \{x\}$), then $n_5(a) \leq 1$. Moreover,*

1. *if $n_5(a) = 1$, then $n_4(x) \leq 1$;*
2. *if $n_5(a) = 0$, then $n_3(x) = n_4(x) = n_5(x) = 0$.*

Proof:
Note that $x \in U^{\iota-1}$. in Algorithm 1 (Lines 18–39), if T^ι_i is not a tree of Type 5 on a, then F^a_i is chosen such that $d_{T^{\iota-1}_i}(x) = 1$ (here $W^a = \{x\}$); if T^ι_i is a tree of Type 5 on a, then F^a_i is chosen such that

$d_{T_i^{\iota-1}}(x) = 2$. Since $|W^a| = 1$, there is at most one T_i^ι of Type 5 on a, and hence $n_5(a) \leq 1$. Furthermore, if $n_5(a) = 1$, then there is exact one $T_i^{\iota-1}$ in $G^{\ell-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$. Hence, $n_4(x) \leq 1$. If $n_5(a) = 0$, then $d_{T_i^{\iota-1}}(x) = 1$ for each $T_i^{\iota-1}$. Therefore, $n_3(x) = n_4(x) = n_5(x) = 0$. □

Claim 3. *Suppose that a, b are two labelled vertices of $v_{root}\overrightarrow{T}u$ and $a \in W^b$, where $a \in U^\iota$ for some $\iota \in [n]$. Then there are at most $\max\{1, n_1(b) + n_3(b) - |R(b)| + 1\}$ T_i^ιs such that $d_{T_i^\iota}(a) = 2$. Furthermore, $n_4(a) \leq \max\{1, n_1(b) + n_3(b) - |R(b)| + 1\}$.*

Proof:
Since $a \in W^b$ and $a \in U^\iota$, it follows that $b \in U^{\iota+1}$. For an $i \in [c]$ with $d_{T_i^\iota}(a) = 2$, we have that either F_i^b is a Hamiltonian path of the clique $H^b[W^b]$, or $a \in V_{b,i} \cap W^b$ for some $i \in [c]$ (recall that $V_{b,i} = \{z \in V(H^b) : z \text{ is an end vertex of some edge in } E_{b,i}\}$, where $E_{b,i}$ is the set of edges in $T_i^{\iota+1}$ incident with b). Hence, if there are h edge-disjoint F_i^bs that are constructed as Hamiltonian paths of $H^b[W^b]$, then there are at most $h+1$ internally disjoint U^ι-Steiner trees T_i^ι such that $d_{T_i^\iota}(a) = 2$. By the definition of $n_4(a)$, we have that $n_4(a) \leq h + 1$.

In Algorithm 1 (Lines 28–36), if $n_1(b)+n_3(b)-|R(b)| > 0$, then there are at most $n_1(b)+n_3(b)-|R(b)|$ F_i^bs that are constructed as Hamiltonian paths in $H^b[W^b]$; if $n_1(b) + n_3(b) - |R(b)| \leq 0$, there is no F_i^b that is constructed as a Hamiltonian path in $H^b[W^b]$. Hence, $h \leq \max\{0, n_1(b) + n_3(b) - |R(b)|\}$. Thus, there are at most $\max\{1, n_1(b) + n_3(b) - |R(b)| + 1\}$ internally disjoint U^ι-Steiner trees T_i^ι such that $d_{T_i^\iota}(a) = 2$, and $n_4(a) \leq \max\{1, n_1(b) + n_3(b) - |R(b)| + 1\}$. □

Claim 4. *Let $a \in V(v_{root}\overrightarrow{T}u)$ be a labelled vertex, where $a \in U^\iota$ for some $\iota \in [n]$. If $n_4(a) \leq 1$ and $2 \leq |W^a| \leq \lceil k/2 \rceil - 1$, then $n_1(a) + n_3(a) - |R(a)| \leq 0$. Moreover, for each vertex $x \in W^a$, there is at most one $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$.*

Proof:
Since $n_4(a) \leq 1$ and $|W^a| \leq \lceil k/2 \rceil - 1$, it follows from Eq. (5) that $n_1(a) + n_3(a) - |R(a)| = |W^a| - \lceil k/2 \rceil + 1 \leq 0$. By Claim 3, for each vertex $x \in W^a$, there is at most one T_i^a such that $d_{T_i^{\iota-1}}(x) \leq 2$. □

Claim 5. *Let $a \in V(v_{root}\overrightarrow{T}u)$ be a labelled vertex, where $a \in U^\iota$ for some $\iota \in [n]$. If $n_4(a) \leq 1$ and $\lceil k/2 \rceil \leq |W^a| \leq k-1$, then $n_1(a) + n_3(a) - |R(a)| \leq \lfloor |W^a|/2 \rfloor$. Moreover, for each $x \in W^a$,*

