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# Note on a Translation from First-Order Logic into the Calculus of Relations Preserving Validity and Finite Validity

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**Abstract.** In this note, we give a linear-size translation from formulas of first-order logic into equations of the calculus of relations preserving validity and finite validity. Our translation also gives a linear-size conservative reduction from formulas of first-order logic into formulas of the three-variable fragment of first-order logic.

**Keywords:** first-order logic, relation algebra

## 1. Introduction

*The calculus of relations* (CoR, for short) [15] is an algebraic system with operations on binary relations. As binary relations appear everywhere in computer science, CoR and relation algebras can be applied to various areas, such as databases and program development and verification [4]. W.r.t. binary relations, CoR has the same expressive power as the three-variable fragment of first-order predicate logic with equality ($\mathrm{FO3}_=$) (where all predicate symbols are binary) [16], so CoR has strictly less expressive power than first-order predicate logic with equality ($\mathrm{FO}_=$). For example, CoR equations cannot characterize the class of structures s.t. "its cardinality is greater than or equal to $4$", whereas it can be characterized by the $\mathrm{FO}_=$ formula $\forall x_1, \forall x_2, \forall x_3, \exists y, (\neg y = x_1) \wedge (\neg y = x_2) \wedge (\neg y = x_3)$ where $x_1, x_2, x_3, y$ are pairwise distinct variables.

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Nevertheless, there is a recursive translation (total recursive function) from $\mathrm{FO}_=$ formulas into CoR equations (resp. $\mathrm{FO3}_=$ formulas) preserving validity [16] (see also [7, 1]).

In this paper, we give another recursive translation from $\mathrm{FO}_=$ formulas into CoR equations preserving validity, slightly refined in that it satisfies both of the following:

1. Our translation preserves both validity and finite validity (so, it also gives a *conservative reduction* [3, Def. 2.1.35]).
2. Our translation is linear-size (i.e., the output size is bounded by a linear function in the input size).

The first refinement is useful, e.g., in finding counter-models (because if there exists a finite counter-model in the pre-translated formula, then there also exists a finite counter-model in the post-translated formula). Such a translation is already known (e.g., [3, Cor. 3.1.8 and Thm. 3.1.9]), but via encodings of Turing-machines and domino problems. Our translation presents a conservative reduction from $\mathrm{FO}_=$ formulas to $\mathrm{FO3}_=$ formulas, directly. Thanks to this, we also have the second refinement, which shows that the validity (resp. finite validity) problem of $\mathrm{FO}_=$ formulas and that of CoR equations are equivalent under linear-size translations, as the converse direction immediately follows from the standard translation from CoR equations into $\mathrm{FO3}_=$ [15] (Prop. 2.2).

Our translation is not so far from known encodings (e.g., [7, 1]) in that they and our translation use pairing (2-tupling) functions, but in our translation, we use *non-nested* $k$-tupling functions where $k$ is an arbitrary natural number, instead of arbitrarily nested pairing functions. For constructions using arbitrarily nested pairing functions, we need infinitely many vertices even if the base universe is finite (as there is no surjective function from $X$ to $X^2$ when $\#X$ is finite and $\#X \geq 2$). Thanks to the modification above, our construction preserves both validity and finite validity. Additionally, to preserve the output size linear in the input size, we apply a cumulative sum technique.

This paper is structured as follows. In Section 2, we give basic definitions of $\mathrm{FO}_=$ and CoR. In Section 3, we give a translation from $\mathrm{FO}_=$ formulas into CoR equations preserving validity and finite validity. In Section 4, we additionally give a Tseitin translation for CoR, which is useful for reducing the number of alternations of operations. Additionally, in Section A, we give a direct translation from $\mathrm{FO}_=$ formulas into $\mathrm{FO3}_=$ formulas, not via CoR, for explicitly writing a transformed $\mathrm{FO3}_=$ formulas; the translation is the same as that given in Section 3.

## 2. Preliminaries

We write $\mathbb{N}$ for the set of all non-negative integers. For a set $A$, we write $\#A$ for the cardinality of $A$.

A *structure* $\mathfrak{A}$ over a set $A$ is a tuple $\langle |\mathfrak{A}|, \{a^{\mathfrak{A}}\}_{a \in A} \rangle$, where

- the universe $|\mathfrak{A}|$ is a non-empty set of vertices,
- each $a^{\mathfrak{A}} \subseteq |\mathfrak{A}|^2$ is a binary relation on $|\mathfrak{A}|$.

We say that a structure $\mathfrak{A}$ is *finite* if $|\mathfrak{A}|$ is finite. For structures $\mathfrak{A}, \mathfrak{B}$ over a set $A$, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* if there is a bijective map $f \colon |\mathfrak{A}| \to |\mathfrak{B}|$ such that for all $x, y \in |\mathfrak{A}|$ and $a \in A$, we have $\langle x, y \rangle \in a^{\mathfrak{A}} \iff \langle f(x), f(y) \rangle \in a^{\mathfrak{B}}$.

## 2.1. First-order logic

Let $\Sigma$ be a countably infinite set of binary predicate symbols and $\mathbf{V}$ be a countably infinite set of *variables*. The set of *formulas* in *first-order predicate logic with equality* ($\mathrm{FO}_=$) is defined by:

$$\varphi, \psi, \rho \quad ::= \quad a(x, y) \mid x = y \mid \neg\psi \mid \psi \wedge \rho \mid \exists x, \psi \qquad (a \in \Sigma \text{ and } x, y \in \mathbf{V})$$

We write $\mathrm{V}(\varphi)$ for the set of free and bound variables occurring in a formula $\varphi$. For $k \geq 0$, we write $\mathrm{FO}k_=$ for the set of all formulas $\varphi$ s.t. $\#\mathrm{V}(\varphi) \leq k$. (In this paper, $\mathrm{FO3}_=$ mostly occurs.) A *sentence* is a formula not having any free variable. We use parentheses in ambiguous situations and use the following notations:

$$\varphi \vee \psi \triangleq \neg((\neg\varphi) \wedge (\neg\psi)) \qquad \varphi \to \psi \triangleq (\neg\varphi) \vee \psi \qquad \mathsf{t} \triangleq \exists x, x = x$$
$$\forall x, \psi \triangleq \neg\exists x, \neg\psi \qquad \varphi \leftrightarrow \psi \triangleq (\varphi \to \psi) \wedge (\psi \to \varphi) \qquad \mathsf{f} \triangleq \neg\mathsf{t}$$

We write $\bigwedge \Gamma$ for the formula $\varphi_1 \wedge \cdots \wedge \varphi_n$ where $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ is a finite set (and $\varphi_1, \ldots, \varphi_n$ are ordered by a total order). The *size* $\|\varphi\| \in \mathbb{N}$ of a formula $\varphi$ is defined by:

$$\|a(x, y)\| \triangleq 1 + 2 \qquad \|x = y\| \triangleq 1 + 2 \qquad \|\neg\psi\| \triangleq 1 + \|\psi\|$$
$$\|\psi \wedge \rho\| \triangleq 1 + \|\psi\| + \|\rho\| \qquad \|\exists x, \psi\| \triangleq 1 + 1 + \|\psi\|$$

For a structure $\mathfrak{A}$ over $\Sigma$, the *semantics* $[\![\varphi]\!]^{\mathfrak{A}} \subseteq |\mathfrak{A}|^{\mathbf{V}}$ of a formula $\varphi$ over $\mathfrak{A}$ is defined as follows (where $|\mathfrak{A}|^{\mathbf{V}}$ denotes the set of functions from $\mathbf{V}$ to $|\mathfrak{A}|$):

$$\begin{aligned}
[\![a(x, y)]\!]^{\mathfrak{A}} &\triangleq \{f \colon \mathbf{V} \to |\mathfrak{A}| \mid \langle f(x), f(y)\rangle \in a^{\mathfrak{A}}\} \\
[\![x = y]\!]^{\mathfrak{A}} &\triangleq \{f \colon \mathbf{V} \to |\mathfrak{A}| \mid f(x) = f(y)\} \\
[\![\neg\psi]\!]^{\mathfrak{A}} &\triangleq |\mathfrak{A}|^{\mathbf{V}} \setminus [\![\psi]\!]^{\mathfrak{A}} \\
[\![\psi \wedge \rho]\!]^{\mathfrak{A}} &\triangleq [\![\psi]\!]^{\mathfrak{A}} \cap [\![\rho]\!]^{\mathfrak{A}} \\
[\![\exists x, \psi]\!]^{\mathfrak{A}} &\triangleq \{f \colon \mathbf{V} \to |\mathfrak{A}| \mid \text{ for some } v \in |\mathfrak{A}|, f[v/x] \in [\![\psi]\!]^{\mathfrak{A}}\}
\end{aligned}$$

Here, $f[v/x]$ denotes the function $f$ in which the value of $x$ has been replaced with $v$.

