Fundamenta Informaticae, Volume 195, Issue 1-4, Article 2, 2025
DOI: 10.46298/fi.12245
https://fi.episciences.org/

The Index and Core of a Relation With Applications to the Axiomatics of Relation Algebra

Roland Backhouse
University of Nottingham, UK

Ed Voermans
Independent Researcher

Abstract. We introduce the general notions of an index and a core of a relation. We postulate a limited form of the axiom of choice —specifically that all partial equivalence relations have an index— and explore the consequences of adding the axiom to standard axiom systems for point-free reasoning. Examples of the theorems we prove are that a core/index of a difunction is a bijection, and that the so-called "all or nothing" axiom used to facilitate pointwise reasoning is derivable from our axiom of choice.

1. Introduction

We introduce the general notions of an "index" and a "core" of a relation. As suggested by the terminology, the practical significance of both notions is to substantially reduce the size of a (possibly very large) binary relation in such a way that the relation can nevertheless easily be recovered. Example 1 illustrates the notions.

Example 1. Fig. 1 depicts a relation (on the left) and two instances of cores of the relation (in the middle and on the right). All are depicted as bipartite graphs. The relation R is a relation on blue and red nodes. The middle figure depicts a core as a relation on squares of blue nodes and squares of red nodes, each square being an equivalence class of the left per domain of R (on the left) or of the right per domain of R (on the right). The rightmost figure depicts a core as a relation on representatives of the equivalence classes: the relation depicted by the thick green edges. The rightmost figure also depicts an index of the relation; the middle does not: although the relations depicted in the middle and rightmost figures are isomorphic, they have different types. □

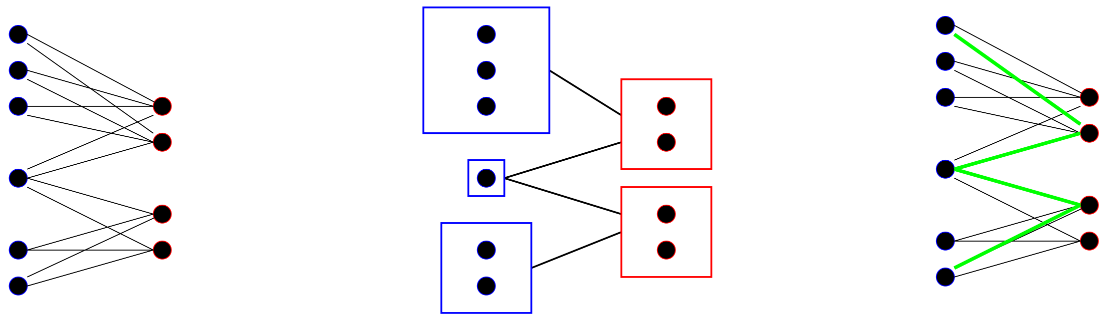

Figure 1. A Relation, a Core and an Index.

Although the notion of the "core" of a relation is more general than the notion of an "index", a significant disadvantage is that a "core" typically has a type that is different from the relation itself; in contrast, an "index" of a relation is a "core" that has the same type as the relation. This is useful for practical purposes, particularly in the context of heterogeneous relations, because it avoids the necessity to introduce type judgements. For this reason, our focus us on the notion of an index.

The paper is divided into three parts. The first part, consisting of Sections 2 and 3, sets up the framework on which our calculations are based. Section 2 summarises the axiom system and the notation we use; section 3 lists a number of derived concepts and their properties. These properties are given without proof.

In Section 4, we formalise the notions of a "core" and an "index" of a relation in the context of point-free relation algebra. We establish a large collection of properties of these notions which form a basis for the third part of the paper. (Because the notions are new, almost all the properties are new. An example of a property that some readers may recognise, albeit expressed differently, is that a difunction has an index that is a bijection.)

Section 5 specialises the notion of an index to partial equivalence relations and difunctions. Section 5.2 concludes by the introduction of a restricted form of the axiom of choice: we postulate that every partial equivalence relation has an index. This is the same as saying that it is possible to choose a representative element of every equivalence class of a partial equivalence relation. Section 5.3 then shows that every relation has an index. Section 6 is about applying our axiom of choice to the derivation of well-known characterisations of pers and difunctions.

Section 7 examines the consequences of adding our axiom of choice to point-free relation algebra in order to facilitate pointwise reasoning. We show that so doing has surprising and remarkable consequences. One such consequence is that we can derive the so-called "all-or-nothing" rule; this is a rule introduced by Glück [Glü17] also as a means of facilitating pointwise reasoning. (See [BDGv22] for examples of how the rule is used in reasoning about graphs.) The main theorem in Section 7 is that, with the addition of our axiom of choice and an extensionality axiom, the type $A{\sim}B$ of relations is isomorphic to the powerset $2^{A\times B}$ (the set of subsets of the cartesian product of A and B). Section 8 presents several models of point-free relation algebra clarifying the relation of the additional axioms to each other.

Section 9 concludes the paper with a discussion of the significance of the notions we have introduced and a pointer to the potential value for practical applications.

2. Axioms of Point-free Relation Algebra

In traditional, pointwise reasoning about relations, it is not the relations themselves that are the focus of interest. Rather, a relation R of type $A{\sim}B$ is defined to be a subset of the cartesian product $A{\times}B$ and the focus of interest is the boolean membership property $(a,b){\in}R$ where a and b are elements of type A and B, respectively. Equality of relations R and S is defined in terms of membership (typically in terms of "if and only if"), leading to a lack of concision (and frequently precision). In point-free relation algebra, the membership relation plays no role, and reasoning is truly about properties of relations.

In this section, we give a brief summary of the axioms of point-free relation algebra. For full details of the axioms, see [BDGv22]. For a summary of our notational conventions, see the appendix (Section 9).

2.1. Summary

Point-free relation algebra comprises three layers with interfaces between the layers plus additional axioms peculiar to relations. The axiom system is typed. For types A and B, $A{\sim}B$ denotes a set; the elements of the set are called *(heterogeneous) relations of type* $A{\sim}B$. Elements of type $A{\sim}A$, for some type A, are called *homogeneous relations*.

The first layer axiomatises the properties of a partially ordered set. We postulate that, for each pair of types A and B, $A{\sim}B$ forms a complete, universally distributive lattice. In anticipation of Section 7, where we add axioms that require $A{\sim}B$ to be a powerset, we use the symbol "$\subseteq$" for the ordering relation, and "$\cup$" and "$\cap$" for the supremum and infimum operators. We assume that this notation is familiar to the reader, allowing us to skip a more detailed account of its properties. However, we use $\perp\!\!\!\perp$ for the least element of the ordering (rather than the conventional $\emptyset$) and $\top\!\!\!\top$ for the greatest element. In keeping with the conventional practice of overloading the symbol "$\emptyset$", both these symbols are overloaded. The symbols "$\perp\!\!\!\perp$" and "$\top\!\!\!\top$" are pronounced "bottom" and "top", respectively. (Strictly we should write something like ${}_A{\perp\!\!\!\perp}_B$ and ${}_A{\top\!\!\!\top}_B$ for the bottom and top elements of type $A{\sim}B$. Of course, care needs to be taken when overloading operators in this way but it is usually the case that elementary type considerations allow the appropriate type to be deduced.)

It is important to note that there is no axiom stating that a relation is a set, and there is no corresponding notion of membership. (In, for example, [ABH$^+$92] and [Voe99], we used the symbols "$\sqsubseteq$", "$\sqcup$" and "$\sqcap$" and the name "spec calculus" rather than "relation algebra" in order to avoid misunderstanding.) The lack of a notion of membership distinguishes point-free relation algebra from pointwise algebra.

The second layer adds a composition operator. If R is a relation of type $A{\sim}B$ and S is a relation of type $B{\sim}C$, the composition of R and S is a relation of type $A{\sim}C$ which we denote by $R{\circ}S$. Composition is associative and, for each type A, there is an identity relation which we denote by $\mathbb{I}_A$. We often overload the notation for the identity relation, writing just $\mathbb{I}$. Occasionally, for greater clarity, we do supply the type information.

The interface between the first and second layers defines a relation algebra to be an instance of a *regular algebra* [Bac06] (also called a *standard Kleene algebra*, or **S**-*algebra* [Con71]). For this paper, the most important aspect of this interface is the existence and properties of the factor operators. These are introduced in Section 2.2. Also, $\perp\!\!\!\perp$ is a zero of composition: for all R, $\perp\!\!\!\perp\circ R = \perp\!\!\!\perp = R\circ\perp\!\!\!\perp$.

The completeness axiom in the first layer allows the reflexive-transitive closure R^* of each element R of type $A{\sim}A$, for some type A, to be defined. For practical applications, this is possibly the most important aspect of regular algebra but such applications are not considered in this paper. For this paper, completeness is only relevant when we add axioms to the algebra that model pointwise reasoning. We do require, however, the existence of $R{\cup}S$ and $R{\cap}S$, for all pairs of relations R and S of the same type, and the usual properties of set union and intersection.

The third layer is the introduction of a *converse* operator. If R is a relation of type $A{\sim}B$, the converse of R, which we denote by $R^{\cup}$ (pronounced R "wok") is a relation of type $B{\sim}A$. The interface with the first layer is that converse is a poset isomorphism (in particular, $\perp\!\!\!\perp^{\cup} = \perp\!\!\!\perp$ and $\top\!\!\!\top^{\cup} = \top\!\!\!\top$), and the interface with the second layer is formed by the two rules $\mathbb{I}^{\cup} = \mathbb{I}$ and, for all relations R and S of appropriate type, $(R\circ S)^{\cup} = S^{\cup}\circ R^{\cup}$.