1. *if $n_4(a) = 1$, then there are at most $|W^a| - \lceil k/2 \rceil + 2$ internally disjoint $U^{\iota-1}$-Steiner trees $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$;*
2. *if $n_4(a) = 0$, then there are at most $|W^a| - \lceil k/2 \rceil + 1$ internally disjoint $U^{\iota-1}$-Steiner trees $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$.*

Proof:
Since $n_4(a) \leq 1$, it follows from Eq. (5) that

$$\begin{aligned} n_1(a) + n_3(a) - |R(a)| &\leq |W^a| - \lceil k/2 \rceil + n_4(a) \\ &\leq |W^a| - \lceil k/2 \rceil + 1 \end{aligned}$$

and the equality indicates $n_4(a) = 1$. If $|W^a| \le k-2$, then

$$n_1(a) + n_3(a) - |R(a)| \le |W^a| - \lceil (|W^a|+2)/2 \rceil + 1 \le \lfloor |W^a|/2 \rfloor ;$$

if $|W^a| = k-1$ and k is even, then

$$n_1(a) + n_3(a) - |R(a)| = |W^a| - \lceil (|W^a|+1)/2 \rceil + 1 = \lfloor (|W^a|+1)/2 \rfloor = \lfloor |W^a|/2 \rfloor ;$$

if $|W^a| = k-1$ and k is odd, then

$$n_1(a) + n_3(a) - |R(a)| = (k-1) - \lceil k/2 \rceil + 1 \le \lfloor k/2 \rfloor = \lfloor (k-1)/2 \rfloor = \lfloor |W^a|/2 \rfloor .$$

Therefore, $n_1(a) + n_3(a) - |R(a)| \le \lfloor |W^a|/2 \rfloor$. By Eq. (5) and Claim 3, if $n_4(a) = 1$, then for each vertex $x \in W^a$, there are at most $n_1(a) + n_3(a) - |R(a)| + 1 = |W^a| - \lceil k/2 \rceil + 2$ internally disjoint $U^{\iota-1}$-Steiner trees $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$; if $n_4(a) = 0$, then for each vertex $x \in W^a$, there are at most $n_1(a) + n_3(a) - |R(a)| + 1 = |W^u| - \lceil k/2 \rceil + 1$ internally disjoint $U^{\iota-1}$-Steiner trees $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$. □

Claim 6. *Suppose $a, b \in V(\overrightarrow{T})$, $a \prec b$ and $a\overrightarrow{T}b = az_1z_2\dots z_pb$, where $a \in U^\iota$ for some $\iota \in [n]$. If $|W^{z_i}| \le \lceil k/2 \rceil - 1$ for each $i \in [p]$, and either $|W^a| = 1$ or $2 \le |W^a| \le \lceil k/2 \rceil - 1$ and $n_4(a) \le 1$, then $n_4(b) \le 1$.*

Proof:
Since $a \in U^\iota$, it follows that $z_q \in U^{\iota-q}$ for each $q \in [p]$ and $b \in U^{\iota-p-1}$. Since $|W^a| = 1$ or $2 \le |W^a| \le \lceil k/2 \rceil - 1$ and $n_4(a) \le 1$, by Claims 2 and 4, there are at most one $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(z_1) = 2$. Hence, $n_4(z_1) \le 1$. Since $|W^{z_1}| \le \lceil k/2 \rceil - 1$ and $n_4(z_1) \le 1$, by Claims 2 and 4, there are at most one $T_i^{\iota-2}$ such that $d_{T_i^{\iota-2}}(z_2) = 2$. Hence, $n_4(z_2) \le 1$. Repeat this progress, we can get that $n_4(z_p) \le 1$. Since $|W^{z_p}| \le \lceil k/2 \rceil - 1$ and $n_4(z_p) \le 1$, by Claims 2 and 4, there are at most one $T_i^{\iota-p-1}$ such that $d_{T_i^{\iota-p-1}}(b) = 2$. Hence, $n_4(b) \le 1$. □

Claim 7. *Suppose $|W^{v_{root}}| \le \lceil k/2 \rceil$, $b \in V(\overrightarrow{T})$ and $v_{root}\overrightarrow{T}b = v_{root}z_1z_2\dots z_pb$. If $|W^{z_i}| = 1$ for each $i \in [p]$, then $n_4(b) = 0$.*