For a formula $\varphi$ and a structure $\mathfrak{A}$, we say that $\varphi$ is *true* on $\mathfrak{A}$, written $\mathfrak{A} \models \varphi$, if $[\![\varphi]\!]^{\mathfrak{A}} = |\mathfrak{A}|^{\mathbf{V}}$. We say that a formula $\varphi$ is *valid* (resp. *finitely valid*) if $[\![\varphi]\!]^{\mathfrak{A}} = |\mathfrak{A}|^{\mathbf{V}}$ holds for all structures (resp. all finite structures) $\mathfrak{A}$. We say that two formulas $\varphi, \psi$ are *semantically equivalent* if the formula $\varphi \leftrightarrow \psi$ is valid. Additionally, we say that a formula $\varphi$ is *satisfiable* (resp. *finitely satisfiable*) if $[\![\varphi]\!]^{\mathfrak{A}} \neq \emptyset$ holds for some structure (resp. finite structure) $\mathfrak{A}$.

**Remark 2.1.** Function and constant symbols can be encoded by predicate symbols with functionality axiom (see, e.g., [2, Sect. 19.4]) and each predicate symbol (of arbitrary arity) can be encoded by binary predicate symbols (see, e.g., [2, Lem. 21.2 (p. 275)], which translations each atomic formula $a(x_1, \ldots, x_k)$ into the formula $\exists z, (\bigwedge_{1 \leq j \leq k} p_j(z, x_j)) \wedge a'(z, z)$ where $z$ is a fresh variable, $p_1, \ldots, p_k$ are fresh binary symbols for expressing projections, and $a'$ is a fresh binary symbol for expressing the relation $a$, respectively. For instance, the formula $a(x, y, x) \wedge a(y, y, x)$ is translated into $(\exists z, p_1(z,x) \wedge p_2(z,y) \wedge p_3(z,x) \wedge a'(z,z)) \wedge (\exists z, p_1(z,y) \wedge p_2(z,y) \wedge p_3(z,x) \wedge a'(z,z)))$. Thus, by well-known facts, we can give a linear-size translation from formulas of first-order logic with predicate and function symbols of arbitrary arity into $\mathrm{FO}_=$ formulas (above) preserving validity and finite validity. Here, the size of a $k$-ary atomic formula is defined as $\|a(x_1,\ldots,x_k)\| = 1+k$. (Note that the translation is not linear-size when the size is defined as $\|a(x_1,\ldots,x_k)\| = 1$ and $k$ is not bounded. Hence, this linearity depends on the size definition.) Hence, we consider the $\mathrm{FO}_=$ above (equality $=$ can also be eliminated, see, e.g., [2, Sect. 19.4], but we introduce it only for convenience).

## 2.2. The calculus of relations

Let $\Sigma$ be a countably infinite set of *(term) variables*. The set of *terms* in *the calculus of relations* (CoR) is defined by:

$$t, s, u \quad ::= \quad a \mid \mathsf{I} \mid s^- \mid s \cap u \mid s \cdot u \mid s^\smile \qquad (a \in \Sigma)$$

We write $\bigcap \Gamma$ for the term $t_1 \cap \cdots \cap t_n$ where $\Gamma = \{t_1,\ldots,t_n\}$ is a finite set (and $t_1,\ldots,t_n$ are ordered by a total order). We use parentheses in ambiguous situations and use the following notations:

$$t \cup s \triangleq (t^- \cap s^-)^- \qquad t \dagger s \triangleq (t^- \cdot s^-)^-$$
$$\top \triangleq \mathsf{I} \cup \mathsf{I}^- \qquad \bot \triangleq \top^-$$

The *size* $\|t\| \in \mathbb{N}$ of a term $t$ is defined by:

$$\|a\| \triangleq 1 \qquad \|\mathsf{I}\| \triangleq 1 \qquad \|s^-\| \triangleq 1 + \|s\|$$
$$\|s \cap u\| \triangleq 1 + \|s\| + \|u\| \qquad \|s \cdot u\| \triangleq 1 + \|s\| + \|u\| \qquad \|s^\smile\| \triangleq 1 + \|s\|$$

The *semantics* $[\![t]\!]^{\mathfrak{A}} \subseteq |\mathfrak{A}|^2$ of a term $t$ over a structure $\mathfrak{A}$ over $\Sigma$ is defined by:

$$
\begin{aligned}
[\![a]\!]^{\mathfrak{A}} &\triangleq \{\langle v, v'\rangle \in |\mathfrak{A}|^2 \mid \langle v, v'\rangle \in a^{\mathfrak{A}}\} \\
[\![\mathsf{I}]\!]^{\mathfrak{A}} &\triangleq \{\langle v, v'\rangle \in |\mathfrak{A}|^2 \mid v = v'\} \\
[\![s^-]\!]^{\mathfrak{A}} &\triangleq |\mathfrak{A}|^2 \setminus [\![s]\!]^{\mathfrak{A}} \\
[\![s \cap u]\!]^{\mathfrak{A}} &\triangleq [\![s]\!]^{\mathfrak{A}} \cap [\![u]\!]^{\mathfrak{A}} \\
[\![s \cdot u]\!]^{\mathfrak{A}} &\triangleq \{\langle v, v'\rangle \in |\mathfrak{A}|^2 \mid \text{ for some } v'' \in |\mathfrak{A}|, \langle v, v''\rangle \in [\![s]\!]^{\mathfrak{A}} \text{ and } \langle v'', v'\rangle \in [\![u]\!]^{\mathfrak{A}}\} \\
[\![s^\smile]\!]^{\mathfrak{A}} &\triangleq \{\langle v, v'\rangle \in |\mathfrak{A}|^2 \mid \langle v', v\rangle \in [\![s]\!]^{\mathfrak{A}}\}
\end{aligned}
$$

We say that a CoR term $t$ and an $\mathrm{FO}_=$ formula $\varphi$ with two distinct free variables $x_1$ and $x_2$ are *semantically equivalent w.r.t. binary relations* if $[\![t]\!]^{\mathfrak{A}} = \{\langle f(x_1), f(x_2)\rangle \mid f \in [\![\varphi]\!]^{\mathfrak{A}}\}$ holds for all structures $\mathfrak{A}$. It is well-known that we can translate CoR terms into $\mathrm{FO3}_=$ formulas.

**Proposition 2.2. (the *standard translation* theorem [15])**
Let $x_1$ and $x_2$ be distinct variables. There is a linear-size translation from CoR terms into $\mathrm{FO3}_=$ formulas with two free variables $x_1$ and $x_2$ preserving the semantic equivalence w.r.t. binary relations.

**Proof** (sketch)**:**
Because we can express each operations in CoR by using $\mathrm{FO3}_=$ formulas (see also [9, Fig. 1]). □

Moreover, the set of *quantifier-free formulas* in CoR is inductively defined as follows:

$$\varphi, \psi, \rho \quad ::= \quad t = s \mid \neg\psi \mid \psi \wedge \rho \qquad (t, s \text{ are terms in CoR})$$

We say that $t = s$ is an *equation*. An *inequation* $t \leq s$ is an abbreviation of the equation $t \cup s = s$. As with Section 2.1, we use the following notations:

$$\varphi \vee \psi \triangleq \neg((\neg\varphi) \wedge (\neg\psi)) \qquad \varphi \to \psi \triangleq (\neg\varphi) \vee \psi \qquad \mathsf{t} \triangleq \mathsf{I} = \mathsf{I}$$
$$\varphi \leftrightarrow \psi \triangleq (\varphi \to \psi) \wedge (\psi \to \varphi) \qquad \mathsf{f} \triangleq \neg\mathsf{t}$$

The *size* $\|\varphi\| \in \mathbb{N}$ of a quantifier-free formula $\varphi$ is defined by:

$$\|t = s\| \triangleq 1 + \|t\| + \|s\| \qquad \|\neg\psi\| \triangleq 1 + \|\psi\| \qquad \|\psi \wedge \rho\| \triangleq 1 + \|\psi\| + \|\rho\|$$

The *semantic relation* $\mathfrak{A} \models \varphi$, where $\varphi$ is a quantifier-free formula and $\mathfrak{A}$ is a structure over $\Sigma$, is defined by:

$$\mathfrak{A} \models t = s \quad \stackrel{\triangle}{\Longleftrightarrow} \quad [\![t]\!]^{\mathfrak{A}} = [\![s]\!]^{\mathfrak{A}}$$
$$\mathfrak{A} \models \neg\psi \quad \stackrel{\triangle}{\Longleftrightarrow} \quad \text{not } (\mathfrak{A} \models \psi)$$
$$\mathfrak{A} \models \psi \wedge \rho \quad \stackrel{\triangle}{\Longleftrightarrow} \quad (\mathfrak{A} \models \psi) \text{ and } (\mathfrak{A} \models \rho)$$

For a quantifier-free formula $\varphi$ and a structure $\mathfrak{A}$, we say that $\varphi$ is *true* on $\mathfrak{A}$ if $\mathfrak{A} \models \varphi$. Similarly for $\mathrm{FO}_=$ formulas, we say that a quantifier-free formula $\varphi$ is *valid* (resp. *finitely valid*) if $\mathfrak{A} \models \varphi$ holds for all structures (resp. all finite structures) $\mathfrak{A}$. We say that two quantifier-free formulas $\varphi, \psi$ are *semantically equivalent* if the quantifier-free formula $\varphi \leftrightarrow \psi$ is valid.