Additional axioms characterise properties peculiar to relations. The modularity rule (aka Dedekind's rule [Rig48]) is that, for all relations R, S and T,

$$R\circ S\cap T \;\subseteq\; R\circ(S \,\cap\, R^{\cup}\circ T) \ . \tag{2}$$

The dual property, obtained by exploiting properties of the converse operator, is, for all relations R, S and T,

$$S\circ R\cap T \;\subseteq\; (S \,\cap\, T\circ R^{\cup})\circ R \ . \tag{3}$$

The modularity rule is necessary to the derivation of some of the properties we state without proof (for example, the properties of the domain operators given in section 3.1). Another rule is the *cone rule*:

$$\langle \forall R \;::\; \top\!\!\!\top\circ R\circ\top\!\!\!\top = \top\!\!\!\top \;\vee\; R = \perp\!\!\!\perp \rangle \ . \tag{4}$$

The cone rule limits consideration to "unary" relation algebras: constructing the cartesian product of two relation algebras to form a relation algebra (whereby the operators are defined pointwise) does not yield an algebra satisfying the cone rule.

2.2. Factors

If R is a relation of type $A{\sim}B$ and S is a relation of type $A{\sim}C$, the relation $R\backslash S$ of type $B{\sim}C$ is defined by the Galois connection, for all T (of type $B{\sim}C$),

$$T \subseteq R\backslash S \;\equiv\; R\circ T \subseteq S \ . \tag{5}$$

Similarly, if R is a relation of type $A{\sim}B$ and S is a relation of type $C{\sim}B$, the relation R/S of type $A{\sim}C$ is defined by the Galois connection, for all T,

$$T \subseteq R/S \;\equiv\; T\circ S \subseteq R \ . \tag{6}$$

The existence of factors is a property of a regular algebra; in relation algebra, factors are also known as "residuals". Factors have the *cancellation* properties:

$$T\circ T\backslash U \;\subseteq\; U \quad\wedge\quad R/S\circ S \;\subseteq\; R \ . \tag{7}$$

The relations $R\backslash R$ (of type $B{\sim}B$ if R has type $A{\sim}B$) and R/R (of type $A{\sim}A$ if R has type $A{\sim}B$) play a central role in what follows. As is easily verified, both are *preorders*. That is, both are *transitive*:

$$R\backslash R \circ R\backslash R \;\subseteq\; R\backslash R \quad\wedge\quad R/R \circ R/R \;\subseteq\; R/R \tag{8}$$

and both are *reflexive*:

$$\mathbb{I} \subseteq R\backslash R \quad\wedge\quad \mathbb{I} \subseteq R/R \ . \tag{9}$$

(The notation "$\mathbb{I}$" is overloaded in the above equation. In the left conjunct, it denotes the identity relation of type $B{\sim}B$ and, in the right conjunct, it denotes the identity relation of type $A{\sim}A$, assuming R has type $A{\sim}B$.) We also have the cancellation property, for all R,

$$R \circ R\backslash R \;=\; R \;=\; R/R \circ R \ . \tag{10}$$

Factors enjoy a rich theory which underlies many of our calculations. However, for space reasons, we omit further details here.

3. Some Definitions

In point-free relation algebra, "coreflexives" of a given type represent sets of elements of that type. A *coreflexive of type* A is a relation p such that $p \subseteq \mathbb{I}_A$. Frequently used properties are that, for all coreflexives p,

$$p = p^{\cup} = p{\circ}p$$

and, for all coreflexives p and q,

$$p{\circ}q \;=\; p \cap q \;=\; q{\circ}p \ .$$

(The proof of these properties relies on the modularity rule.) In the literature, coreflexives have several different names, usually depending on the application area in question. Examples are "monotype", "pid" (short for "partial identity") and "test".

3.1. The Domain Operators

The "domain operators" (see eg. [BH93]) play a dominant and unavoidable role. We exploit their properties frequently in calculations, so much so that we assume great familiarity with them.

Definition 11. (Domain Operators)

Given relation R of type $A{\sim}B$, the *left domain* $R{<}$ of R is a relation of type A defined by the equation

$$R{<} \;=\; \mathbb{I}_A \cap R \circ R^{\cup}$$

and the *right domain* $R{>}$ of R is a relation of type B is defined by the equation

$$R{>} \;=\; \mathbb{I}_B \cap R^{\cup} \circ R \ .$$

□

The name "domain operator" is chosen because of the fundamental properties: for all R and all coreflexives p,

$$R = R{\circ}p \quad\equiv\quad R{>} = R{>}\circ p \tag{12}$$

and

$$R = p{\circ}R \quad\equiv\quad R{<} = p\circ R{<} \ . \tag{13}$$

A simple, often-used consequence of (12) and (13) is the property:

$$R{<}\circ R \;=\; R \;=\; R\circ R{>} \ . \tag{14}$$

In words, $R{>}$ is the least coreflexive p such that restricting the "domain" of R on the right has no effect on R. It is in this sense that $R{<}$ and $R{>}$ represent the set of points on the left and on the right on which the relation R is "defined", i.e. its left and right "domains".

By instantiating p to $\perp\!\!\!\perp$ in (12) and (13) we get

$$(R{<} = \perp\!\!\!\perp) \;=\; (R = \perp\!\!\!\perp) \;=\; (R{>} = \perp\!\!\!\perp) \ . \tag{15}$$

Additional properties used frequently below are as follows.

Theorem 16. For all relations R and coreflexives p,

$$R{>} \subseteq p \;\equiv\; R \subseteq \top\!\!\!\top{\circ}p \;\text{ and }\; R{<} \subseteq p \;\equiv\; R \subseteq p{\circ}\top\!\!\!\top \ , \tag{17}$$

$$R{>} \subseteq p \;\equiv\; R \subseteq R{\circ}p \;\text{ and }\; R{<} \subseteq p \;\equiv\; R = p{\circ}R \ . \tag{18}$$

□

Theorem 19. For all relations R and S,

(a) $\top\!\!\!\top\circ R{>} = \top\!\!\!\top{\circ}R$ and $R{<}\circ\top\!\!\!\top = R{\circ}\top\!\!\!\top$,

(b) $(R^{\cup}){>} = R{<}$ and $(R^{\cup}){<} = R{>}$, and

(c) $(R{\circ}S){>} = (R{>}\circ S){>}$ and $(R{\circ}S){<} = (R\circ S{<}){<}$. □

We also use the fact that the domain operators are monotonic (as is evident from Definition 11).

3.2. Pers and Per Domains

Given relations R of type $A{\sim}B$ and S of type $A{\sim}C$, the symmetric *right-division* is the relation $R\backslash\!\backslash S$ of type $B{\sim}C$ defined in terms of *right* factors as

$$R\backslash\!\backslash S \;=\; R\backslash S \cap (S\backslash R)^{\cup} \ . \tag{20}$$

Dually, given relations R of type $B{\sim}A$ and S of type $C{\sim}A$, the symmetric *left-division* is the relation $R/\!/S$ of type $B{\sim}C$ defined in terms of left factors as

$$R/\!/S \;=\; R/S \;\cap\; (S/R)^{\cup} \ . \tag{21}$$

The relation $R\backslash\!\backslash R$ is an equivalence relation[1]. Voermans [Voe99] calls it the "greatest right domain" of R. Riguet [Rig48] calls $R\backslash\!\backslash R$ the "noyau" of R (but defines it using nested complements). Others (see [Oli18] for references) call it the "kernel" of R.

As remarked elsewhere [Oli18], the *symmetric left-division* inherits a number of (left) cancellation properties from the properties of factorisation in terms of which it is defined. For our purposes, the only cancellation property we use is the following (inherited from (10)). For all R,

$$R\circ R\backslash\!\backslash R \;=\; R \;=\; R/\!\!/R\circ R \;. \tag{22}$$

In this section the focus is on the left and right "per domains" introduced by Voermans [Voe99].

Definition 23. (Right and Left Per Domains)
The *right per domain* of relation R, denoted $R\succ$, is defined by the equation

$$R\succ \;=\; R{>}\circ R\backslash\!\backslash R \;. \tag{24}$$

Dually, the *left per domain* of relation R, denoted $R\prec$, is defined by the equation

$$R\prec \;=\; R/\!\!/R\circ R{<} \;. \tag{25}$$

□

The left and right per domains are "pers" where "per" is an abbreviation of "partial equivalence relation".

Definition 26. (Partial Equivalence Relation (per))
A relation is a *partial equivalence relation* iff it is symmetric and transitive. That is, R is a partial equivalence relation iff

$$R = R^{\cup} \;\wedge\; R\circ R\subseteq R \;.$$

Henceforth we abbreviate partial equivalence relation to *per*. □

That $R\prec$ and $R\succ$ are indeed pers is a direct consequence of the symmetry and transitivity of $R\backslash\!\backslash R$.

The left and right per domains are called "domains" because, like the coreflexive domains, we have the properties: for all relations R and pers P,

$$R = R\circ P \;\;\equiv\;\; R\succ = R\succ\circ P \;\;\text{, and} \tag{27}$$

$$R = P\circ R \;\;\equiv\;\; R\prec = P\circ R\prec \;. \tag{28}$$

As with the coreflexive domains, we also have:

$$R\prec\circ R \;=\; R \;=\; R\circ R\succ \;. \tag{29}$$

[1] This is a well-known fact: the relation $R\backslash\!\backslash R$ is the symmetric closure of the preorder $R\backslash R$. The easy proof is left to the reader.

(The second of these equalities is an immediate consequence of (22) and the properties of (coreflexive) domains; the first is symmetric.) Indeed, $R{\prec}$ and $R{\succ}$ are the "least" pers that satisfy the equalities (29). See [Voe99] for details of the ordering relation on pers.

The right per domain $R{\succ}$ can be defined equivalently by the equation

$$R{\succ} \;=\; R\backslash\!\backslash R \circ R{>} \;\;. \tag{30}$$

Moreover,

$$(R{\succ}){<} \;=\; R{>} \;=\; (R{\succ}){>} \;\;. \tag{31}$$

(See [Bac21] for the proofs of these properties.) Symmetrical properties hold of $R{\prec}$.

The following lemma extends [Rig48, Corollaire, p.134] from equivalence relations to pers.

Lemma 32. For all relations R, the following statements are all equivalent.

(i) R is a per (i.e. $R = R^{\cup} \wedge R{\circ}R \subseteq R$) ,

(ii) $R = R^{\cup} \circ R$,

(iii) $R = R{\prec}$,

(iv) $R = R{\succ}$. □

For further properties of pers and per domains, see [Voe99].