Proof:
Since $|W^{v_{root}}| \le \lceil k/2 \rceil$, by Claim 1, $d_{T_i^{n-1}}(z_1) = 1$ for each U^{n-1}-Steiner tree T_i^{n-1}. Hence, $n_3(z_1) = n_4(z_1) = n_5(z_1) = 0$. Since $|W^{z_1}| = 1$ and $n_5(z_1) = 0$, by the second statement of Claim 2, $n_3(z_2) = n_4(z_2) = n_5(z_2) = 0$. Since $|W^{z_2}| = 1$ and $n_5(z_2) = 0$, by the second statement of Claim 2, $n_3(z_3) = n_4(z_3) = n_5(z_3) = 0$. Repeat this process, we get that $n_4(b) = 0$. □

Claim 8. *Suppose that a, b are two labelled vertices and $a \in W^b$, where $a \in U^\iota$ for some $\iota \in [n]$. If k is even and $|W^b| = |W^a| = k/2$, then $n_1(a) + n_3(a) - |R(a)| \le \lfloor |W^a|/2 \rfloor$ and the following hold.*

1. *If $b = v_{root}$, then for each vertex $x \in W^a$, there is at most one $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$.*

2. *If $b \neq v_{root}$, then for each vertex $x \in W^a$, there are at most two $T_i^{\iota-1}$s such that $d_{T_i^{\iota-1}}(x) = 2$.*

Proof:
Suppose $b = v_{root}$. Then by Claim 1, $n_4(a) = 0$. Thus,

$$n_1(a) + n_3(a) - |R(a)| = n_4(a) + |W^a| - k/2 = |W^a| - k/2 = 0.$$

By Claim 3, for each vertex $x \in W^a$, there is at most one $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$.

Now assume that $b \neq v_{root}$. Without loss of generality, suppose $v_{root}\overrightarrow{T}b = v_{root}v_1v_2\ldots v_pb$. By Fact 1, we have that $|W^{root}| = 2 \leq k/2$ and $|W^{v_i}| = 1$ for each $i \in [p]$. By Claim 7, $n_4(b) = 0$. By Claim 3,

$$\begin{aligned} n_4(a) &\leq \max\{1, n_1(b) + n_3(b) - |R(b)| + 1\} \\ &\leq \max\{1, n_4(b) + |W^b| - \lceil k/2 \rceil + 1\} \\ &= 1 \leq \lfloor |W^a|/2 \rfloor. \end{aligned}$$

By Claim 3 again, for each vertex $x \in W^a$, there are at most two $T_i^{\iota-1}$ such that $d_{T_i^{\iota-1}}(x) = 2$. □

With the above preparations, we now prove Lemma 2.1. Recall that $u \in V(G_s)$ is a labelled vertex with $|W^u| \geq 2$. Then each vertex of $V(v_{root}\overrightarrow{T}u)$ is also a labelled vertex. Suppose that v^* is the maximum vertex of $v_{root}\overrightarrow{T}u$ such that one of the following holds (if such vertex v^* exists).

(i) $|W^{v^*}| = 1$,

(ii) $2 \leq |W^{v^*}| \leq \lceil k/2 \rceil - 1$ and $n_4(v^*) \leq 1$.

We distinguish the following two cases to show this lemma, that is, to prove Ineq. (4) holds (recall that the Ineq. (4) is $n_1(u) + n_3(u) - |R(u)| \leq \lfloor |W^u|/2 \rfloor$).

Case 1. *v^* exists.*

Let $\overrightarrow{P} = v^*\overrightarrow{T}u = v^*z_1z_2\ldots z_pu$. By the maximality of v^*, we have that $|W^{z_i}| \geq 2$ for each $i \in [p]$. If $|\overrightarrow{P}| = 1$, then $v^* = u$. Since $|W^u| \geq 2$, it follows from (ii) that $2 \leq |W^u| \leq \lceil k/2 \rceil - 1$ and $n_4(u) \leq 1$. By Claim 4, we have that $n_1(u) + n_3(u) - |R(u)| \leq 0$, and Ineq. (4) holds. Thus, we assume that $|\overrightarrow{P}| \geq 2$ below.

By Claim 6, we have that

$$n_4(z_1) \leq 1. \tag{6}$$

Thus, by the maximality of v^*, we have that $|W^{z_1}| \geq \lceil k/2 \rceil$. Since $|W^{v_{root}}| + |W^{z_1}| \leq k + 1$ (by Fact 1) and $|W^{v_{root}}| \geq 2$, it follows that $|W^{z_1}| \leq k - 1$. Hence,

$$\lceil k/2 \rceil \leq |W^{z_1}| \leq k - 1. \tag{7}$$

Subcase 1.1. $p = 0$.