It is also well-known that we can translate CoR quantifier-free formulas into CoR equations, preserving the semantic equivalence.

**Proposition 2.3.** **(*Schröder-Tarski translation* theorem [15])**
There is a linear-size translation from a given quantifier-free formula $\varphi$ in CoR into a term $t$ such that $\varphi$ and $(t = \top)$ are semantically equivalent.

**Proof:**
[The proof is from [15].] First, by using $(s = u) \leftrightarrow ((s \cap u) \cup (s^- \cap u^-) = \top)$, we translate a given quantifier-free formula into a quantifier-free formula s.t. each equation is of the form $t = \top$. Second, by using the following two semantic equivalences, we eliminate logical connectives:

$$\neg(s = \top) \leftrightarrow \top \cdot s^- \cdot \top = \top \qquad (s = \top) \wedge (u = \top) \leftrightarrow s \cap u = \top$$

We then have obtained the desired equation of the form $t = \top$. □

**Remark 2.4.** There is also a translation from $\mathrm{FO3}_=$ formulas (with two free variables) into CoR terms preserving the semantic equivalence w.r.t. binary relations (i.e., the converse direction of Prop. 2.2) [16, 4, 9, 10], but the best known translation is an exponential-size translation and it is open whether there is a subexponential-size translation [9, 10]. This paper's translation given in Sect. 3 only preserves validity and finite validity and does not preserve the semantic equivalence w.r.t. binary relations, but it is a linear-size translation.

## 3. A translation from first-order logic into CoR

We consider the following structure transformation. Based on this transformation, we will give a translation from $\mathrm{FO}_=$ formulas into CoR equations.

**Definition 3.1.** (***$k$-tuple structure***)
Let $\mathfrak{A}$ be a structure over $\Sigma$. For $k \geq 1$, the $k$-tuple structure of $\mathfrak{A}$, written $\mathfrak{A}^{(k)}$, is the structure over $\Sigma^{(k)} \triangleq \Sigma \cup \{U\} \cup \{\pi_i, Q_i, E_{[1,i]}, E_{[i,k]} \mid 1 \leq i \leq k\}$ defined as follows:

$$
\begin{aligned}
|\mathfrak{A}^{(k)}| &= |\mathfrak{A}|^k \\
a^{\mathfrak{A}^{(k)}} &= \{\langle\langle v, \dots, v\rangle, \langle w, \dots, w\rangle\rangle \mid \langle v, w\rangle \in a^{\mathfrak{A}}\} \text{ for } a \in \Sigma \\
U^{\mathfrak{A}^{(k)}} &= \{\langle\langle v, \dots, v\rangle, \langle v, \dots, v\rangle\rangle \mid v \in |\mathfrak{A}|\} \\
\pi_i^{\mathfrak{A}^{(k)}} &= \{\langle\langle v_1, \dots, v_i, \dots, v_k\rangle, \langle v_i, \dots, v_i\rangle\rangle \mid v_1, \dots, v_k \in |\mathfrak{A}|\} \\
Q_i^{\mathfrak{A}^{(k)}} &= \{\langle\langle v_1, \dots, v_k\rangle, \langle v'_1, \dots, v'_k\rangle\rangle \in |\mathfrak{A}|^k \times |\mathfrak{A}|^k \mid v_j = v'_j \text{ for } 1 \leq j \leq k \text{ s.t. } j \neq i\} \\
E_{[i,i']}^{\mathfrak{A}^{(k)}} &= \{\langle\langle v_1, \dots, v_k\rangle, \langle v'_1, \dots, v'_k\rangle\rangle \in |\mathfrak{A}|^k \times |\mathfrak{A}|^k \mid v_j = v'_j \text{ for } i \leq j \leq i'\}
\end{aligned}
$$

Intuitively, in $k$-tuple structures $\mathfrak{A}^{(k)}$, we reflect each vertex $v$ on $\mathfrak{A}$ to the vertex $\langle v, \dots, v\rangle$ on $\mathfrak{A}^{(k)}$. The predicate $U$ denotes the set of such vertices on $\mathfrak{A}^{(k)}$ (coded into a binary identity relation). Each $k$-tuple $\langle v_1, \dots, v_i, \dots, v_k\rangle$ denotes the values of $k$ variables. By using the predicates $\pi_i$, we can map the tuple to the tuple $\langle v_i, \dots, v_i\rangle$ (so, each $\pi_i$ behaves as a projection), which is the vertex indicated by the $i$-th variable. The predicate $Q_i$ relates two $k$-tuples if their $j$-th elements are equal except when $j = i$; we will use $Q_i$ to denote the existential quantifier "$\exists x_i$" where $x_i$ denotes the $i$-th variable. The predicate $E_{[i,i']}$ relates two $k$-tuples if their $j$-th elements are equal for $i \leq j \leq i'$; we will use $E_{[i,i']}$ for succinctly defining $Q_i$.

We write $k$-TUPLE for the class of all $k$-tuple structures. Fig. 1 gives a graphical example of $k$-tuple structures when $k = 2$. Here, each 2-tuple $\langle v, v'\rangle$ is abbreviated to $vv'$ and the relations of $U$, $Q_i$, $E_{[i,i']}$ on $\mathfrak{A}^{(2)}$ are omitted where

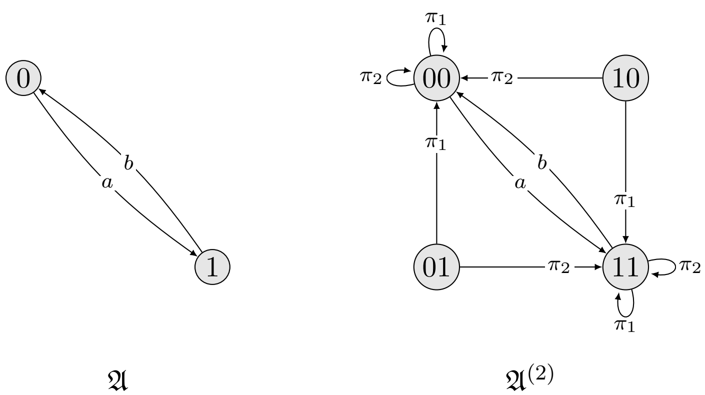


Figure 1. Example of $k$-tuple structures (when $k = 2$)

- $U^{\mathfrak{A}^{(2)}} = \{\langle 00, 00\rangle, \langle 11, 11\rangle\}$,
- $Q_1^{\mathfrak{A}^{(2)}} = \{\langle 00, 00\rangle, \langle 00, 10\rangle, \langle 01, 01\rangle, \langle 01, 11\rangle, \langle 10, 00\rangle, \langle 10, 10\rangle, \langle 11, 01\rangle, \langle 11, 11\rangle\}$,
- $Q_2^{\mathfrak{A}^{(2)}} = \{\langle 00, 00\rangle, \langle 00, 01\rangle, \langle 01, 00\rangle, \langle 01, 01\rangle, \langle 10, 10\rangle, \langle 10, 11\rangle, \langle 11, 10\rangle, \langle 11, 11\rangle\}$,
- $E_{[1,1]}^{\mathfrak{A}^{(2)}} = Q_2^{\mathfrak{A}^{(2)}}$, $E_{[2,2]}^{\mathfrak{A}^{(2)}} = Q_1^{\mathfrak{A}^{(2)}}$, and $E_{[1,2]}^{\mathfrak{A}^{(2)}} = [\![\mathsf{I}]\!]^{\mathfrak{A}^{(2)}}$.

This construction preserves finiteness, so the following holds.

**Proposition 3.2.** For all structures $\mathfrak{A}$ and $k \geq 1$, $\mathfrak{A}$ is finite if and only if $\mathfrak{A}^{(k)}$ is finite.

Additionally note that in $k$-tuple structures, $U, Q_i, E_{[i,i']}$ can be defined by using $\pi_i$ as follows:

$$U = \bigcap_{1 \leq j \leq k} \pi_j \qquad E_{[i,i']} = \bigcap_{i \leq j \leq i'} \pi_j \cdot \pi_j^{\smile} \qquad Q_i = \bigcap_{1 \leq j \leq k; j \neq i} \pi_j \cdot \pi_j^{\smile}$$

Thus, $U, Q_i, E_{[i,i']}$ does not change the expressive power; they are introduced to reduce the output size to be linear. By using $E_{[1,i-1]}$ and $E_{[i+1,k]}$, we can succinctly express $Q_i$ as $Q_i = E_{[1,i-1]} \cap E_{[i+1,k]}$.