3.3. Functionality

A relation R of type $A{\sim}B$ is said to be *left-functional* iff $R \circ R^{\cup} = R{<}$. Equivalently, R is *left-functional* iff $R \circ R^{\cup} \subseteq \mathbb{I}_A$. It is said to be *right-functional* iff $R^{\cup} \circ R = R{>}$ (equivalently, $R^{\cup} \circ R \subseteq \mathbb{I}_B$). A relation R is said to be a *bijection* iff it is both left- and right-functional.

Rather than left- and right-functional, the more common terminology is "functional" and "injective" but publications differ on which of left- or right-functional is "functional" or "injective". We choose to abbreviate "left-functional" to *functional* and to use the term *injective* instead of right-functional. Typically, we use f and g to denote functional relations, and Greek letters to denote bijections (although the latter is not always the case). Other authors make the opposite choice.

3.4. Difunctions

Formally, relation R is *difunctional* iff

$$R \circ R^{\cup} \circ R \;\subseteq\; R \;\;. \tag{33}$$

As for pers, there are several equivalent definitions of "difunctional". We begin with the point-free definitions:

Theorem 34. For all R, the following statements are all equivalent.

(i) R is difunctional (i.e. $R \circ R^{\cup} \circ R \subseteq R$) ,

(ii) $R = R \circ R^{\cup} \circ R$,

(iii) $R{>} \circ R\backslash R = R^{\cup} \circ R$,

(iv) $R{\succ} = R^{\cup} \circ R$,

(v) $R/R \circ R{<} = R \circ R^{\cup}$,

(vi) $R{\prec} = R \circ R^{\cup}$,

(vii) $R = R \cap (R\backslash R/R)^{\cup}$. □

The equivalence of 34(i) and 34(ii) is well-known and due to Riguet [Rig48, Rig50]; the equivalence of 34(i), (iv) and (vi) is due to Voermans [Voe99]. The equivalence of 34(i), (iii) and (v) is formally stronger: a consequence is that, if R is difunctional,

$$R{\succ} = R{>} \circ R\backslash R \quad \wedge \quad R{\prec} = R/R \circ R{<} \ . \tag{35}$$

(Cf. (24).)

Riguet [Rig51] calls the right side of 34(vii) the "différence" of relation R, although he defines it using nested complements rather than factors. The name "différence" is motivated by his definition. Since we avoid the use of complements (and for other reasons) we prefer the term "diagonal"; see [BV22, BV23] for further discussion.

The equivalence of (i) and (vii) is a straightforward but beautiful application of the Galois connections defining intersection, converse and factors.

Definition (33) is the most useful when it is required to establish that a particular relation is difunctional, whereas properties 34(ii)-(vii) are more useful when it is required to exploit the fact that a particular relation is difunctional.

The combination of Theorem 34 (in particular 34(ii) and 34(iv) with Lemma 32) allows one to prove that a per is a symmetric difunction. (We leave the easy calculation to the reader.) This property is sometimes used to specialise properties of difunctions to properties of pers.

3.5. Squares and Rectangles

We now introduce the notions of a "rectangle" and a "square"; rectangles are typically heterogeneous whilst squares are, by definition, homogeneous relations. Squares are rectangles; properties of squares are typically obtained by specialising properties of rectangles. (Riguet [Rig48] uses the terms "rectangle" and "carré".)

Definition 36. (Rectangle and Square)
A relation R is a *rectangle* iff $R = R \circ \mathbb{T} \circ R$. A relation R is a *square* iff R is a symmetric rectangle. □

It is easily shown that a rectangle is a difunction and a square is a per.

Lemma 37. For all relations R and S, $R{\circ}\mathbb{T}{\circ}S$ is a rectangle. It follows that $R{\circ}T{\circ}S$ is a rectangle if T is a rectangle.

Proof:
Because the proof is based on the cone rule, a case analysis is necessary. In the case that either R or S is the empty relation, the lemma clearly holds (because $R{\circ}\mathbb{T}{\circ}S$ is the empty relation, and the empty relation is a rectangle). Suppose now that both R and S are non-empty. Then

$$
\begin{array}{ll}
 & R{\circ}\mathbb{T}{\circ}S{\circ}\mathbb{T}{\circ}R{\circ}\mathbb{T}{\circ}S \\
= & \{ \quad \text{cone rule: (4) (applied twice), assumption: } R{\neq}\perp\!\!\!\perp \text{ and } S{\neq}\perp\!\!\!\perp \quad \} \\
 & R{\circ}\mathbb{T}{\circ}S \ .
\end{array}
$$

If T is a rectangle, $R{\circ}T{\circ}S = R{\circ}T{\circ}\mathbb{T}{\circ}T{\circ}S$; thus $R{\circ}T{\circ}S$ is a rectangle. □

3.6. Isomorphic Relations

The (yet-to-be-defined) cores and indexes of a given relation are not unique; in common mathematical jargon, they are unique "up to isomorphism". In order to make this precise, we need to define the notion of isomorphic relation and establish a number of properties.

Definition 38. Suppose R and S are two relations (not necessarily of the same type). Then we say that R and S are *isomorphic* and write $R{\cong}S$ iff

$$
\begin{array}{ll}
\langle \exists\, \phi,\psi & \\
\quad : & \phi\circ\phi^{\cup} = R{<} \;\wedge\; \phi^{\cup}\circ\phi = S{<} \;\wedge\; \psi\circ\psi^{\cup} = R{>} \;\wedge\; \psi^{\cup}\circ\psi = S{>} \\
\quad : & R = \phi\circ S\circ\psi^{\cup} \\
\rangle \ . &
\end{array}
$$

□

The relation between R and S in definition 38 can be strengthened to the conjunction

$$R = \phi\circ S\circ\psi^{\cup} \quad\wedge\quad \phi^{\cup}\circ R\circ\psi = S \ . \tag{39}$$

Alternatively, the leftmost conjunct can be replaced by the rightmost conjunct. This is a consequence of the following lemma.

Lemma 40. For all ϕ, ψ, R and S,

$$
\begin{array}{ll}
 & (R = \phi\circ S\circ\psi^{\cup} \;\equiv\; \phi^{\cup}\circ R\circ\psi = S) \\
\Leftarrow & \phi\circ\phi^{\cup} = R{<} \;\wedge\; \phi^{\cup}\circ\phi = S{<} \;\wedge\; \psi\circ\psi^{\cup} = R{>} \;\wedge\; \psi^{\cup}\circ\psi = S{>} \ .
\end{array}
$$

□

We often choose one or other of the conjuncts in (39), whichever being most convenient at the time.

Lemma 41. The relation $\cong$ is reflexive, transitive and symmetric. That is, $\cong$ is an equivalence relation. □

The task of proving that two relations are isomorphic involves constructing ϕ and ψ that satisfy the conditions of the existential quantification in Definition 38; we call the constructed values *witnesses* to the isomorphism.

Note that the requirement on ϕ in Definition 38 is that it is both functional and injective; thus it is required to "witness" a (1–1) correspondence between the points in the left domain of R and the points in the left domain of S. Similarly, the requirement on ψ is that it "witnesses" a (1–1) correspondence between the points in the right domain of R and the points in the right domain of S. Formally, $R{<}$ and $S{<}$ are isomorphic as "witnessed" by ϕ and $R{>}$ and $S{>}$ are isomorphic as "witnessed" by ψ:

Lemma 42. Suppose R and S are relations such that $R \cong S$. Then $R{<} \cong S{<}$ and $R{>} \cong S{>}$. Specifically, if ϕ and ψ witness the isomorphism $R \cong S$,

$$R{<} = \phi \circ S{<} \circ \phi^{\cup} \quad \wedge \quad R{>} = \psi \circ S{>} \circ \psi^{\cup} \ .$$

Proof:
Suppose ϕ and ψ are such that

$$\phi \circ \phi^{\cup} = R{<} \ \wedge \ \phi^{\cup} \circ \phi = S{<} \ \wedge \ \psi \circ \psi^{\cup} = R{>} \ \wedge \ \psi^{\cup} \circ \psi = S{>} \ .$$

Then

$$\begin{array}{ll}
 & R{<} \\
= & \qquad \{ \quad R{<} \text{ is a coreflexive} \quad \} \\
 & R{<} \circ R{<} \\
= & \qquad \{ \quad \text{assumption} \quad \} \\
 & \phi \circ \phi^{\cup} \circ \phi \circ \phi^{\cup} \\
= & \qquad \{ \quad \text{assumption} \quad \} \\
 & \phi \circ S{<} \circ \phi^{\cup} \ .
\end{array}$$

That is $R{<} = \phi \circ S{<} \circ \phi^{\cup}$. Similarly, $R{>} = \psi \circ S{>} \circ \psi^{\cup}$. But also (because the domain operators are closure operators),

$$\phi \circ \phi^{\cup} = (R{<}){<} \ \wedge \ \phi^{\cup} \circ \phi = (S{<}){<} \ \wedge \ \psi \circ \psi^{\cup} = (R{>}){>} \ \wedge \ \psi^{\cup} \circ \psi = (S{>}){>} \ .$$

Applying Definition 38 with $R{,}S{,}\phi{,}\psi := R{<}\,, S{<}\,, \phi\,, \phi$ and $R{,}S{,}\phi{,}\psi := R{>}\,, S{>}\,, \psi\,, \psi$, the lemma is proved. □

The property of the left and right domains stated in Lemma 42 is also valid for the left and right per domains:

Lemma 43. Suppose R and S are relations such that $R \cong S$. Then $R{\prec} \cong S{\prec}$ and $R{\succ} \cong S{\succ}$. Specifically, if ϕ and ψ witness the isomorphism $R \cong S$,

$$R{\prec} = \phi \circ S{\prec} \circ \phi^{\cup} \quad \wedge \quad R{\succ} = \psi \circ S{\succ} \circ \psi^{\cup} \ .$$