Then $\overrightarrow{P} = v^*\overrightarrow{T}u = v^*u$ and $z_1 = u$. Since $n_4(u) \leq 1$ and $\lceil k/2 \rceil \leq |W^u| \leq k-1$ by Ineqs. (6) and (7), it follows from Claim 5 that $n_1(u) + n_3(u) - |R(u)| \leq \lfloor |W^u|/2 \rfloor$, Ineq. (4) holds.

Subcase 1.2. $p = 1$.

Then $\overrightarrow{P} = v^*\overrightarrow{T}u = v^*z_1u$ and $u = z_2$. By Fact 1, we have that

$$|W^{z_1}| + |W^u| + |W^{v_{root}}| - 2 \leq k. \tag{8}$$

Hence $|W^{z_1}| + |W^u| \leq k$. Recall that $|W^{z_1}| \geq \lceil k/2 \rceil$. If $|W^u| \geq \lceil k/2 \rceil$, then k is even and $|W^{z_1}| = |W^u| = k/2$. By Claim 8, we have that $n_1(u) + n_3(u) - |R(u)| \leq \lfloor |W^u|/2 \rfloor$. Therefore, Ineq. (4) holds. Now, we assume that $2 \leq |W^u| \leq \lceil k/2 \rceil - 1$. Since $n_4(z_1) \leq 1$ and $k - 1 \geq |W^{z_1}| \geq \lceil k/2 \rceil$, by Claim 5, there are at most $|W^{z_1}| - \lceil k/2 \rceil + 2$ internally disjoint U^s-Steiner trees T_i^s such that $d_{T_i^s}(u) = 2$ (note that $u \in G_s$). Hence, $n_4(u) \leq |W^{z_1}| - \lceil k/2 \rceil + 2$. Thus, by Eq. (5),

$$n_1(u) + n_3(u) - |R(u)| = n_4(u) + |W^u| - \lceil k/2 \rceil \leq |W^u| + |W^{z_1}| - 2\lceil k/2 \rceil + 2.$$

If $|W^u| + |W^{z_1}| \leq k-1$ or k is odd, then $n_1(u) + n_3(u) - |R(u)| \leq 1 \leq \lfloor |W^u|/2 \rfloor$, Ineq. (4) holds. Thus, assume that $|W^u| + |W^{z_1}| = k$ (recall that $|W^u| + |W^{z_1}| \leq k$) and k is even below ($k \geq 4$). Since $|W^{z_1}| \geq \lceil k/2 \rceil$, it follows that $|W^u| \leq k/2$. Suppose that $v_{root}\overrightarrow{T}z_p = v_{root}w_1w_2 \ldots w_qv^*z_p$. Since $|W^{v_{root}}| \geq 2$, it follows from Ineq. (8) and Fact 1 that $|W^{v_{root}}| = 2 \leq k/2$ and $|W^{w_1}| = \ldots = |W^{w_q}| = |W^{v^*}| = 1$. According to Claim 7, we have that $n_4(u) = 0$. Hence,

$$n_1(u) + n_3(u) - |R(u)| = n_4(u) + |W^u| - \lceil k/2 \rceil \leq 0 \leq \lfloor |W^u|/2 \rfloor,$$

the Ineq. (4) holds.

Subcase 1.3. $p \geq 2$.

Then $u \succeq z_3$. Recall that $|W^{z_i}| \geq 2$ for each $i \in [p]$. Since

$$|W^{v_{root}}| + |W^{z_1}| + |W^{z_2}| + |W^{z_3}| - 3 \leq k \tag{9}$$

and $|W^{z_1}| \geq \lceil k/2 \rceil$, it follows that $|W^{z_2}|, |W^{z_3}| \leq \lfloor k/2 \rfloor - 1$ and $|W^{z_1}| + |W^{z_2}| \leq k - 1$. On the other hand, since $|W^{z_3}| \geq 2$, it follows that $k \geq 6$.

Without loss of generality, suppose $z_2 \in U^\iota$. Recall Ineqs. (6) and (7), and combine with Claim 5, there are at most $|W^{z_1}| - \left\lceil \frac{k}{2} \right\rceil + 2$ internally disjoint U^ι-Steiner trees T_i^ι such that $d_{T_i^\iota}(z_2) = 2$. Hence, $n_4(z_2) \leq |W^{z_1}| - \left\lceil \frac{k}{2} \right\rceil + 2$. By Claim 3, $n_4(z_3) \leq \max\{1, n_4(z_2) + |W^{z_2}| - \lceil k/2 \rceil + 1\}$. However, by the maximality of v^*, we have that $n_4(z_3) \geq 2$. Hence,

$$n_4(z_3) \leq n_4(z_2) + |W^{z_2}| - \lceil k/2 \rceil + 1 \leq |W^{z_1}| + |W^{z_2}| - 2\lceil k/2 \rceil + 3. \tag{10}$$