We show that the class of (the isomorphism closure of) $k$-tuple structures can be characterized by using equations in CoR. Let $\Gamma^{(k)}$ be the following finite set of equations where $i$ ranges over $1 \leq i \leq k$ and $E_{[1,0]}$ and $E_{[k+1,k]}$ are the notations for denoting the term $\top$:

$$U = \bigcap_{1 \leq j \leq k} \pi_j \tag{1}$$
$$E_{[1,i]} = E_{[1,i-1]} \cap (\pi_i \cdot \pi_i^{\smile}) \tag{2}$$
$$E_{[i,k]} = E_{[i+1,k]} \cap (\pi_i \cdot \pi_i^{\smile}) \tag{3}$$
$$Q_i = E_{[1,i-1]} \cap E_{[i+1,k]} \tag{4}$$
$$U \leq \mathsf{I} \tag{5}$$
$$\pi_i^{\smile} \cdot \pi_i \leq \mathsf{I} \tag{6}$$
$$\mathsf{I} \leq \pi_i \cdot U \cdot \pi_i^{\smile} \tag{7}$$
$$\top \cdot U \leq Q_i \cdot \pi_i \tag{8}$$
$$\mathsf{I} = E_{[1,k]} \tag{9}$$
$$\top \cdot U \cdot \top = \top \tag{10}$$
$$a \leq U \cdot \top \cdot U \tag{11}$$

Equations (1) to (4) define $U$, $E_{[i,i']}$, and $Q_i$, respectively. Equation (5) expresses that $U$ is a subset of the identity relation. Equations (6) and (7) express that $\pi_i$ is a left-total function into $U$, namely $\pi_i$ is a function relation (Equation (6) implies that $\pi_i$ is functional and Equation (7) implies that $\pi_i$ is left-total) and its range is a subset of $U$. Equation (8) means that the vertex $\langle u_1, \dots, u_{i-1}, u'_i, u_{i+1}, \dots, u_k\rangle$ exists for every vertex $\langle u_1, \dots, u_k\rangle$ and every $u'_i$ in $U$. Equation (9) implies that if each $\pi_j$-images of two vertices are the same, then the vertices themselves are the same. Equation (10) expresses that the relation $U$ is not empty. Equation (11) expresses that the domain and the range of the relation $a$ are subsets of $U$.

For a class $\mathcal{C}$ of structures, we write $\mathrm{I}(\mathcal{C})$ for the *isomorphism closure* of $\mathcal{C}$: the minimal class $\mathcal{C}'$ subsuming $\mathcal{C}$ such that, if $\mathfrak{A} \in \mathcal{C}$ and $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$, then $\mathfrak{B} \in \mathcal{C}'$. The set $\Gamma^{(k)}$ can characterize the class of the isomorphism closure of $k$-tuple structures as follows:

**Lemma 3.3.** Let $\mathfrak{A}$ be a structure over $\Sigma^{(k)}$. Then we have:

$$\mathfrak{A} \models \bigwedge \Gamma^{(k)} \quad \Longleftrightarrow \quad \mathfrak{A} \in \mathrm{I}(k\text{-TUPLE}).$$

**Proof:**

($\Longleftarrow$): This direction can be shown by checking that each equation holds on $k$-tuple structures.

($\Longrightarrow$): Let $U_0 = \{v \mid \langle v, v\rangle \in U^{\mathfrak{A}}\}$. Combining $\pi_i^{\smile} \cdot \pi_i \leq \mathsf{I}$ (6), $\mathsf{I} \leq \pi_i \cdot U \cdot \pi_i^{\smile}$ (7), and $U \leq \mathsf{I}$ (5) yields that $\pi_i^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|$ to $U_0$. Let $f\colon |\mathfrak{A}| \to U_0^k$ be the function defined by $f(v) = \langle \pi_1^{\mathfrak{A}}(v), \dots, \pi_k^{\mathfrak{A}}(v)\rangle$. Then $f$ is bijective as follows. Let $v_0 \in |\mathfrak{A}|$ be an arbitrary vertex. Let $w_1, \dots, w_k$ be s.t. $f(v_0) = \langle w_1, \dots, w_k\rangle$. For any $w_1' \in U_0$, by $\pi_1 \cdot \top \cdot U \leq Q_1 \cdot \pi_1$ (8), $\langle v_0, w_1\rangle \in \pi_1^{\mathfrak{A}}$, $\langle w_1, w_1'\rangle \in [\![\top]\!]^{\mathfrak{A}}$, and $w_1' \in U_0$, there is some $v_1$ such that $\langle v_0, v_1\rangle \in Q_1^{\mathfrak{A}}$ and $\langle v_1, w_1'\rangle \in \pi_1^{\mathfrak{A}}$. Then by $Q_1 = E_{[2,k]}$ (4), we have $f(v_1) = \langle w_1', w_2, \dots, w_k\rangle$. Similarly, for any $w_2' \in U_0$, by $\pi_2 \cdot \top \cdot U \leq Q_2 \cdot \pi_2$ (8), $\langle v_1, w_2\rangle \in \pi_2^{\mathfrak{A}}$, $\langle w_2, w_2'\rangle \in [\![\top]\!]^{\mathfrak{A}}$, and $w_2' \in U_0$, there is some $v_2$ such that $\langle v_1, v_2\rangle \in Q_2^{\mathfrak{A}}$ and $\langle v_2, w_2'\rangle \in \pi_2^{\mathfrak{A}}$. Then by $Q_2 = E_{[1,1]} \cap E_{[3,k]}$ (4), we have $f(v_2) = \langle w_1', w_2', w_3, \dots, w_k\rangle$. By applying this method iteratively, we have that for any $w_1', \dots, w_k' \in U_0$, there is some $v$ such that $f(v) = \langle w_1', \dots, w_k'\rangle$. Hence, $f$ is surjective. Also, if $f(v) = \langle w_1, \dots, w_k\rangle = f(v')$, then by $\mathsf{I} = E_{[1,k]}$ (9), we have $v = v'$. Hence, $f$ is injective. Therefore, $f$ is bijective.

Note that for each $v \in |\mathfrak{A}|$, $v \in U_0$ iff $\pi_1^{\mathfrak{A}}(v) = \dots = \pi_k^{\mathfrak{A}}(v)$ (by $U = \bigcap_{1 \leq j \leq k} \pi_j$ (1)) iff $f(v) = \langle w, \dots, w\rangle$ for some $w$ (by the definition of $f$) iff $f(v) = \langle v, \dots, v\rangle$ (by $\bigcap_{1 \leq j \leq k} \pi_j \leq \mathsf{I}$ (1)(5)). Thus, $v \in U_0$ iff $f(v) = \langle w, \dots, w\rangle$ for some $w$ iff $f(v) = \langle v, \dots, v\rangle$. We now define $\mathfrak{B}$ as the structure over $\Sigma$, where

$$|\mathfrak{B}| = U_0 \qquad\qquad a^{\mathfrak{B}} = a^{\mathfrak{A}} \text{ for } a \in \Sigma.$$

Here, $\mathfrak{B}$ is indeed an structure, because $U_0$ is not empty by $\top \cdot U \cdot \top = \top$ (10) and $a^{\mathfrak{B}} \subseteq U_0^2$ by $a \leq U \cdot \top \cdot U$ (11). Then the bijection $f$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}^{(k)}$, as follows.

- For $a \in \Sigma$: Let $v, v' \in |\mathfrak{A}|$ be arbitrary vertices. We distinguish the following cases.
  - Case $v \notin U_0$ or $v' \notin U_0$: By $U \leq \mathsf{I}$ (5) and $a \leq U \cdot \top \cdot U$ (11), we have $\langle v, v'\rangle \notin a^{\mathfrak{A}}$. Also, in this case, $f(v)$ or $f(v')$ is not of the form $\langle w, \dots, w\rangle$ for any $w$ (by $v \notin U_0$ or $v' \notin U_0$); thus by the construction of $\mathfrak{B}^{(k)}$, we have $\langle f(v), f(v')\rangle \notin a^{\mathfrak{B}^{(k)}}$.
  - Case $v, v' \in U_0$: Then we have $f(v) = \langle v, \dots, v\rangle$ and $f(v') = \langle v', \dots, v'\rangle$. Thus, we have: $\langle v, v'\rangle \in a^{\mathfrak{A}}$ iff $\langle v, v'\rangle \in a^{\mathfrak{B}}$ iff $\langle f(v), f(v')\rangle \in a^{\mathfrak{B}^{(k)}}$ (by the construction of $\mathfrak{B}^{(k)}$).

  Hence, $\langle v, v'\rangle \in a^{\mathfrak{A}}$ iff $\langle f(v), f(v')\rangle \in a^{\mathfrak{B}^{(k)}}$.
- For $\pi_i$: Let $v, v' \in |\mathfrak{A}|$ be arbitrary vertices. We distinguish the following cases.