Proof:
Suppose ϕ and ψ witness the isomorphism $R \cong S$. We show that the pair (ψ, ψ) witnesses the isomorphism $R{\succ} \cong S{\succ}$. By assumption, $\psi \circ \psi^{\cup} = R{>}$, $\psi^{\cup} \circ \psi = S{>}$. Moreover, for all R,

$$(R{\succ}){>} = (R{\succ}){<} = R{>};$$

thus $\psi \circ \psi^{\cup} = (R{\succ}){>}$ and $\psi^{\cup} \circ \psi = (S{\succ}){>}$. So it remains to show that $R{\succ} \;=\; \psi \circ S{\succ} \circ \psi^{\cup}$. Now

$$\begin{array}{ll}
 & R{\succ} \;=\; \psi \circ S{\succ} \circ \psi^{\cup} \\
\Leftarrow & \quad \{ \quad \text{transitivity} \quad \} \\
 & R{\succ} \;=\; R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \;=\; \psi \circ S{\succ} \circ \psi^{\cup} \;\; .
\end{array}$$

The calculation thus splits into two steps: the proof of the leftmost equality and the proof of the rightmost equality. The leftmost equality proceeds as follows.

$$\begin{array}{ll}
 & R{\succ} \;=\; R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \\
= & \quad \{ \quad (27),\ \psi \circ S{\succ} \circ \psi^{\cup} \text{ is a per (see below)} \quad \} \\
 & R \;=\; R \circ \psi \circ S{\succ} \circ \psi^{\cup} \;\; .
\end{array}$$

Continuing with the right hand side:

$$\begin{array}{ll}
 & R \circ \psi \circ S{\succ} \circ \psi^{\cup} \\
= & \quad \{ \quad R = \phi \circ S \circ \psi^{\cup} \quad \} \\
 & \phi \circ S \circ \psi^{\cup} \circ \psi \circ S{\succ} \circ \psi^{\cup} \\
= & \quad \{ \quad \psi^{\cup} \circ \psi = S{>},\ \text{domains: (14) and (29)} \quad \} \\
 & \phi \circ S \circ \psi^{\cup} \\
= & \quad \{ \quad \text{see Lemma 41} \quad \} \\
 & R \;\; .
\end{array}$$

Combining the two calculations, we have established that

$$R{\succ} \;=\; R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \;\; .$$

Now, for the rightmost equality, we have:

$$\begin{array}{ll}
 & R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \;=\; \psi \circ S{\succ} \circ \psi^{\cup} \\
= & \quad \{ \quad (R{\succ}){<} = R{>},\ \text{domains: (14)} \quad \} \\
 & R{>} \circ R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \;=\; \psi \circ S{\succ} \circ \psi^{\cup} \\
= & \quad \{ \quad R{>} = \psi \circ \psi^{\cup} \quad \} \\
 & \psi \circ \psi^{\cup} \circ R{\succ} \circ \psi \circ S{\succ} \circ \psi^{\cup} \;=\; \psi \circ S{\succ} \circ \psi^{\cup} \\
\Leftarrow & \quad \{ \quad \text{Leibniz} \quad \} \\
 & \psi^{\cup} \circ R{\succ} \circ \psi \circ S{\succ} \;=\; S{\succ} \\
= & \quad \{ \quad \text{converse (noting that } R{\succ} \text{ and } S{\succ} \text{ are symmetric)} \quad \} \\
 & S{\succ} \circ \psi^{\cup} \circ R{\succ} \circ \psi \;=\; S{\succ} \\
= & \quad \{ \quad (27),\ \psi^{\cup} \circ R{\succ} \circ \psi \text{ is a per (see below)} \quad \} \\
 & S \circ \psi^{\cup} \circ R{\succ} \circ \psi \;=\; S \\
= & \quad \{ \quad \text{as above, with } R,S,\psi := S\,,R\,,\psi^{\cup} \quad \} \\
 & \mathsf{true} \;\; .
\end{array}$$

Note that the usage of (27) relies on the fact that both $\psi \circ S{\succ} \circ \psi^{\cup}$ and $\psi^{\cup} \circ R{\succ} \circ \psi$ are pers. The straightforward proof is omitted. □

Lemma 44. A relation R is isomorphic to a coreflexive iff R is a bijection.

Proof:
The proof is by mutual implication. Suppose first that R is a bijection. That is,

$$R \circ R^{\cup} = R{<} \quad \wedge \quad R^{\cup} \circ R = R{>} \ \ .$$

We prove that R is isomorphic to $R{<}$. (Symmetrically, R is isomorphic to $R{>}$.) For the witnesses we take $R{<}$ and R. Instantiating Definition 38, we have to verify that

$$R{<} \circ (R{<})^{\cup} = R{<} \ \wedge \ (R{<})^{\cup} \circ R{<} = R{<} \ \wedge \ R \circ R^{\cup} = (R{<}){>} \ \wedge \ R^{\cup} \circ R = R{>}$$

and

$$R{<} = R{<} \circ R \circ R^{\cup} \ \ .$$

The verification is a straightforward application of properties of the left domain operator.

Now suppose that coreflexive p is isomorphic to R. Suppose the witnesses are ϕ and ψ. That is,

$$\phi \circ \phi^{\cup} \ = \ p \ \wedge \ \phi^{\cup} \circ \phi = R{<} \ \wedge \ \psi^{\cup} \circ \psi = R{>} \tag{45}$$

and

$$p = \phi \circ R \circ \psi^{\cup} \ \ . \tag{46}$$

Then

$$\begin{array}{ll}
 & R{<} \\
= & \{ \quad \phi^{\cup} \circ \phi = R{<} = R{<} \circ R{<} \quad \} \\
 & \phi^{\cup} \circ \phi \circ \phi^{\cup} \circ \phi \\
= & \{ \quad \phi \circ \phi^{\cup} = p = p \circ p^{\cup} \quad \} \\
 & \phi^{\cup} \circ p \circ p^{\cup} \circ \phi \\
= & \{ \quad (46) \quad \} \\
 & \phi^{\cup} \circ \phi \circ R \circ \psi^{\cup} \circ (\phi \circ R \circ \psi^{\cup})^{\cup} \circ \phi \\
= & \{ \quad \text{converse} \quad \} \\
 & \phi^{\cup} \circ \phi \circ R \circ \psi^{\cup} \circ \psi \circ R^{\cup} \circ \phi^{\cup} \circ \phi \\
= & \{ \quad (45) \quad \} \\
 & R{<} \circ R \circ R{>} \circ R^{\cup} \circ R{<} \\
= & \{ \quad \text{domains: (14)} \quad \} \\
 & R \circ R^{\cup} \ \ .
\end{array}$$

We conclude that $R{<} = R \circ R^{\cup}$. Symmetrically, $R{>} = R^{\cup} \circ R$. That is, R is a bijection. □

Theorem 47. Suppose P is a per. Then,

$$P{<} = P \quad \Leftarrow \quad P{<} \cong P \ .$$

In particular, for all R,

$$R{<} = R{\prec} \quad \Leftarrow \quad R{<} \cong R{\prec} \ .$$

Symmetrically, for all R,

$$R{>} = R{\succ} \quad \Leftarrow \quad R{>} \cong R{\succ} \ .$$

Proof:
This is an instance of Lemma 44. Specifically, assuming that $P{<} \cong P$, we may apply the instantiation $p{,}R := P{<}\,,P$ in Lemma 44 to deduce that P is a bijection. That is, $P \circ P^{\cup} = P{<}$. But P is a per (i.e. $P = P \circ P^{\cup}$). So we conclude that

$$P = P{<} \ .$$

That, for all R, $R{<} = R{\prec}$ if $R{<} \cong R{\prec}$ now follows by making the instantiation $P := R{\prec}$ and using the fact that $(R{\prec}){<} = R{<}$. The symmetric property of the right domain operators follows by making the instantiation $P := R{\succ}$ and using the fact that $(R{\succ}){<} = R{>}$. □

4. Indexes and Core Relations

This section introduces the notions of "index" and "core" of a relation. An "index" is a special case of a "core" of a relation but, in general, it is more useful. The properties of both notions are explored in depth.

4.1. Indexes

Recall Fig. 1. We said that the middle and rightmost figures depict "core relations". The property that is common to both is captured by the following definition.

Definition 48. (Core Relation)
A relation R is a *core relation* iff $R{<} = R{\prec}$ and $R{>} = R{\succ}$. □

The rightmost figure of Fig. 1 is what we call an "index" of the relation depicted by the leftmost figure. The definition of an "index" of a relation is as follows.

Definition 49. (Index)
An *index* of a relation R is a relation J that has the following properties:

(a) $J \subseteq R$,

(b) $R{\prec} \circ J \circ R{\succ} \ = \ R$,

(c) $J{<} \circ R{\prec} \circ J{<} \ = \ J{<}$,

(d) $J{>} \circ R{\succ} \circ J{>} \ = \ J{>}$. □

Note particularly requirement 49(a). A consequence of this requirement is that an index of a relation has the same type as the relation. This means that the relation depicted by the middle figure of Fig. 1 is *not* an index of the relation depicted by the leftmost figure because the relations have different types.

An obvious property is that a core relation is an index of itself:

Theorem 50. Suppose R is a core relation. Then R is an index of R.

Proof:
Straightforward application of Definitions 48 and 49 together with the properties of (coreflexive and per) domains. □

In general, the existence of an index of an arbitrary relation is *not* derivable in systems that axiomatise point-free relation algebra. In Section 5.2 we add a limited form of the axiom of choice that guarantees the existence of indexes of arbitrary pers; we also show that this then guarantees the existence of indexes for arbitrary relations. For the moment, we establish a number of properties of indexes assuming they exist. For example, we show that all indexes of a given relation are isomorphic: see Theorem 60.

Lemma 51. If J is an index of the relation R then

$$J{\prec} \subseteq R{\prec} \quad \wedge \quad J{\succ} \subseteq R{\succ} \ .$$

It follows that

$$J{<} = J{\prec} \quad \wedge \quad J{>} = J{\succ} \ .$$

That is, an index is a core relation.