Since $|W^{z_1}| + |W^{z_2}| \leq k-1$, it follows that if k is odd or $|W^{z_1}| + |W^{z_2}| \leq k-2$, then $n_4(z_3) \leq 1$, a contradiction. Hence, k is even and $|W^{z_1}| + |W^{z_2}| = k-1$. This implies that $|W^{z_3}| = n_4(z_3) = 2$

by Ineqs. (9) and (10). Since $k \geq 6$, it follows that $n_4(z_3) + |W^{z_3}| - \lceil k/2 \rceil \leq 1 \leq \lfloor |W^{z_3}|/2 \rfloor$. Recall that $u \succeq z_3$. If $v_3 = u$, then

$$n_1(u) + n_3(u) - |R(u)| = n_4(u) + |W^u| - \lceil k/2 \rceil \leq \lfloor |W^u|/2 \rfloor,$$

Ineq. (4) holds. If $u \succ z_3$, then since

$$|W^{v_{root}}| + |W^{z_1}| + |W^{z_2}| + |W^{z_3}| + |W^u| - 4 \leq k,$$

we have that

$$|W^u| \leq k + 4 - (|W^{v_{root}}| + |W^{z_1}| + |W^{z_2}| + |W^{z_3}|) = 1,$$

a contradiction.

Case 2. v^* *does not exist.*

Let $\overrightarrow{P} = v_{root}\overrightarrow{T}u = v_{root}z_1 \ldots, z_t u$. Since v^* does not exist, for each vertex $z \in V(\overrightarrow{P})$, either $2 \leq |W^z| \leq \lceil k/2 \rceil - 1$ and $n_4(z) \geq 2$, or $|W^z| \geq \lceil k/2 \rceil$. Since $n_4(v_{root}) = 0$, it follows that $|W^{v_{root}}| \geq \lceil k/2 \rceil$. By Claim 1, $n_4(z_1) \leq |W^{v_{root}}| - \lceil k/2 \rceil$.

Subcase 2.1. $|W^{z_1}| \geq \lceil k/2 \rceil$.

Since $|W^{v_{root}}|, |W^{z_1}| \geq \lceil k/2 \rceil$, we have that $t \leq 1$. Otherwise,

$$\sum_{z \in V(\overrightarrow{P})} |W^z| - (|\overrightarrow{P}| - 1) > k,$$

which contradicts Fact 1. Moreover, if $t = 1$, then k is even, $|W^{v_{root}}| = |W^{z_1}| = k/2$ and $|W^u| = 2$.

Suppose that $t = 0$. Then $u = z_1$, and hence $n_4(u) \leq |W^{v_{root}}| - \lceil k/2 \rceil$. Thus

$$\begin{aligned} n_1(u) + n_3(u) - |R(u)| &= n_4(u) + |W^u| - \lceil k/2 \rceil \\ &\leq |W^{v_{root}}| + |W^u| - 2\lceil k/2 \rceil \\ &\leq (k+1) - 2\lceil k/2 \rceil \ (by\ Fact\ 1) \\ &\leq 1 \leq \lfloor |W^u|/2 \rfloor, \end{aligned}$$

Ineq. (4) holds.

Suppose $t = 1$. Then $\overrightarrow{P} = v_{root}z_1u$. Hence, k is even ($k \geq 4$), $|W^{v_{root}}| = |W^{z_1}| = k/2$ and $|W^u| = 2$. By the first statement of Claim 8 (here, we regard v_{root} and z_1 as the vertices b and a in Claim 8, respectively, and then u can be regarded as the vertex x in Claim 8), we have that $n_4(u) \leq 1$. Hence,

$$n_1(u) + n_3(u) - |R(u)| = n_4(u) + |W^u| - \lceil k/2 \rceil \leq 1 \leq \lfloor |W^u|/2 \rfloor,$$

Ineq. (4) holds.

Subcase 2.2. $2 \leq |W^{z_1}| \leq \lceil k/2 \rceil - 1$.

Recall that $n_4(z_1) = |W^{v_{root}}| - \lceil k/2 \rceil$. Since $n_4(z_1) \geq 2$, $|W^{v_{root}}| \geq \lceil k/2 \rceil + 2$. If $|W^{v_{root}}| + |W^{z_1}| = k + 1$, then $v = v_{root}$, $u = z_1$ and

$$n_1(u) + n_3(u) - |R(u)| = n_4(u) + |W^u| - \lceil k/2 \rceil \leq 1 \leq \lfloor |W^u|/2 \rfloor.$$

Thus, suppose $|W^{v_{root}}| + |W^{z_1}| \leq k$. Then

$$n_1(z_1) + n_3(z_1) - |R(z_1)| = n_4(z_1) + |W^{z_1}| - \lceil k/2 \rceil \leq 0.$$