- Case $v' \notin U_0$: By $U \leq \mathsf{I}$ (5) and $\mathsf{I} \leq \pi_i \cdot U \cdot \pi_i^{\smile}$ (7), we have $\langle v, v' \rangle \notin \pi_i^{\mathfrak{A}}$. Also, in this case, $f(v')$ is not of the form $\langle w, \ldots, w \rangle$ for any $w$ (by $v' \notin U_0$); thus by the construction of $\mathfrak{B}^{(k)}$, $f(v')$ is not in the range of $\pi_i^{\mathfrak{B}^{(k)}}$. Hence, $\langle f(v), f(v') \rangle \notin \pi_i^{\mathfrak{B}^{(k)}}$.
- Case $v' \in U_0$: Then we have $f(v') = \langle v', \ldots, v' \rangle$. Thus, we have: $\langle v, v' \rangle \in \pi_i^{\mathfrak{A}}$ iff the $i$-th element of $f(v)$ is $v'$ (by the definition of $f$) iff $\langle f(v), \langle v', \ldots, v' \rangle \rangle \in \pi_i^{\mathfrak{B}^{(k)}}$ (by the construction of $\mathfrak{B}^{(k)}$) iff $\langle f(v), f(v') \rangle \in \pi_i^{\mathfrak{B}^{(k)}}$ (by $\langle v', \ldots, v' \rangle = f(v')$).

Hence, $\langle v, v' \rangle \in \pi_i^{\mathfrak{A}}$ iff $\langle f(v), f(v') \rangle \in \pi_i^{\mathfrak{B}^{(k)}}$.

- For $U, E_{[i,i']}, Q_i$: By Equations (1) to (4) with the fact that $f$ is an isomorphism w.r.t. $\pi_i$ as above.

Hence, this completes the proof. □

Using $k$-tuple structures, we can give the following translation.

**Definition 3.4.** Let $k \geq 1$ and $X = \{x_1, \ldots, x_k\}$ where $x_1, \ldots, x_k$ are pairwise distinct variables. For each formula $\varphi$ of $\mathrm{V}(\varphi) \subseteq X$, the term $\mathrm{T}^{(k)}(\varphi)$ is inductively defined as follows:

$$\begin{aligned} \mathrm{T}^{(k)}(a(x_i, x_j)) &\triangleq (\pi_i \cdot a \cdot \pi_j^{\smile}) \cap \mathsf{I} \\ \mathrm{T}^{(k)}(\neg\psi) &\triangleq (\mathrm{T}^{(k)}(\psi))^{-} \\ \mathrm{T}^{(k)}(\psi \wedge \rho) &\triangleq \mathrm{T}^{(k)}(\psi) \cap \mathrm{T}^{(k)}(\rho) \\ \mathrm{T}^{(k)}(\exists x_i, \psi) &\triangleq (Q_i \cdot \mathrm{T}^{(k)}(\psi) \cdot Q_i^{\smile}) \cap \mathsf{I}. \end{aligned}$$

**Lemma 3.5.** Let $k \geq 1$ and $X = \{x_1, \ldots, x_k\}$ where $x_1, \ldots, x_k$ are pairwise distinct variables. Let $\mathfrak{A}$ be a structure. For all formulas $\varphi$ of $\mathrm{V}(\varphi) \subseteq X$ and all $v_1, \ldots, v_k \in |\mathfrak{A}|$, we have:

$$[\![\mathrm{T}^{(k)}(\varphi)]\!]^{\mathfrak{A}^{(k)}} \quad = \quad \{\langle\langle v_1, \ldots, v_k\rangle, \langle v_1, \ldots, v_k\rangle\rangle \mid \{x_1 \mapsto v_1, \ldots, x_k \mapsto v_k\} \in [\![\varphi]\!]^{\mathfrak{A}} \restriction X\}.$$

Here, $[\![\varphi]\!]^{\mathfrak{A}} \restriction X$ denotes the set $\{f \restriction X \mid f \in [\![\varphi]\!]^{\mathfrak{A}}\}$ where $f \restriction X$ is the restriction of $f$ to $X$.

**Proof:**
By induction on the structure of $\varphi$.

Case $\varphi = a(x_i, x_j)$: Since $\mathrm{T}^{(k)}(\varphi) = (\pi_i \cdot a \cdot \pi_j^{\smile}) \cap \mathsf{I}$, we have:

$$\begin{aligned} &[\![(\pi_i \cdot a \cdot \pi_j^{\smile}) \cap \mathsf{I}]\!]^{\mathfrak{A}^{(k)}} \\ &= \{\langle\langle v_1, \ldots, v_k\rangle, \langle v_1, \ldots, v_k\rangle\rangle \mid \langle\langle v_i, \ldots, v_i\rangle, \langle v_j, \ldots, v_j\rangle\rangle \in a^{\mathfrak{A}^{(k)}}\} \\ &= \{\langle\langle v_1, \ldots, v_k\rangle, \langle v_1, \ldots, v_k\rangle\rangle \mid \langle v_i, v_j\rangle \in a^{\mathfrak{A}}\} && \text{(Def. of } a^{\mathfrak{A}^{(k)}}) \\ &= \{\langle\langle v_1, \ldots, v_k\rangle, \langle v_1, \ldots, v_k\rangle\rangle \mid \{x_1 \mapsto v_1, \ldots, x_k \mapsto v_k\} \in [\![a(x_i, x_j)]\!]^{\mathfrak{A}} \restriction X\} \end{aligned}$$

Case $\varphi = \neg\psi$: Since $\mathrm{T}^{(k)}(\varphi) = \mathrm{T}^{(k)}(\psi)^{-}$, we have:

$$
\begin{aligned}
&[\![\mathrm{T}^{(k)}(\psi)^{-}]\!]\\
&= |\mathfrak{A}^{(k)}|^2 \setminus [\![\mathrm{T}^{(k)}(\psi)]\!]\\
&= |\mathfrak{A}^{(k)}|^2 \setminus \{\langle\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle\rangle \mid \{x_1 \mapsto v_1,\dots,x_k \mapsto v_k\} \in [\![\psi]\!]^{\mathfrak{A}} \restriction X\} \qquad \text{(IH)}\\
&= \{\langle\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle\rangle \mid \{x_1 \mapsto v_1,\dots,x_k \mapsto v_k\} \notin [\![\psi]\!]^{\mathfrak{A}} \restriction X\}\\
&= \{\langle\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle\rangle \mid \{x_1 \mapsto v_1,\dots,x_k \mapsto v_k\} \in [\![\neg\psi]\!]^{\mathfrak{A}} \restriction X\}
\end{aligned}
$$

Case $\varphi = \psi \wedge \rho$: Since $\mathrm{T}^{(k)}(\varphi) = \mathrm{T}^{(k)}(\psi) \cap \mathrm{T}^{(k)}(\rho)$, we have:

$$
\begin{aligned}
&[\![\mathrm{T}^{(k)}(\psi) \cap \mathrm{T}^{(k)}(\rho)]\!]\\
&= [\![\mathrm{T}^{(k)}(\psi)]\!]^{\mathfrak{A}^{(k)}} \cap [\![\mathrm{T}^{(k)}(\rho)]\!]^{\mathfrak{A}^{(k)}}\\
&= \{\langle\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle\rangle \mid \{x_1 \mapsto v_1,\dots,x_k \mapsto v_k\} \in ([\![\psi]\!]^{\mathfrak{A}} \restriction X) \cap ([\![\psi]\!]^{\mathfrak{A}} \restriction X)\} \qquad \text{(IH)}\\
&= \{\langle\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle\rangle \mid \{x_1 \mapsto v_1,\dots,x_k \mapsto v_k\} \in [\![\psi \wedge \rho]\!]^{\mathfrak{A}} \restriction X\}
\end{aligned}
$$

Case $\varphi = \exists x_i, \psi$: Since $\mathrm{T}^{(k)}(\varphi) = (Q_i \cdot \mathrm{T}^{(k)}(\psi) \cdot Q_i^{\smile}) \cap \mathsf{I}$, we have:

$$
\begin{aligned}
&[\![(Q_i \cdot \mathrm{T}^{(k)}(\psi) \cdot Q_i^{\smile}) \cap \mathsf{I}]\!]^{\mathfrak{A}^{(k)}}\\
&= \{\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle \mid \langle\langle v'_1,\dots,v'_k\rangle,\langle v'_1,\dots,v'_k\rangle\rangle \in [\![\mathrm{T}^{(k)}(\psi)]\!]^{\mathfrak{A}^{(k)}}\\
&\qquad \text{for some } v'_1,\dots,v'_k \in |\mathfrak{A}| \text{ s.t. } v'_j = v_j \text{ for } 1 \le j \le k \text{ s.t. } j \neq i\}\\
&\qquad\qquad \text{(note that } [\![\mathrm{T}^{(k)}(\psi)]\!]^{\mathfrak{A}^{(k)}} \subseteq [\![\mathsf{I}]\!]^{\mathfrak{A}^{(k)}} \text{ by IH)}\\
&= \{\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle \mid \{x_1 \mapsto v'_1,\dots,x_k \mapsto v'_k\} \in [\![\psi]\!]^{\mathfrak{A}^{(k)}} \restriction X\\
&\qquad \text{for some } v'_1,\dots,v'_k \in |\mathfrak{A}| \text{ s.t. } v'_j = v_j \text{ for } 1 \le j \le k \text{ s.t. } j \neq i\} \qquad \text{(IH)}\\
&= \{\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle \mid \{x_1 \mapsto v_1,\dots,x_k \mapsto v_k\} \in [\![\exists x_i, \psi]\!]^{\mathfrak{A}^{(k)}} \restriction X\}
\end{aligned}
$$

Hence, this completes the proof. □

Combining the two above yields the following main lemma.