Proof:
We first prove that $J{\prec} \subseteq R{\prec}$.

$$\begin{array}{cl}
 & R{\prec} \\
= & \quad \{ \quad \text{definition} \quad \} \\
 & R/\!\!/R \circ R{<} \\
\supseteq & \quad \{ \quad \text{49(a) and monotonicity} \quad \} \\
 & R/\!\!/R \circ J{<} \\
\supseteq & \quad \{ \quad \text{see below} \quad \} \\
 & J{\prec} \ .
\end{array}$$

The last step in the above calculation proceeds as follows.

$$\begin{array}{cl}
 & J{\prec} \ \subseteq \ R/\!\!/R \circ J{<} \\
\Leftarrow & \quad \{ \quad (J{\prec}){>} = J{<} \text{ (so } J{\prec} = J{\prec} \circ J{<}\text{) and } J{<} \circ J{<} = J{<} \\
 & \qquad\quad \text{monotonicity} \quad \} \\
 & J{\prec} \ \subseteq \ R/\!\!/R
\end{array}$$

$$
\begin{array}{ll}
= & \{ \quad \text{definition of } R/\!\!/R \quad \} \\
& J_{\prec} \ \subseteq\ R/R \cap (R/R)^{\cup} \\
= & \{ \quad J_{\prec} = (J_{\prec})^{\cup} \quad \} \\
& J_{\prec} \ \subseteq\ R/R \\
= & \{ \quad \text{shunting} \quad \} \\
& J_{\prec} \circ R \ \subseteq\ R \ .
\end{array}
$$

We continue with the lefthand side of the above inclusion.

$$
\begin{array}{ll}
& J_{\prec} \circ R \\
= & \{ \quad 49\text{(b)} \quad \} \\
& J_{\prec} \circ R_{\prec} \circ J \circ R_{\succ} \\
= & \{ \quad (J_{\prec})_{>} = J_{<} \text{ and domains: (14)} \quad \} \\
& J_{\prec} \circ J_{<} \circ R_{\prec} \circ J_{<} \circ J \circ R_{\succ} \\
= & \{ \quad 49\text{(c)} \quad \} \\
& J_{\prec} \circ J_{<} \circ J \circ R_{\succ} \\
= & \{ \quad \text{(corefexive and per) domains: (14) and (29)} \quad \} \\
& J \circ R_{\succ} \\
\subseteq & \{ \quad 49\text{(a)} \quad \} \\
& R \circ R_{\succ} \\
= & \{ \quad \text{per domains: (29)} \quad \} \\
& R \ .
\end{array}
$$

We conclude that $J_{\prec} \subseteq R_{\prec}$. The equation $J_{\prec} = J_{<}$ uses anti-symmetry.

$$
\begin{array}{ll}
& J_{\prec} \\
\supseteq & \{ \quad \text{per domains: (25), and reflexivity of } J/\!\!/J \quad \} \\
& J_{<} \\
= & \{ \quad 49\text{(c)} \quad \} \\
& J_{<} \circ R_{\prec} \circ J_{<} \\
\supseteq & \{ \quad J_{\prec} \subseteq R_{\prec} \text{ (see above), composition of coreflexives is idempotent} \quad \} \\
& J_{<} \ .
\end{array}
$$

The other two properties are symmetrical. □

An immediate corollary of Lemma 51 is the following theorem.

Theorem 52. If J is an index (of some relation) then J is an index of J.

Proof:
Suppose J is an index of R. Then we have to prove the properties 49(a), (b), (c) and (d) with $R := J$. These are the properties:

(e) $J \subseteq J$,

(f) $J_{\prec} \circ J \circ J_{\succ} \ = \ J$,

(g) $J{<}\circ J{\prec}\circ J{<} \;=\; J{<}$,

(h) $J{>}\circ J{\succ}\circ J{>} \;=\; J{>}$.

Properties (e) and (f) are true of all relations J. Properties (g) and (h) follow from Lemma 51 and the fact that composition of coreflexives is idempotent. □

The indexes of a relation are uniquely defined by their left and right domains. See Corollary 54, which is an immediate consequence of the following lemma.

Lemma 53. Suppose J is an index of the relation R. Then

$$J \;=\; J{<}\circ R\circ J{>} \;\; .$$

Proof:

$$\begin{array}{ll}
 & J \\
= & \{ \quad \text{domains: (14)} \quad \} \\
 & J{<}\circ J\circ J{>} \\
= & \{ \quad \text{49(c) and (d)} \quad \} \\
 & J{<}\circ R{\prec}\circ J{<}\circ J\circ J{>}\circ R{\succ}\circ J{>} \\
= & \{ \quad \text{domains: (14)} \quad \} \\
 & J{<}\circ R{\prec}\circ J\circ R{\succ}\circ J{>} \\
= & \{ \quad \text{49(b)} \quad \} \\
 & J{<}\circ R\circ J{>} \;\; .
\end{array}$$

□

Corollary 54. Suppose J and K are both indexes of the relation R. Then

$$J = K \quad\equiv\quad J{<} = K{<} \;\wedge\; J{>} = K{>} \;\; .$$

Proof:
Implication is an immediate consequence of Leibniz's rule. For the "if" part, we assume that $J{<} = K{<}$ and $J{>} = K{>}$. Then

$$\begin{array}{ll}
 & J \\
= & \{ \quad J \text{ is an index of } R\text{, Lemma 53} \quad \} \\
 & J{<}\circ R\circ J{>} \\
= & \{ \quad \text{assumption: } J{<} = K{<} \;\wedge\; J{>} = K{>} \quad \} \\
 & K{<}\circ R\circ K{>} \\
= & \{ \quad K \text{ is an index of } R\text{, Lemma 53 with } J := K \quad \} \\
 & K \;\; .
\end{array}$$

□

The following lemma becomes relevant when we study indexes of difunctions. (See Section 5.1.)

Lemma 55. Suppose J is an index of R. Then

$$R \circ J^{\cup} \circ R \;=\; R \circ R^{\cup} \circ R \;\;.$$

Proof:

$$\begin{array}{ll} & R \circ J^{\cup} \circ R \\ = & \quad \{ \quad \text{per domains: (24) and (25)} \quad \} \\ & R \circ R{\succ} \circ J^{\cup} \circ R{\prec} \circ R \\ = & \quad \{ \quad \text{49(b) and converse} \quad \} \\ & R \circ R^{\cup} \circ R \;\;. \end{array}$$

□

We now formulate a couple of lemmas that lead to Lemma 58 which, in turn, leads to Theorem 59.

Lemma 56. Suppose J is an index of R. Then $R{\prec} \circ J{<} \circ R{\prec}$ and $R{\succ} \circ J{>} \circ R{\succ}$ are pers.

Proof:
We prove that

$$R{\prec} \circ J{<} \circ R{\prec} \;=\; R{\prec} \circ J{<} \circ R{\prec} \circ (R{\prec} \circ J{<} \circ R{\prec})^{\cup} \;\;.$$

We have:

$$\begin{array}{ll} & R{\prec} \circ J{<} \circ R{\prec} \circ (R{\prec} \circ J{<} \circ R{\prec})^{\cup} \\ = & \quad \{ \quad R{\prec} \text{ is a per, } J{<} \text{ is a coreflexive, converse} \quad \} \\ & R{\prec} \circ J{<} \circ R{\prec} \circ J{<} \circ R{\prec} \\ = & \quad \{ \quad \text{49(c)} \quad \} \\ & R{\prec} \circ J{<} \circ R{\prec} \;\;. \end{array}$$

□

Lemma 57. Suppose J is an index of R. Then

$$(R{\prec} \circ J{<} \circ R{\prec}){<} \;=\; R{<} \;\;.$$

Symmetrically,

$$(R{\succ} \circ J{>} \circ R{\succ}){>} \;=\; R{>} \;\;.$$

Proof:

$$\begin{array}{ll} & (R{\prec} \circ J{<} \circ R{\prec}){<} \\ = & \quad \{ \quad \text{domains: Theorem 19(c), } (R{\prec}){<} \;=\; R{<} \quad \} \\ & (R{\prec} \circ J{<} \circ R{<}){<} \\ = & \quad \{ \quad \text{by 49(a), } J{<} \subseteq R{<}\text{, domains} \quad \} \\ & (R{\prec} \circ J){<} \\ = & \quad \{ \quad \text{by 49(a), } J{>} \subseteq R{>}\text{, domains} \quad \} \end{array}$$

$$
\begin{array}{ll}
 & (R{\prec} \circ J \circ R{>}){<} \\
= & \quad \{ \quad \text{domains: Theorem 19(c), } (R{\succ}){<} \;=\; R{>} \quad \} \\
 & (R{\prec} \circ J \circ R{\succ}){<} \\
= & \quad \{ \quad \text{49(b)} \quad \} \\
 & R{<} \;\; .
\end{array}
$$

□

Lemma 58. Suppose J is an index of R. Then

(a) $R{\prec} \circ J{<} \circ R{\prec} \;=\; R{\prec}$,

(b) $R{\succ} \circ J{>} \circ R{\succ} \;=\; R{\succ}$.

Proof:

$$
\begin{array}{ll}
 & R{\prec} \\
= & \quad \{ \quad R{\prec} \text{ is a per} \quad \} \\
 & R{\prec} \circ R{\prec} \circ R{\prec} \\
\supseteq & \quad \{ \quad R{\prec} \supseteq R{<} \quad \} \\
 & R{\prec} \circ R{<} \circ R{\prec} \\
\supseteq & \quad \{ \quad J \text{ is an index of } R\text{; Definition 49(a) and monotonicity} \quad \} \\
 & R{\prec} \circ J{<} \circ R{\prec} \\
= & \quad \{ \quad R{\prec} \text{ is a per} \quad \} \\
 & R{\prec} \circ J{<} \circ R{\prec} \circ R{\prec} \\
\supseteq & \quad \{ \quad \text{Lemma 56: } R{\prec} \circ J{<} \circ R{\prec} \text{ is a per} \quad \} \\
 & (R{\prec} \circ J{<} \circ R{\prec}){<} \,\circ\, R{\prec} \\
= & \quad \{ \quad \text{Lemma 57} \quad \} \\
 & R{<} \,\circ\, R{\prec} \\
= & \quad \{ \quad (R{\prec}){<} \;=\; R{<} \quad \} \\
 & R{\prec} \;\; .
\end{array}
$$

By anti-symmetry of the subset relation we have proved (a). Property (b) is symmetrical. □

Theorem 59. Suppose J is an index of R. Then $J{<}$ is an index of $R{\prec}$ and $J{>}$ is an index of $R{\succ}$.

Proof:
We prove that $J{<}$ is an index of $R{\prec}$. That $J{>}$ is an index of $R{\succ}$ is symmetrical.