If $z_1 = u$, then $n_1(u) + n_3(u) - |R(u)| \leq 0$, Ineq. (4) holds. Now, assume that $z_1 \neq u$. Then z_2 exists. By Claim 3, $n_4(z_2) \leq 1$. Since z_2 is not a candidate of v^*, it follows that $|W^{z_2}| \geq \lceil k/2 \rceil$. Since $|W^{v_{root}}| \geq \lceil k/2 \rceil + 2$, $|W^{z_2}| \geq \lceil k/2 \rceil$ and $|W^{z_1}| \geq 2$, it follows that $|W^{v_{root}}| + |W^{z_1}| + |W^{z_2}| - 2 \geq k + 2$, which contradicts Fact 1. □

With the conclusion of Lemma 2.1, the proof of Theorem 2.1 is completed.

2.3. Proof of Theorem 1.5

We first consider $3 \leq k \leq \ell$. The lower bound $\ell - \lceil k/2 \rceil \leq \kappa_k(S(n, \ell)) \leq \lambda_k(S(n, \ell))$ can be obtained from Theorem 2.1 directly. For the upper bounds of $\kappa_k(S(n, \ell))$ and $\lambda_k(S(n, \ell))$, consider the graph G_{n-1}. Let $V(G_{n-1}) = \{u_1, \ldots, u_\ell\}$, $U = \{x_1, \ldots, x_k\}$ and $x_i \in A_{n-1,u_i}$, where $k \leq \ell$. Suppose there are p edge-disjoint U-Steiner trees $T'_1, \ldots, T'_p$ of $G = S(n, \ell)$ and $\mathcal{P} = \{A_{n-1,u} : u \in V(G_{n-1})\}$. Let $T^*_i = T'_i/\mathcal{P}$ for $i \in [p]$. Then $T^*_1, \ldots, T^*_p$ are edge-disjoint connected graphs of G_{n-1} containing $\{u_1, \ldots, u_k\}$. Thus, $p \leq \lambda_k(G_{n-1}) = \lambda_k(K_\ell) = \ell - \lceil k/2 \rceil$. Therefore, $\kappa_k(S(n, \ell)) \leq \lambda_k(S(n, \ell)) \leq \ell - \lceil k/2 \rceil$, the upper bound follows.

Now consider the case $\ell + 1 \leq k \leq \ell^n$. By Theorem 1.2, $\lambda_k(S(n, \ell)) \geq \lfloor \ell/2 \rfloor$. Since $\lambda_k(S(n, \ell)) \leq \lambda_\ell(S(n, \ell)) = \lfloor \ell/2 \rfloor$, it follows that $\kappa_k(S(n, \ell)) \leq \lambda_k(S(n, \ell)) \leq \lfloor \ell/2 \rfloor$. Therefore, $\lambda_k(S(n, \ell)) = \lfloor \ell/2 \rfloor$ and $\kappa_k(S(n, \ell)) \leq \lfloor \ell/2 \rfloor$. The proof is completed.

3. Some network properties

Generalized connectivity is a graph parameter to measure the stability of networks. In the following part, we will give the following other properties of Sierpiński graphs.

The Sierpiński graph(networks) is obtained after t iteration as $S(t, \ell)$ that has N_t nodes and E_t edges, where $t = 0, 1, 2, \cdots, T-1$, and T is the total number of iterations, and our generation process can be illustrated as follows.

Step 1: Initialization. Set $t = 0$, G_1 is a complete graph of order ℓ, and thus $N_1 = \ell$ and $E_1 = \binom{\ell}{2}$. Set $G_1 = S(1, \ell)$.

Step 2: Generation of G_{t+1} from G_t. Let $S^1(t, \ell), S^2(t, \ell), \ldots, S^\ell(t, \ell)$ be all Sierpiński graphs added at Step t, where $S^i(t, \ell) \cong S(t, \ell)(1 \leq i \leq \ell)$. At Step $t + 1$, we add one edge (*bridge edge*) between $S^i(t, \ell)$ and $S^j(t, \ell)$, $i \neq j$, namely the edge between vertices $\langle ij \cdots j \rangle$ and $\langle ji \cdots i \rangle$.