**Lemma 3.6.** Let $k \geq 1$ and $X = \{x_1, \dots, x_k\}$ where $x_1, \dots, x_k$ are pairwise distinct variables. Let $\varphi$ be an $\mathrm{FO}_{=}$ formula of $\mathrm{V}(\varphi) \subseteq X$. Then

$$(\textstyle\bigwedge \Gamma^{(k)}) \to \mathrm{T}^{(k)}(\varphi) \geq \mathsf{I} \text{ is [finitely] valid} \iff \varphi \text{ is [finitely] valid.}$$

**Proof:** We have:

$$
\begin{aligned}
&(\textstyle\bigwedge \Gamma^{(k)}) \to \mathrm{T}^{(k)}(\varphi) \geq \mathsf{I} \text{ is valid}\\
\iff\ & \langle v, v\rangle \in [\![\mathrm{T}^{(k)}(\varphi)]\!]^{\mathfrak{A}} \text{ for all } \mathfrak{A} \text{ s.t. } \mathfrak{A} \models \textstyle\bigwedge \Gamma^{(k)} \text{ and all } v \in |\mathfrak{A}|\\
\iff\ & \langle\langle v_1,\dots,v_k\rangle,\langle v_1,\dots,v_k\rangle\rangle \in [\![\mathrm{T}^{(k)}(\varphi)]\!]^{\mathfrak{B}^{(k)}} \text{ for all } \mathfrak{B} \text{ and } v_1,\dots,v_k \in |\mathfrak{B}| \qquad \text{(Lem. 3.3)}\\
\iff\ & \{x_1 \mapsto v_1,\dots,x_k \mapsto v_k\} \in [\![\varphi]\!]^{\mathfrak{B}} \restriction X \text{ for all } \mathfrak{B} \text{ and } v_1,\dots,v_k \in |\mathfrak{B}| \qquad \text{(Lem. 3.5)}\\
\iff\ & \varphi \text{ is valid.}
\end{aligned}
$$

For finite validity, it is shown in the same way because $\mathfrak{B}$ is finite iff $\mathfrak{B}^{(k)}$ is finite (Prop. 3.2). □

**Theorem 3.7.** There is a linear-size translation from $\mathrm{FO}_=$ formulas into CoR equations preserving validity and finite validity.

**Proof:** By Lem. 3.6 with the Schröder-Tarski translation (Prop. 2.3). □

**Remark 3.8.** We do not know whether our translation works for the equational theory of (possibly non-representable) relation algebras. This is because our construction is not compatible with *quasi-projective relation algebras*—relation algebras having elements $p$ and $q$ s.t. $p^{\smile} \cdot p \leq \mathsf{I}$, $q^{\smile} \cdot q \leq \mathsf{I}$, and $p^{\smile} \cdot q = \top$. (As quasi-projective relation algebras are representable [16], this class is useful to show that a given translation works also for the equational theory of relation algebras, see, e.g., [1].)

### 3.1. Reducing to a more restricted syntax of CoR

We recall that we can eliminate converse $\_^{\smile}$ and identity $\mathsf{I}$ by using translations given in [8].

**Proposition 3.9. ([8, Lem. 7, 9])**
There is a linear-size translation from CoR equations into CoR equations without $\_^{\smile}$ nor $\mathsf{I}$ preserving validity and finite validity.

**Proposition 3.10. ([8, Lem. 7, 9, 11, 16])**
There is a polynomial-size translation from CoR equations into CoR equations with one variable and without $\_^{\smile}$ nor $\mathsf{I}$ preserving validity and finite validity.

**Remark 3.11.** The translation in [8, Lem. 11] (for reducing the number of variables to one) is not a linear-size translation, as the output size is not bounded in linear (bounded in quadratic) to the input size.

By Thm. 3.7 with the two propositions above, we also have the following:

**Corollary 3.12.** There is a linear-size translation from $\mathrm{FO}_=$ formulas into CoR equations without $\_^{\smile}$ nor $\mathsf{I}$ preserving validity and finite validity.

**Corollary 3.13.** There is a polynomial-size translation from $\mathrm{FO}_=$ formulas into CoR equations with one variable and without $\_^{\smile}$ nor $\mathsf{I}$ preserving validity and finite validity.

Additionally, by the standard translation (Prop. 2.2), we also have obtained the following.

**Corollary 3.14.** There is a polynomial-size translation from $\mathrm{FO}_=$ formulas into FO3 formulas (without equality) with one binary predicate symbol preserving validity and finite validity.

# 4. Tseitin translation for CoR

By a similar argument as the *Tseitin translation* [17], which is a translation from propositional formulas into conjunctive normal form preserving validity in proposition logic (see also the Plaisted-Greenbaum translation [13] for $\mathrm{FO}_=$ and the translation from $\mathrm{FO2}_=$ into the Scott class [14, 5]), we can translate into CoR terms with bounded alternation of operations.

For each term $t$, we introduce a fresh variable $a_t$. Then for a term $t$, we define the set of equations $\Gamma_t$ as follows:

$$\begin{array}{lll} \Gamma_b \triangleq \{a_b = b\} & \Gamma_{s^-} = \Gamma_s \cup \{a_{s^-} = a_s^-\} & \Gamma_{s\cap u} = \Gamma_s \cup \Gamma_u \cup \{a_{s\cap u} = a_s \cap a_u\} \\ \Gamma_{\mathsf{I}} \triangleq \{a_{\mathsf{I}} = \mathsf{I}\} & \Gamma_{s^{\smile}} = \Gamma_s \cup \{a_{s^{\smile}} = a_s^{\smile}\} & \Gamma_{s\cdot u} = \Gamma_s \cup \Gamma_u \cup \{a_{s\cdot u} = a_s \cdot a_u\} \end{array}$$

Then we have the following:

**Lemma 4.1.** For all CoR terms $t$, we have:

$$t = \top \text{ is [finitely] valid} \quad \Longleftrightarrow \quad (\bigwedge \Gamma_t) \to a_t = \top \text{ is [finitely] valid.}$$

**Proof:** For all structures $\mathfrak{A}$ s.t. $\mathfrak{A} \models \bigwedge \Gamma_t$, we have $\mathfrak{A} \models s = a_s$ for all subterms $s$ of $t$, by straightforward induction on $s$. Thus it suffices to prove that $t = \top$ is [finitely] valid $\Longleftrightarrow$ $\bigwedge \Gamma_t \to t = \top$ is [finitely] valid. Both directions are shown as follows.

($\Longrightarrow$): Trivial.

($\Longleftarrow$): Since $a_s$ is not occurring in $t$, we can easily transform a structure $\mathfrak{A}$ s.t. $\mathfrak{A} \not\models t = \top$ into a structure $\mathfrak{A}'$ s.t. $\mathfrak{A}' \models \bigwedge \Gamma_t$ and $\mathfrak{A}' \not\models t = \top$ by only modifying $a_s^{\mathfrak{A}}$ appropriately. Hence this completes the proof. □

**Example 4.2.** The equation $((b \cdot c)^- \cdot d)^- = \top$ is translated into the following quantifier-free formula preserving validity and finite validity (we omit equations for each variables $b, c, d$, as they are verbose):

$$\bigwedge \left\{ \begin{array}{c} a_{b\cdot c} = a_b \cdot a_c, \quad a_{(b\cdot c)^-} = a_{b\cdot c}^-, \\ a_{(b\cdot c)^-\cdot d} = a_{(b\cdot c)^-} \cdot a_d, \quad a_{((b\cdot c)^-\cdot d)^-} = a_{(b\cdot c)^-\cdot d}^- \end{array} \right\} \to a_{((b\cdot c)^-\cdot d)^-} = \top$$

This is semantically equivalent to the following equation:

$$(\top \cdot \bigcup \left\{ \begin{array}{c} (a_{b\cdot c} \cap (a_b^- \dagger a_c^-)), \ (a_{b\cdot c}^- \cap (a_b \cdot a_c)), \\ (a_{(b\cdot c)^-} \cap a_{b\cdot c}), \ (a_{(b\cdot c)^-}^- \cap a_{b\cdot c}^-), \\ (a_{(b\cdot c)^-\cdot d} \cap (a_{(b\cdot c)^-}^- \dagger a_d^-)), \ (a_{(b\cdot c)^-\cdot d}^- \cap (a_{(b\cdot c)^-} \cdot a_d)), \\ (a_{((b\cdot c)^-\cdot d)^-} \cap a_{(b\cdot c)^-\cdot d}), \ (a_{((b\cdot c)^-\cdot d)^-}^- \cap a_{(b\cdot c)^-\cdot d}^-) \end{array} \right\} \cdot \top) \cup a_{((b\cdot c)^-\cdot d)^-} = \top$$

By using the translation above (and replacing complemented variables $b^-$ with fresh variables $c$ and introducing the axiom $b^- = c$), we can translate each CoR equation without $\_^{\smile}$ nor $\mathsf{I}$ into an equation of the form $(\top \cdot (\bigcup \Gamma) \cdot \top) \cup a = \top$, where $\Gamma$ is a finite set of terms of one of the following forms:

$$b \cap c \qquad b^- \cap c^- \qquad b \cap (c \dagger d) \qquad b \cap (c \cdot d) \qquad b \cap (c \cap d)$$

In this form, the number of alternations of operations, particularly the operations $\cdot$ and $\dagger$ (and similarly, $\cdot$ and $\_^-$), is reduced. Hence, we have obtained the following:

**Theorem 4.3.** There is a linear-size translation from CoR equations into equations of the form $t = \top$ preserving validity and finite validity, where $t$ is in the level $\Sigma_2^{\mathrm{CoR}}$ of the *dot-dagger alternation hierarchy* [10] and $t$ does not contain $\_^{\smile}$ nor $\mathsf{I}$.