Instantiating Definition 49 with $R,J \;:=\; R{\prec}\,,\,J{<}$, our task is to prove the four properties:

(a) $J{<} \subseteq R{\prec}$,

(b) $(R{\prec}){\prec} \circ (J{<}){<} \circ (R{\succ}){\prec} \;=\; R{\prec}$,

(c) $(J{<}){<} \circ (R{\prec}){\prec} \circ (J{<}){<} \;=\; (J{<}){<}$,

(d) $(J{<}){>} \circ (R{\prec}){\succ} \circ (J{<}){>} \;=\; (J{<}){>}$.

The proof of property (a) is straightforward:

$$
\begin{array}{ll}
 & J{<} \subseteq R{\prec} \\
\Leftarrow & \quad \{ \quad R{<} \subseteq R{\prec}\text{, transitivity} \quad \} \\
 & J{<} \subseteq R{<} \\
\Leftarrow & \quad \{ \quad \text{monotonicity} \quad \} \\
 & J \subseteq R \\
= & \quad \{ \quad J \text{ is an index of } R\text{, 49(a)} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

Property (b) simplifies using the fact that $(R{\prec}){\prec} = R{\prec}$, $(R{\succ}){\prec} = R{\succ}$ and $J{<} = (J{<}){<}$ to:

(b') $R{\prec} \circ J{<} \circ R{\succ} \ = \ R{\prec}$,

This is the first of the two properties proved in Lemma 58. Using the fact that $(R{\prec}){\prec} = R{\prec}$ and $J{<} = (J{<}){<}$, property (c) is the same as property (c) of Definition 49; similarly, using the fact that $R{\prec} = (R{\prec}){\succ}$, and $J{<} = (J{<}){>}$, property (d) is also the same as property (c) of Definition 49. □

We show later that the converse of Theorem 59 is a prescription for constructing an index of an arbitrary relation. See Theorem 76.

Theorem 60. If R and S are isomorphic relations then indexes of R and S are also isomorphic. In particular, indexes of a relation R are isomorphic.

Proof:
Suppose ϕ and ψ witness the isomorphism $R \cong S$ and J is an index of R and K is an index of S. We verify that λ and ρ defined by

$$\lambda = J{<} \circ R{\prec} \circ \phi \circ S{\prec} \circ K{<} \quad \wedge \quad \rho = J{>} \circ R{\succ} \circ \psi \circ S{\succ} \circ K{>}$$

witness the isomorphism $J \cong K$.

The task is to verify that

$$J{<} = \lambda \circ \lambda^{\cup} \quad \wedge \quad \lambda^{\cup} \circ \lambda = K{<} \quad \wedge \quad \rho \circ \rho^{\cup} = J{>} \quad \wedge \quad \rho^{\cup} \circ \rho = K{>}$$

and

$$J \ = \ \lambda \circ K \circ \rho^{\cup} \ .$$

The four domain properties are all essentially the same so we only verify the first conjunct:

$$
\begin{array}{ll}
 & \lambda \circ \lambda^{\cup} \\
= & \quad \{ \quad \text{definition, converse} \quad \} \\
 & J{<} \circ R{\prec} \circ \phi \circ S{\prec} \circ K{<} \circ S{\prec} \circ \phi^{\cup} \circ R{\prec} \circ J{<} \\
= & \quad \{ \quad K \text{ is an index of } S\text{, Lemma 56 with } J{,}R := K{,}S \quad \} \\
 & J{<} \circ R{\prec} \circ \phi \circ S{\prec} \circ \phi^{\cup} \circ R{\prec} \circ J{<} \\
= & \quad \{ \quad \text{Theorem 60} \quad \} \\
 & J{<} \circ R{\prec} \circ R{\prec} \circ R{\prec} \circ J{<} \\
= & \quad \{ \quad R{\prec} \text{ is a per, } J \text{ is an index of } R\text{, Definition 49(c)} \quad \} \\
 & J{<} \ .
\end{array}
$$

Finally,

$$
\begin{array}{ll}
 & \lambda \circ K \circ \rho^{\cup} \\
= & \quad \{ \quad \text{definition, converse} \quad \} \\
 & J{<} \circ R{\prec} \circ \phi \circ S{\prec} \circ K{<} \circ K \circ K{>} \circ S{\succ} \circ \psi^{\cup} \circ R{\succ} \circ J{>} \\
= & \quad \{ \quad \text{domains: (14)} \quad \} \\
 & J{<} \circ R{\prec} \circ \phi \circ S{\prec} \circ K \circ S{\succ} \circ \psi^{\cup} \circ R{\succ} \circ J{>} \\
= & \quad \{ \quad K \text{ is an index of } S\text{, Definition 49(b)} \quad \} \\
 & J{<} \circ R{\prec} \circ \phi \circ S \circ \psi^{\cup} \circ R{\succ} \circ J{>} \\
= & \quad \{ \quad R = \phi \circ S \circ \psi^{\cup} \quad \} \\
 & J{<} \circ R{\prec} \circ R \circ R{\succ} \circ J{>} \\
= & \quad \{ \quad \text{per domains: (29)} \quad \} \\
 & J{<} \circ R \circ J{>} \\
= & \quad \{ \quad J \text{ is an index of } R\text{, Definition 49(b)} \quad \} \\
 & J \ \ .
\end{array}
$$

That the indexes of a relation R are isomorphic follows because R is isomorphic to itself (with witnesses $R{<}$ and $R{>}$), i.e. the isomorphism relation is reflexive. □

The construction of the witnesses λ and ρ looks very much like the proverbial rabbit out of a hat! In fact, they were calculated using the type judgements formulated in Voermans' thesis [Voe99]. We hope at a later date to exploit Voermans' calculus in order to make the process of constructing witnesses much more methodical.

4.2. Core Relations

Indexes are a special case of what we call "core" relations. (Recall Definition 48.) This section is about the properties of a "core" of a given relation R, first introduced in [BO23].

Definition 61. (Core)
Suppose R is an arbitrary relation and suppose C is a relation such that

$$C \;=\; \lambda \circ R \circ \rho^{\cup}$$

for some relations λ and ρ satisfying

$$R{\prec} \;=\; \lambda^{\cup} \circ \lambda \quad \wedge \quad \lambda{<} = \lambda \circ \lambda^{\cup} \quad \wedge \quad R{\succ} \;=\; \rho^{\cup} \circ \rho \quad \wedge \quad \rho{<} = \rho \circ \rho^{\cup} \ \ .$$

Then C is said to be a *core of* R *as witnessed by* λ *and* ρ. □

(The terminology just introduced anticipates Theorem 65 which establishes that a core of a relation is indeed a core relation according to Definition 48.)

The existence of a core of a given relation R has a constructive element: it is necessary to construct the "witnesses" λ and ρ. In general, given a per P, a functional relation f with the property that P equals $f^{\cup} \circ f$ is called a "splitting" of P. Constructing a core of relation R thus involves "splitting " the pers $R{\prec}$ and $R{\succ}$ into functional relations λ and ρ. As with indexes, the existence of cores is not derivable in point-free relation algebra. However, just as for indexes, all cores of a given relation are isomorphic in the sense of Definition 38. See Section 6 for further discussion of the construction of cores of pers.

Immediately obvious is that an index of a relation is a core of the relation:

Theorem 62. Suppose R is an arbitrary relation and suppose J is an index of R. Then J is a core of R as witnessed by $J{<} \circ R{\prec}$ and $J{>} \circ R{\succ}$.

Proof: First,

$$
\begin{array}{ll}
 & J \\
= & \quad \{ \quad \text{Lemma 53} \quad \} \\
 & J{<} \circ R \circ J{>} \\
= & \quad \{ \quad \text{per domains: (29)} \quad \} \\
 & J{<} \circ R{\prec} \circ R \circ R{\succ} \circ J{>} \\
= & \quad \{ \quad \text{converse, domains are coreflexive} \quad \} \\
 & (J{<} \circ R{\prec}) \circ R \circ (J{>} \circ R{\succ})^{\cup} \quad .
\end{array}
$$

This establishes the required property of C in Definition 61, with $C := J$. (The parentheses in the last line of the calculation indicate the definitions of the splittings λ and ρ.) Second,

$$
\begin{array}{ll}
 & (J{<} \circ R{\prec})^{\cup} \circ J{<} \circ R{\prec} \\
= & \quad \{ \quad \text{converse, } (R{\prec})^{\cup} = R{\prec} \text{ and } (J{<})^{\cup} \circ J{<} \;=\; J{<} \quad \} \\
 & R{\prec} \circ J{<} \circ R{\succ} \\
= & \quad \{ \quad \text{Lemma 58} \quad \} \\
 & R{\prec} \quad .
\end{array}
$$

Third,

$$
\begin{array}{ll}
 & J{<} \circ R{\prec} \circ (J{<} \circ R{\prec})^{\cup} \\
= & \quad \{ \quad \text{converse, } (J{<})^{\cup} = J{<} \text{ and } R{\prec} \circ (R{\prec})^{\cup} \;=\; R{\prec} \quad \} \\
 & J{<} \circ R{\prec} \circ J{<} \\
= & \quad \{ \quad J \text{ is an index of } R\text{, Definition 49(c)} \quad \} \\
 & J{<} \\
= & \quad \{ \quad \text{Theorem 59; in particular, } J{<} \subseteq R{<} \quad \} \\
 & (J{<} \circ R{<}){<} \\
= & \quad \{ \quad (R{\prec}){<} = R{<}\text{, domains: Theorem 19(c)} \quad \} \\
 & (J{<} \circ R{\prec}){<} \quad .
\end{array}
$$

This establishes the required properties of λ in Definition 61 (with $\lambda := J{<} \circ R{\prec}$). The properties of ρ in Definition 61 (with $\rho := J{>} \circ R{\succ}$) are established similarly. $\square$

Fig. 2 illustrates Theorem 62 applied to the relation introduced in Fig. 1. The index J is depicted by the green edges in the lower bipartite graph. The decomposition of the relation in the definition of