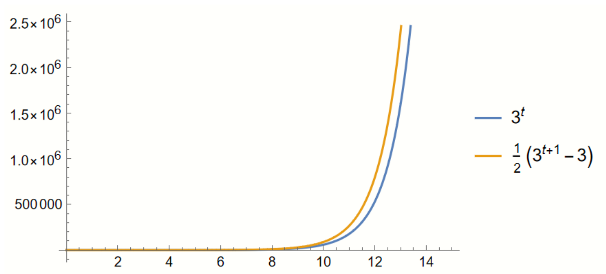

Figure 7: The Function of $E_t = 3^t$ and $N_t = \frac{3^{t+1}-3}{2}$

Table 2: The size and order of G_t for $\ell = 3$

t	1	2	3	4	5	6	t
N_t	3	9	27	81	242	729	ℓ^t
E_t	3	12	39	120	363	1092	$\frac{\ell^{t+1}-\ell}{2}$

For Sierpiński graphs $G_t = S(t, \ell)$, its order and size are $N_t = \ell^t$ and $E_t = \frac{\ell^{t+1}-\ell}{2}$, respectively; see Table 2 and Figure 7(for $\ell = 3$).

The degree distribution for t times are

$$\left(\underbrace{\ell-1, \ell-1, \cdots, \ell-1}_{\ell}, \underbrace{\ell, \ell, \ell, \cdots, \ell}_{\ell^t-\ell}\right). \tag{11}$$

From Equation 11, the instantaneous degree distribution is $P(\ell-1, t) = 1/\ell^{t-1}$ for $t = 2, \cdots, T$ and $P(\ell, t) = (\ell^t - \ell)/\ell^t$ for $t = 2, \cdots, T$. Note that the density of Sierpiński graphs is $\rho = E_t/\binom{N_t}{2} \to 0$ for $t \to +\infty$. For large enough ℓ and any $1 \leq k \leq \ell$, we have $|\{v \in V(G)|d_{S(t,\ell)}(v)| \geq k\} \approx |V(S(t, \ell))|$.

Theorem 3.1. [11] *If $n \in \mathbb{N}$ and G is a graph, then $\kappa(S(n, G)) = \kappa(G)$ and $\lambda(S(n, G)) = \lambda(G)$.*

From Theorem 3.1, we have $\kappa(S(n, \ell)) = \kappa(K_\ell) = \ell - 1$ and $\lambda(S(n, \ell)) = \lambda(K_\ell) = \ell - 1$. Note that $\lambda_k(S(n, \ell)) = \ell - \lceil k/2 \rceil$ and $\kappa_k(S(n, \ell)) = \ell - \lceil k/2 \rceil$; see Figure 8.

The number of spanning tree of G denoted by $\tau(G)$. Let

$$\rho(G) = \lim_{V(G) \longrightarrow \infty} \frac{\ln|\tau(G)|}{|V(G)|}, \tag{12}$$

where $\rho(G)$ is called the entropy of spanning trees or the asymptotic complexity [2, 7].

As an application of generalized (edge-)connectivity, similarly to the Equation 12, it can describe the fault tolerance of a graph or network, a common metric is called the entropy of spanning trees.

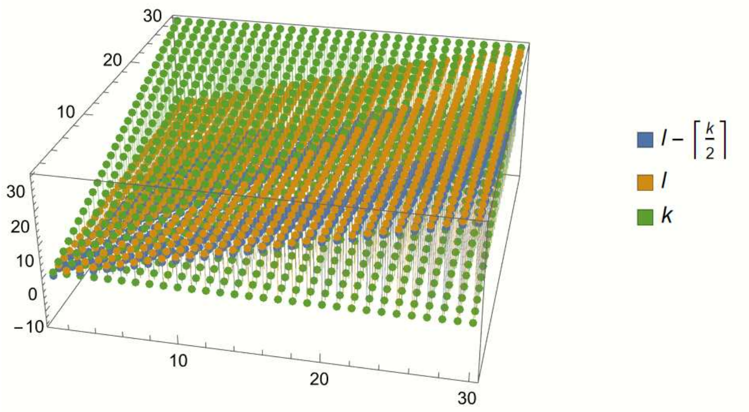

Figure 8: The generalized (edge-)connectivity of $S(n, \ell)$

we give the entropy of the k-Steiner tree of a graph G can be defined as

$$\rho_k(G) = \lim_{|V(G)| \longrightarrow \infty} \frac{\ln |\kappa_k(G)|}{|V(G)|}$$

The entropy of the $3, 6, 9$-Steiner tree of Sierpiński graph $S(8, \ell)$ can be seen in Figure 9.