**Proof:**
By Prop. 3.9, there is a linear-size translation from CoR equations into CoR equations without $\_^{\smile}$ nor $\mathsf{I}$. Then, by applying the translation of Lem. 4.1 (with the Schröder-Tarski translation (Prop. 2.3)) as above, this completes the proof. □

Hence, the equational theory of the form $t = \top$, where $t$ is in the level $\Sigma_2^{\mathrm{CoR}}$ of the dot-dagger alternation hierarchy, is also undecidable, cf. [11, Prop. 24][12, Appendix A].

## 4.1. Linear-size conservative reduction to Gödel's class $[\forall^3\exists^*, (0,\omega), (0)]$

Additionally, we note that by using the argument above, we can give a linear-size translation from $\mathrm{FO}_=$ formulas into $[\forall^3\exists^*, (0,\omega), (0)]$ sentences (i.e., sentences of the form $\forall x, \forall y, \forall z, \exists w_1, \dots, \exists w_n, \varphi$ where $n \geq 0$ and $\varphi$ is quantifier-free, has only binary predicate symbols and does not have constant symbols, function symbols or non-binary predicate symbols, see e.g., [3] for the notation of the prefix-vocabulary class. $\mathrm{FO}_=$ in this paper corresponds to the class $[\mathrm{all}, (0,\omega), (0)]_=$) [6][3, p. 440] preserving satisfiability and finite satisfiability.

For example, let us recall the translated equation in Example 4.2. By the standard translation (Prop. 2.2), this equation is semantically equivalent w.r.t. binary relations to the following FO3 sentence (without equality):

$$\exists x, \exists y, \bigvee \left\{ \begin{array}{l} (a_{b\cdot c}(x,y) \wedge (\forall w_1, \neg a_b(x,w_1) \vee \neg a_c(w_1,y))), \\ \quad (\neg a_{b\cdot c}(x,y) \wedge (\exists z, a_b(x,z) \wedge a_c(z,y))), \\ a_{(b\cdot c)^-}(x,y) \wedge a_{b\cdot c}(x,y),\ \neg a_{(b\cdot c)^-}(x,y) \wedge \neg a_{b\cdot c}(x,y), \\ ((a_{(b\cdot c)^-\cdot d}(x,y) \wedge (\forall w_2, \neg a_{(b\cdot c)^-}(x,w_2) \vee \neg a_d(w_2,y))), \\ \quad (\neg a_{(b\cdot c)^-\cdot d}(x,y) \wedge (\exists z, a_{(b\cdot c)^-}(x,z) \wedge a_d(z,y)))), \\ a_{((b\cdot c)^-\cdot d)^-}(x,y) \wedge a_{(b\cdot c)^-\cdot d}(x,y),\ \neg a_{((b\cdot c)^-\cdot d)^-}(x,y) \wedge \neg a_{(b\cdot c)^-\cdot d}(x,y) \end{array} \right\}$$
$$\vee\ (\forall w_3, \forall w_4, a_{((b\cdot c)^-\cdot d)^-}(w_3,w_4))$$

By taking the prenex normal form of the sentence above in the ordering of $x, y, z, w_1, w_2, w_3, w_4$, we can obtain an $[\exists^3\forall^*, (0,\omega), (0)]$ sentence (note that $(\exists z, \psi \vee \rho) \leftrightarrow ((\exists z, \psi) \vee (\exists z, \rho))$). Thus, as a corollary of Thm. 4.3, we can translate CoR equations without $\mathsf{I}$ into $[\exists^3\forall^*, (0,\omega), (0)]$ sentences preserving validity and finite validity. Hence, we also have the following:

**Corollary 4.4.** There is a linear-size conservative reduction from $\mathrm{FO}_=$ formulas into $[\forall^3\exists^*, (0,\omega), (0)]$ sentences.

**Proof:**
By Thm. 4.3 with the translation above, there is a linear-size translation from $\mathrm{FO}_=$ formulas into $[\exists^3\forall^*, (0,\omega), (0)]$ sentences preserving validity and finite validity. Hence, this completes the proof (by considering negated formulas). □

## References


[1] Andréka H, Németi I. Reducing First-order Logic to Df3, Free Algebras. In: Cylindric-like Algebras and Algebraic Logic, Bolyai Society Mathematical Studies, pp. 15–35. Springer. ISBN 978-3-642-35025-2, 2013. doi:10.1007/978-3-642-35025-2_2.

[2] Boolos GS, Burgess JP, Jeffrey RC. Computability and Logic. Cambridge University Press, 5 edition, 2007. ISBN 978-0-521-87752-7. doi:10.1017/CBO9780511804076.

[3] Börger E, Grädel E, Gurevich Y. The Classical Decision Problem. Springer, 1997. ISBN 3-540-42324-9. URL https://link.springer.com/9783540423249.

[4] Givant S. The calculus of relations. In: Introduction to Relation Algebras, volume 1, pp. 1–34. Springer International Publishing, 2017. doi:10.1007/978-3-319-65235-1_1.

[5] Grädel E, Kolaitis PG, Vardi MY. On the Decision Problem for Two-Variable First-Order Logic. *The Bulletin of Symbolic Logic*, 1997. **3**(1):53–69. doi:10.2307/421196.

[6] Kurt G. Zum Entscheidungsproblem des logischen Funktionenkalkiils. *Monatshefte f. Mathematik u. Physik*, 1933. pp. 40:433–443.

[7] Maddux RD. Finitary Algebraic Logic. *Mathematical Logic Quarterly*, 1989. **35**(4):321–332. doi:10.1002/malq.19890350405.

[8] Nakamura Y. The Undecidability of FO3 and the Calculus of Relations with Just One Binary Relation. In: ICLA, volume 11600 of *LNTCS*. Springer, 2019 pp. 108–120. doi:10.1007/978-3-662-58771-3_11.

[9] Nakamura Y. Expressive Power and Succinctness of the Positive Calculus of Relations. In: RAMiCS, volume 12062 of *LNTCS*. Springer, 2020 pp. 204–220. doi:10.1007/978-3-030-43520-2_13.

[10] Nakamura Y. Expressive power and succinctness of the positive calculus of binary relations. *Journal of Logical and Algebraic Methods in Programming*, 2022. **127**:100760. doi:10.1016/j.jlamp.2022.100760.

[11] Nakamura Y. On the Finite Variable-Occurrence Fragment of the Calculus of Relations with Bounded Dot-Dagger Alternation. In: MFCS, volume 272 of *LIPIcs*. Schloss Dagstuhl, 2023 p. 69:1–69:15. doi:10.4230/LIPIcs.MFCS.2023.69.

[12] Nakamura Y. On the Finite Variable-Occurrence Fragment of the Calculus of Relations with Bounded Dot-Dagger Alternation, 2023. doi:10.48550/arXiv.2307.05046.

[13] Plaisted DA, Greenbaum S. A Structure-preserving Clause Form Translation. *Journal of Symbolic Computation*, 1986. **2**(3):293–304. doi:10.1016/S0747-7171(86)80028-1.

[14] Scott D. A decision method for validity of sentences in two variables. *Journal of Symbolic Logic*, 1962. **27**(337).

[15] Tarski A. On the Calculus of Relations. *The Journal of Symbolic Logic*, 1941. **6**(3):73–89. doi:10.2307/2268577.

[16] Tarski A, Givant S. A Formalization of Set Theory without Variables, volume 41. American Mathematical Society, 1987. ISBN 978-0-8218-1041-5. doi:10.1090/coll/041.

[17] Tseitin GS. On the Complexity of Derivation in Propositional Calculus. In: Automation of Reasoning: 2: Classical Papers on Computational Logic 1967–1970, Symbolic Computation, pp. 466–483. Springer. ISBN 978-3-642-81955-1, 1983. doi:10.1007/978-3-642-81955-1_28.