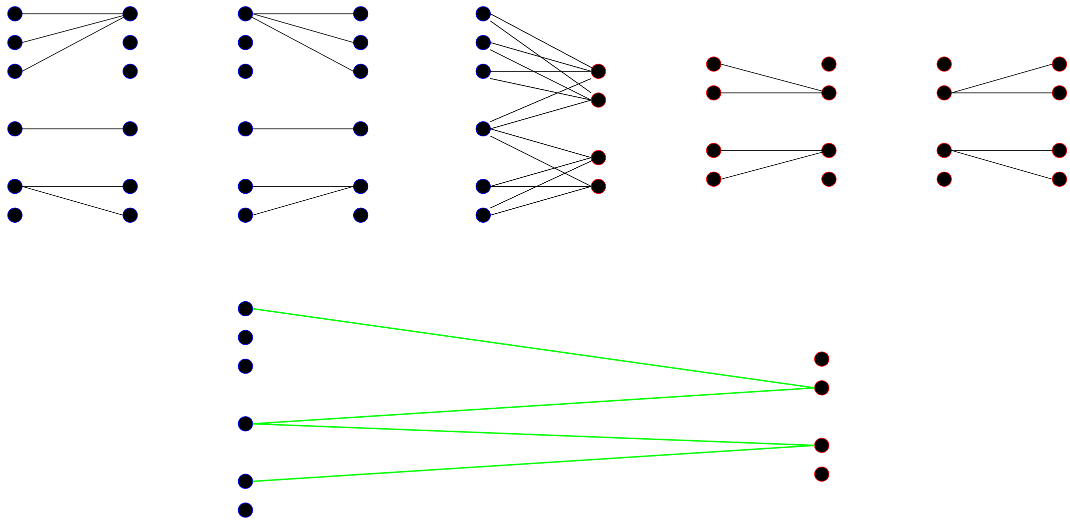

Figure 2. Decomposition of a Relation into a Core and Witnesses

a core is illustrated by the row of bipartite graphs at the top; the relations depicted are, in order, $\lambda^{\cup}$, λ, R, $\rho^{\cup}$ and ρ. The composition of the middle three figures is the index J.

A number of properties of indexes are derived from the fact that indexes are cores. The remainder of this section catalogues such properties.

The name "core" in Definition 61 anticipates Theorem 65 where we show that the relation C is a core relation as defined by Definition 48. Some preliminary lemmas are needed first.

For later use, we calculate the left and right domains of the core of a relation.

Lemma 63. Suppose R, λ, ρ and C are as in Definition 61. Then

$$R{<} = \lambda{>} \;\;\wedge\;\; C{<} = \lambda{<} \;\;\wedge\;\; R{>} = \rho{>} \;\;\wedge\;\; C{>} = \rho{<} \;\; .$$

Proof: We prove the middle two equations. First,

$$\begin{array}{ll}
 & R{>} \\
= & \qquad \{ \quad (31) \quad \} \\
 & (R{\succ}){<} \\
= & \qquad \{ \quad \text{Definition 61} \quad \} \\
 & (\rho^{\cup} \circ \rho){<} \\
= & \qquad \{ \quad \text{domains} \quad \} \\
 & \rho{>} \;\; .
\end{array}$$

The dual equation, $R{<} = \lambda{>}$, is proved similarly. Second,

$$\begin{array}{ll}
 & C{<} \\
= & \qquad \{ \quad \text{Definition 61} \quad \}
\end{array}$$

$$
\begin{array}{ll}
 & (\lambda \circ R \circ \rho^{\cup})< \\
= & \quad \{ \quad R> = \rho> \text{ (just proved) } \} \\
 & (\lambda \circ R \circ R>)< \\
= & \quad \{ \quad \text{domains: (14) } \} \\
 & (\lambda \circ R<)< \\
= & \quad \{ \quad R< = \lambda> \text{ (see above) } \} \\
 & \lambda< \ .
\end{array}
$$

The final equation is also proved similarly. □

Lemma 64. Suppose R, λ, ρ and C are as in Definition 61. Suppose also that J is an index of R. Then $C \cong J$ as witnessed by $\lambda \circ J<$ and $\rho \circ J>$.

Proof: We construct the witnesses as follows.

$$
\begin{array}{ll}
 & C \\
= & \quad \{ \quad \text{Definition 61 } \} \\
 & \lambda \circ R \circ \rho^{\cup} \\
= & \quad \{ \quad J \text{ is an index of } R\text{, Definition 49(b) } \} \\
 & \lambda \circ R\prec \circ J \circ R\succ \circ \rho^{\cup} \\
= & \quad \{ \quad \text{Definition 61 } \} \\
 & \lambda \circ \lambda^{\cup} \circ \lambda \circ J \circ \rho^{\cup} \circ \rho \circ \rho^{\cup} \\
= & \quad \{ \quad \lambda \text{ and } \rho \text{ are functional,} \\
 & \quad\quad\ \text{so } \lambda< = \lambda \circ \lambda^{\cup} \text{ and } \rho< = \rho \circ \rho^{\cup} \ \} \\
 & \lambda \circ J \circ \rho^{\cup} \\
= & \quad \{ \quad \text{domains: (14) and converse } \} \\
 & \lambda \circ J< \circ J \circ (\rho \circ J>)^{\cup} \ .
\end{array}
$$

Comparing the last line with the definition of an isomorphism of relations (Definition 38 with the instantiation $R,S,\phi,\psi := C\ ,\ J\ ,\ \lambda \circ J<\ ,\ \rho \circ J>$), we postulate $\lambda \circ J<$ and $\rho \circ J>$ as witnesses to the isomorphism.

It remains to show that $\lambda \circ J<$ and $\rho \circ J>$ are bijections on the appropriate domains. First,

$$
\begin{array}{ll}
 & (\rho \circ J>)^{\cup} \circ \rho \circ J> \\
= & \quad \{ \quad \text{converse } \} \\
 & J> \circ \rho^{\cup} \circ \rho \circ J> \\
= & \quad \{ \quad \text{definition 61 } \} \\
 & J> \circ R\succ \circ J> \\
= & \quad \{ \quad J \text{ is an index of } R\text{, Definition 49(d) } \} \\
 & J> \ .
\end{array}
$$

Symmetrically,

$$(\lambda \circ J<)^{\cup} \circ \lambda \circ J< \ = \ J< \ .$$

Finally,

$(\rho \circ J{>}){<}$

$=$ { ρ is functional, and $\rho^{\cup} \circ \rho = R{\succ}$,
i.e. $\rho = \rho \circ \rho^{\cup} \circ \rho = \rho \circ R{\succ}$ }

$(\rho \circ R{\succ} \circ J{>}){<}$

$=$ { $J{>} \subseteq R{>}$ and $R{>} = (R{\succ}){>}$ }

$(\rho \circ R{\succ} \circ J{>} \circ (R{\succ}){>}){<}$

$=$ { domains: Theorem 19(b) and (c), $R{\succ} = (R{\succ})^{\cup}$ }

$(\rho \circ R{\succ} \circ J{>} \circ R{\succ}){<}$

$=$ { domains: Theorem 19(c) }

$(\rho \circ (R{\succ} \circ J{>} \circ R{\succ}){<}){<}$

$=$ { Lemmas 56 and 57(b) }

$(\rho \circ R{>}){<}$

$=$ { (31) and domains: Theorem 19(c) }

$(\rho \circ R{\succ}){<}$

$=$ { $\rho = \rho \circ R{\succ}$ (see first step) }

$\rho{<}$

$=$ { Lemma 63 }

$C{>}$.

Symmetrically, $(\lambda \circ J{<}){<} = C{<}$.

Putting all the calculations together, we conclude that $\lambda \circ J{<}$ and $\rho \circ J{>}$ are bijections; the left domain of $\lambda \circ J{<}$ is $C{<}$ and its right domain is $J{<}$; the left domain of $\rho \circ J{>}$ is $C{>}$ and its right domain is $J{>}$. □

We now prove the theorem alluded to by the nomenclature of Definition 61, namely any core of a given relation R is a core relation in the sense of Definition 48.

Theorem 65. Suppose C is a core of R. Then, if R has an index,

$$C{\succ} \;=\; C{>} \ \text{, and} \tag{66}$$

$$C{\prec} \;=\; C{<} \ . \tag{67}$$

That is, if R has an index, any core C of R is a core relation. (See definition 48.)

Proof:
Assume that J is an index of R. The proof is a combination of several preceding lemmas and theorems.

$C{\prec} \;=\; C{<}$

$\Leftarrow$ { Theorem 47 }

$C{\prec} \;\cong\; C{<}$

$\Leftarrow$ { Leibniz }

$J{\prec} = J{<} \;\wedge\; C{\prec} \cong J{\prec} \;\wedge\; J{<} \cong C{<}$

$\Leftarrow$ { index J is a core relation (Lemma 51) }

$$
\begin{array}{ll}
 & C{\prec} \cong J{\prec} \;\wedge\; J{<} \cong C{<} \\
\Leftarrow & \{ \quad \text{Lemmas 43 and 42} \quad \} \\
 & C \cong J \\
= & \{ \quad \text{Lemma 64} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

□

Note. Theorem 65 assumes that relation R has an index J. Likewise, a corollary of Lemma 64 is that, assuming relation R has an index, all cores of R are isomorphic. It is straightforward to prove that all cores of R are isomorphic without the assumption that R has an index. Similarly, Theorem 65 can be proved without this assumption but the proof is quite long and complex. See [Bac21] for full details.

We argue later that this assumption has no practical significance: in Section 5.3 we show that every relation R has an index if both its per domains have an index. This means that, for a given R, it is necessary to calculate indices of $R{\prec}$ and $R{\succ}$; however, in practice, this is not an issue. **End of Note**

5. Indexes of Difunctions and Pers

5.1. Indexes of Difunctions

We now specialise the notion of index to difunctions.

Lemma 68. Suppose J is an index of relation R and J is difunctional. Then R is difunctional.