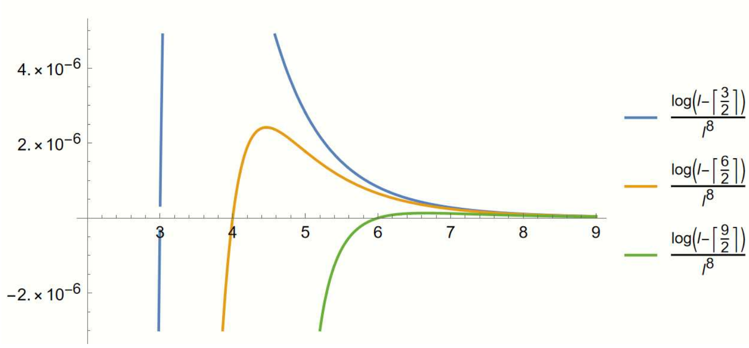

Figure 9: Entropy of $S(n, \ell)$for $n = 8$, $k = 3, 6, 9$ and $\ell \to +\infty$

The definition of clustering coefficient can be found in [3]. Let $N_v(t)$ be the number of edges in G_t among neighbors of v, which is the number of triangles connected to the vertex v. The clustering coefficient of a graph is based on a local clustering coefficient for each vertex

$$C[v] = \frac{N_v(t)}{d_G(v)(d_G(v) - 1)/2},$$

If the degree of node v is 0 or 1, then we can set $C[v] = 0$. By definition, we have $0 \leq C[v] \leq 1$ for $v \in V(G)$.

The clustering coefficient for the whole graph G is the average of the local values $C(v)$

$$C(G) = \frac{1}{|V(G)|}\left(\sum_{v\in V(G)} C[v]\right).$$

The clustering coefficient of a graph is closely related to the transitivity of a graph, as both measure the relative frequency of triangles[22, 24].

Proposition 3.1. *The clustering coefficient of generalized Sierpiński graph $S(n,\ell)$ is*

$$C(S(n,\ell)) = \frac{\ell^{-n}\left(2\ell - 2\ell^n + \ell^{n+1}\right)}{\ell}.$$

Proof:

For any $v \in V(S(n,\ell))$, if v is a extremal vertex, then $d_{S(n,\ell)}(v) = \ell - 1$ and $G[\{N(v)\}] \cong K_{\ell-1}$, and hence

$$C[v] = \frac{N_v(t)}{d_G(v)(d_G(v)-1)/2} = \binom{\ell-1}{2}/\binom{\ell-1}{2} = 1.$$

If v is not a extremal vertex, then $d_{S(n,\ell)}(v) = \ell$ and $G[\{N(v)\}] \cong K_\ell + e$, where $K_\ell + e$ is graph obtained from a complete graph K_ℓ by adding a pendent edge. Hence, we have $C[v] = \binom{\ell-1}{2}/\binom{\ell}{2} = \frac{\ell-2}{\ell}$.

Since there exists ℓ extremal vertices in Sierpiński graph $S(n,\ell)$, it follows that

$$C(S(n,\ell)) = \frac{1}{|V(G)|}\left(\sum_{v\in V(G)} C[v]\right) = \frac{1}{\ell^n}\left(\ell \times 1 + (\ell^n - \ell)\frac{\ell-2}{\ell}\right) = \frac{\ell^{-n}\left(2\ell - 2\ell^n + \ell^{n+1}\right)}{\ell}$$

□

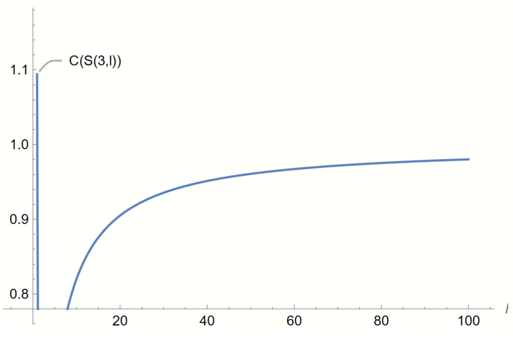

Figure 10: The Function of $C_{S(3,l)}$

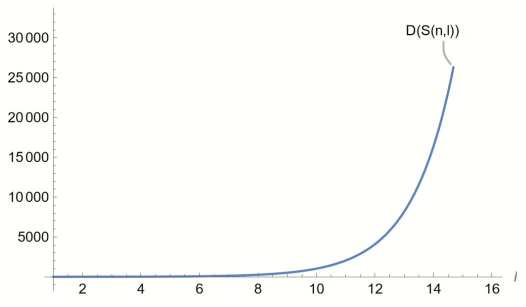

Figure 11: The diameter of $S(n,\ell)$

Theorem 3.2. [21] *The diameter of $S(n,\ell)$ is $Diam(S(n,\ell)) = 2^\ell - 1$;*

For network properties of Sierpiński graph $S(n,\ell)$, the the diameter function can be seen in Figure 11 and its clustering coefficient is closely related to 1 when $\ell \longrightarrow \infty$; see Figure 10, which implies that the Sierpiński graph $S(n,\ell)$ is a hight transitivity graph.

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