# A. A construction from FO to FO3 not via CoR

In the following, we give a direct translation from $\mathrm{FO}_=$ formulas into $\mathrm{FO3}_=$ formulas not via CoR (this is almost immediately obtained from Sect. 3 with the standard translation from CoR terms into $\mathrm{FO3}_=$ formulas (Prop. 2.2)).

Let $\Gamma^{(k)}_{\mathrm{FO3}_=}$ be the following finite set of $\mathrm{FO3}_=$ formulas where $i$ ranges over $1 \leq i \leq k$, $E_{[1,0]}(x, y)$ and $E_{[k+1,k]}(x, y)$ are the notations for denoting the "true" formula $\mathtt{t}$ (Section 2.1), and $x, y, z$ are pairwise distinct variables:

$$\forall x, \forall y, (U(x, y) \leftrightarrow \bigwedge_{1 \leq j \leq k} \pi_j(x, y)) \tag{1'}$$

$$\forall x, \forall y, (E_{[1,i]}(x, y) \leftrightarrow (E_{[1,i-1]}(x, y) \wedge (\exists z, (\pi_i(x, z) \wedge \pi_i(y, z))))) \tag{2'}$$

$$\forall x, \forall y, (E_{[i,k]}(x, y) \leftrightarrow (E_{[i+1,k]}(x, y) \wedge (\exists z, (\pi_i(x, z) \wedge \pi_i(y, z))))) \tag{3'}$$

$$\forall x, \forall y, (Q_i(x, y) \leftrightarrow (E_{[1,i-1]}(x, y) \wedge E_{[i+1,k]}(x, y))) \tag{4'}$$

$$\forall x, \forall y, (U(x, y) \rightarrow x = y) \tag{5'}$$

$$\forall x, \forall y, \forall z, ((\pi_i(x, y) \wedge \pi_i(x, z)) \rightarrow y = z) \tag{6'}$$

$$\forall x, \exists y, (\pi_i(x, y) \wedge U(y, y)) \tag{7'}$$

$$\forall x, \forall y, (U(y, y) \rightarrow (\exists z, Q_i(x, z) \wedge \pi_i(z, y))) \tag{8'}$$

$$\forall x, \forall y, (x = y) \leftrightarrow E_{[1,k]}(x, y) \tag{9'}$$

$$\exists x, U(x, x) \tag{10'}$$

$$\forall x, \forall y, a(x, y) \rightarrow (U(x, x) \wedge U(y, y)) \tag{11'}$$

**Lemma A.1.** Let $\mathfrak{A}$ be a structure over $\Sigma^{(k)}$. Then

$$\mathfrak{A} \models \bigwedge \Gamma^{(k)}_{\mathrm{FO3}_=} \quad \Longleftrightarrow \quad \mathfrak{A} \in \mathrm{I}(k\text{-TUPLE}).$$

**Proof:**
We can check that $(\bigwedge \Gamma^{(k)}_{\mathrm{FO3}_=})$ and $(\bigwedge \Gamma^{(k)})$ are semantically equivalent. Thus, by Lem. 3.3, this completes the proof. □

**Definition A.2.** Let $k \geq 3$ and $X = \{x_1, \dots, x_k\}$ where $x_1, \dots, x_k$ are pairwise distinct variables. For each formula $\varphi$ of $\mathrm{V}(\varphi) \subseteq X$ and $z \in \{x_1, x_2, x_3\}$, the FO3 formula $\mathrm{T}^{(k)}_z(\varphi)$ of $\mathrm{V} \subseteq \{x_1, x_2, x_3\}$ is inductively defined as follows, where $z' = \min(\{x_1, x_2, x_3\} \setminus \{z\})$ and $z'' = \min(\{x_1, x_2, x_3\} \setminus \{z, z'\})$ under the ordering $x_1 < x_2 < x_3$:

$$\begin{aligned}
\mathrm{T}^{(k)}_z(a(x_i, x_j)) &\triangleq \exists z', \exists z'', \pi_i(z, z') \wedge a(z', z'') \wedge \pi_j(z, z'') \\
\mathrm{T}^{(k)}_z(\neg\psi) &\triangleq \neg \mathrm{T}^{(k)}_z(\psi) \\
\mathrm{T}^{(k)}_z(\psi \wedge \rho) &\triangleq \mathrm{T}^{(k)}_z(\psi) \wedge \mathrm{T}^{(k)}_z(\rho) \\
\mathrm{T}^{(k)}_z(\exists x_i, \psi) &\triangleq \exists z', Q_i(z, z') \wedge \mathrm{T}^{(k)}_{z'}(\psi).
\end{aligned}$$

**Lemma A.3.** Let $k \geq 3$ and $X = \{x_1, \dots, x_k\}$ where $x_1, \dots, x_k$ are pairwise distinct variables. Let $\mathfrak{A}$ be a structure. For all formulas $\varphi$ of $\mathrm{V}(\varphi) \subseteq X$, all $z \in \{x_1, x_2, x_3\}$, and all $u_1, \dots, u_k \in |\mathfrak{A}|$, we have:

$$\{z \mapsto \langle u_1, \dots, u_k \rangle\} \in [\![\mathrm{T}_z^{(k)}(\varphi)]\!]^{\mathfrak{A}^{(k)}} \restriction \{z\} \quad \Longleftrightarrow \quad \{x_1 \mapsto u_1, \dots, x_k \mapsto u_k\} \in [\![\varphi]\!]^{\mathfrak{A}} \restriction X.$$

**Proof:** By induction on the structure of $\varphi$ (similarly for Lem. 3.5). □

Combining the two above, we have obtained the following main lemma.

**Lemma A.4.** Let $k \geq 3$ and $X = \{x_1, \dots, x_k\}$ where $x_1, \dots, x_k$ are pairwise distinct variables. Let $\varphi$ be a formula of $\mathrm{V}(\varphi) \subseteq X$ and $z \in \{x_1, x_2, x_3\}$. Then

$$(\bigwedge \Gamma_{\mathrm{FO3}_=}^{(k)}) \to \mathrm{T}_z^{(k)}(\varphi) \text{ is [finitely] valid} \quad \Longleftrightarrow \quad \varphi \text{ is [finitely] valid.}$$

**Proof:** We have:

$$\begin{aligned}
&(\bigwedge \Gamma_{\mathrm{FO3}_=}^{(k)}) \to \mathrm{T}_z^{(k)}(\varphi) \text{ is valid} \\
\Longleftrightarrow\quad & \{z \mapsto v\} \in [\![\mathrm{T}_z^{(k)}(\varphi)]\!]^{\mathfrak{A}} \restriction \{z\} \text{ for all } \mathfrak{A} \text{ s.t. } \mathfrak{A} \models \bigwedge \Gamma_{\mathrm{FO3}_=}^{(k)} \text{ and all } v \in |\mathfrak{A}| \\
\Longleftrightarrow\quad & \{z \mapsto \langle u_1, \dots, u_k \rangle\} \in [\![\mathrm{T}_z^{(k)}(\varphi)]\!]^{\mathfrak{B}^{(k)}} \restriction \{z\} \text{ for all } \mathfrak{B} \text{ and } u_1, \dots, u_k \in |\mathfrak{B}| \quad \text{(Lem. A.1)} \\
\Longleftrightarrow\quad & \{x_1 \mapsto u_1, \dots, x_k \mapsto u_k\} \in [\![\varphi]\!]^{\mathfrak{B}} \restriction X \text{ for all } \mathfrak{B} \text{ and } u_1, \dots, u_k \in |\mathfrak{B}| \quad \text{(Lem. A.3)} \\
\Longleftrightarrow\quad & \varphi \text{ is valid.}
\end{aligned}$$

For finite validity, it is shown in the same way, because $\mathfrak{B}$ is finite iff $\mathfrak{B}^{(k)}$ is finite. □

Additionally, note that equality can be eliminated in $\mathrm{FO3}_=$.

**Proposition A.5.** There is a linear-size conservative reduction from $\mathrm{FO3}_=$ formulas into FO3 formulas without equality.

**Proof** (sketch)**:**
See, e.g., [2, Prop. 19.13] from $\mathrm{FO}_=$ to FO. This can be proved by replacing each occurrence of equality $=$ with a fresh binary predicate symbol $E$ and then adding axioms of that $E$ is an equivalence relation and of that each binary predicate $a$ satisfies the congruence law w.r.t. $E$:

$$\forall x, \forall x', \forall y, \forall y', (E(x, x') \land E(y, y')) \to (a(x, y) \leftrightarrow a(x', y'))$$

While the formula above is not in FO3, the construction in [2, Prop. 19.13] still works for FO3 by replacing the formula with the conjunction of the following two formulas:

$$\forall x, \forall x', \forall y, E(x, x') \to (a(x, y) \leftrightarrow a(x', y))$$
$$\forall x, \forall y, \forall y', E(y, y') \to (a(x, y) \leftrightarrow a(x, y'))$$

□

**Theorem A.6.** There is a linear-size conservative reduction from $\mathrm{FO}_=$ formulas into FO3 formulas (without equality).

**Proof:** By Lem. A.4 with Prop. A.5. □