Proof:

$$
\begin{array}{ll}
 & R \circ R^{\cup} \circ R \\
= & \{ \quad J \text{ is an index of } R\text{, Lemma 55} \quad \} \\
 & R \circ J^{\cup} \circ R \\
= & \{ \quad J \text{ is an index of } R\text{, 49(b)} \quad \} \\
 & R{\prec} \circ J \circ R{\succ} \circ J^{\cup} \circ R{\prec} \circ J \circ R{\succ} \\
= & \{ \quad \text{domains: (14) and Theorem 19(b)} \quad \} \\
 & R{\prec} \circ J \circ J{>} \circ R{\succ} \circ J{>} \circ J^{\cup} \circ J{<} \circ R{\prec} \circ J{<} \circ J \circ R{\succ} \\
= & \{ \quad J \text{ is an index of } R\text{, 49(d) and (c)} \quad \} \\
 & R{\prec} \circ J \circ J{>} \circ J^{\cup} \circ J{<} \circ J \circ R{\succ} \\
= & \{ \quad \text{domains: (14) and Theorem 19(b), and } J \text{ is difunctional (i.e. } J = J \circ J^{\cup} \circ J) \quad \} \\
 & R{\prec} \circ J \circ R{\succ} \\
= & \{ \quad \text{49(b)} \quad \} \\
 & R \ .
\end{array}
$$

□

The property that R is a difunction is equivalent to $R{\prec} = R \circ R^{\cup}$ (and symmetrically to $R{\succ} = R^{\cup} \circ R$). Also, since $R = R \circ R^{\cup} \circ R$, the right side of Lemma 55 simplifies to R. In this way, the definition of an index of a difunction can be restated as follows.

Definition 69. (Difunction Index)
An index of a difunction R is a relation J that has the following properties:

(a) $J \subseteq R$,

(b) $R \circ J^{\cup} \circ R \;=\; R$.

(c) $J{<} \circ R \circ R^{\cup} \circ J{<} \;=\; J{<}$,

(d) $J{>} \circ R^{\cup} \circ R \circ J{>} \;=\; J{>}$. □

Lemma 70. An index J of a difunction R is a bijection between $J{<}$ and $J{>}$.

Proof:

$$
\begin{array}{ll}
 & J{<} \\
= & \quad \{ \quad 69(c) \quad \} \\
 & J{<} \circ R^{\cup} \circ R \circ J{<} \\
\supseteq & \quad \{ \quad 69(a) \quad \} \\
 & J{<} \circ J^{\cup} \circ J \circ J{<} \\
= & \quad \{ \quad \text{domains: (14) and Theorem 19(b)} \quad \} \\
 & J^{\cup} \circ J \\
\supseteq & \quad \{ \quad \text{domains: Definition 11} \quad \} \\
 & J{<} \ .
\end{array}
$$

Thus, by anti-symmetry,

$$J{<} \;=\; J^{\cup} \circ J \ .$$

Symmetrically, $J{>} \;=\; J \circ J^{\cup}$. That is, J is a bijection. □

Corollary 71 formulates a method to determine whether a relation is a difunction: compute an index of the relation and then determine whether it is a difunction. By 49(a), the second step in this process will be no less efficient than determining difunctionality directly and, in many cases, may be substantially more efficient. (There is, however, no guarantee of improved efficiency since the inequality in 49(a) may be an equality.)

Corollary 71. Suppose J is an index of relation R. Then R is a difunction iff J is a difunction.

Proof: Lemma 68 establishes "if". Lemma 70 establishes "only if" (since a bijection is a difunction). □

5.2. Indexes of Pers

That every difunction has an index is a desirable property but it is not provable in standard axiomatic formulations of relation algebra. Rather than postulate its truth, we shall postulate that all pers have an index, and then show that a consequence of the postulate is that all difunctions have an index.

A relation R is a per iff $R = R{\prec} = R{\succ}$. Using this property, the definition of index can be simplified for pers. Specifically, an index J of per R has the following properties. (Cf. definition 49.)

(a) $J \subseteq R$,

(b) $R{\circ}J{\circ}R = R$,

(c) $J{<} \circ R \circ J{<} \;=\; J{<}$,

(d) $J{>} \circ R \circ J{>} \;=\; J{>}$.

Lemmas 72 and 73 prepare the way for Definition 74.

Lemma 72. If a per has an index, then it has an index that is a coreflexive.

Proof:
Suppose R is a per and J is an index of R. The lemma is proved if we show that $J{<}$ is an index of R. We thus have to prove that

(e) $J{<} \subseteq R$,

(f) $R \circ J{<} \circ R \;=\; R$,

(g) $(J{<}){<} \circ R \circ (J{<}){<} \;=\; (J{<}){<}$,

(h) $(J{>}){>} \circ R \circ (J{>}){>} \;=\; (J{>}){>}$,

assuming the properties (a), (b), (c) and (d) above.

Of the four properties, only (f) is non-trivial. (Properties (g) and (h) follow because $J{<} = (J{<}){<}$ and $J{>} = (J{>}){>}$. Property (e) follows because, since R is a per, $R{<} \subseteq R$.)

Property (f) is proved as follows.

$$
\begin{array}{ll}
 & R \circ J{<} \circ R \\
= & \quad \{ \quad \text{by Lemma 70, } J \circ J^{\cup} = J{<} \quad \} \\
 & R \circ J \circ J^{\cup} \circ R \\
= & \quad \{ \quad \text{domains: (14)} \quad \} \\
 & R \circ J \circ J{>} \circ J^{\cup} \circ R \\
= & \quad \{ \quad \text{(d)} \quad \} \\
 & R \circ J \circ J{>} \circ R \circ J{>} \circ J^{\cup} \circ R \\
= & \quad \{ \quad \text{domains: (14)} \quad \} \\
 & R \circ J \circ R \circ J^{\cup} \circ R \\
= & \quad \{ \quad \text{(b)} \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & R \circ J^{\cup} \circ R \\
= & \quad \{ \quad R \text{ is a per, so } R = R^{\cup}\text{; converse} \quad \} \\
 & (R{\circ}J{\circ}R)^{\cup} \\
= & \quad \{ \quad R \text{ is a per, so } R = R^{\cup}\text{; (b) and converse} \quad \} \\
 & R \ \ .
\end{array}
$$

□

Lemma 73. For all pers R, if R has an index then there is a relation J such that

(a) $J \subseteq R{<}$,

(b) $J{\circ}R{\circ}J = J$,

(c) $R{\circ}J{\circ}R = R$.

Conversely, for all pers R, if relation J satisfies the properties (a), (b) and (c) above, then J is an index of R.

Proof:
First, suppose R is a per that has an index. By Lemma 72, R has a coreflexive index. Let J be such a coreflexive index of R. We must show that properties (a), (b) and (c) hold. We have

$$
\begin{array}{ll}
 & J \subseteq R{<} \\
\Leftarrow & \quad \{ \quad \text{49(a) and monotonicity} \quad \} \\
 & J = J{<} \\
= & \quad \{ \quad J \text{ is a coreflexive} \quad \} \\
 & \mathsf{true} \ \ .
\end{array}
$$

This proves (a). Now for (b):

$$
\begin{array}{ll}
 & J{\circ}R{\circ}J \\
= & \quad \{ \quad J \text{ is a coreflexive, so } J = J{<}, \\
 & \qquad\quad R \text{ is a per, so } R = R{\prec} \quad \} \\
 & J{<} \circ R{\prec} \circ J{<} \\
= & \quad \{ \quad \text{49(c)} \quad \} \\
 & J{<} \\
= & \quad \{ \quad J \text{ is a coreflexive, so } J = J{<} \quad \} \\
 & J \ \ .
\end{array}
$$

Finally, (c):

$$
\begin{array}{ll}
 & R{\circ}J{\circ}R \\
= & \quad \{ \quad R \text{ is a per, so } R = R{\prec} \quad \} \\
 & R{\prec} \circ J \circ R{\prec} \\
= & \quad \{ \quad \text{49(b)} \quad \} \\
 & R \ \ .
\end{array}
$$

For the converse, assume R is a per and J satisifies the properties (a), (b) and (c) above. We have to check the four properties listed in Definition 49. First, 49(a):

$$
\begin{array}{ll}
 & J \\
\subseteq & \{ \quad \text{assumption: (a) above} \quad \} \\
 & R{\scriptstyle <} \\
\subseteq & \{ \quad R \text{ is per} \quad \} \\
 & R \ .
\end{array}
$$

The properties 49(b), (c) and (d) follow because $J = J{\scriptstyle <} = J{\scriptstyle >}$ and $R = R{\scriptstyle \prec} = R{\scriptstyle \succ}$. □

As a consequence of Lemma 73, we postulate the following definition of an index of a per.

Definition 74. (Index of a Per)
Suppose P is a per. Then a (*coreflexive*) *index* of P is a relation J such that

(a) $J \subseteq P{\scriptstyle <}$,

(b) $J \circ P \circ J = J$,

(c) $P \circ J \circ P = P$. □

We also postulate that every per has a coreflexive index. We call this the *axiom of choice*.

Axiom 75. (Axiom of Choice)
Every per has a coreflexive index. □

5.3. From Pers To Relations

It is a desirable property that every relation has an index. However, as mentioned earlier, this can't be proved in standard relation algebra. It can be proved if we assume that every per has an index. The construction is suggested by Theorem 59.

Theorem 76. Suppose J and K are (coreflexive) indices of $R{\scriptstyle \prec}$ and $R{\scriptstyle \succ}$, respectively. Then $J \circ R \circ K$ is an index of R.

Proof:
For convenience, we list the properties of J and K. These are obtained by instantiating Definition 74 with $J{,}R := J\,,R{\scriptstyle \prec}$ and $J{,}R := K\,,R{\scriptstyle \succ}$. (Domain properties have been used to simplify (a) and (d).)

(a) $J \subseteq R{\scriptstyle <}$,

(b) $J \circ R{\scriptstyle \prec} \circ J = J$,

(c) $R{\scriptstyle \prec} \circ J \circ R{\scriptstyle \prec} = R{\scriptstyle \prec}$,

(d) $K \subseteq R{\scriptstyle >}$,

(e) $K \circ R_{\succ} \circ K \;=\; K$,

(f) $R_{\succ} \circ K \circ R_{\succ} \;=\; R_{\succ}$.

We have to prove the four properties 49(a)-(d) with the instantiation $J{,}R := J{\circ}R{\circ}K\,{,}R$. By (a), $J = J^{\cup} = J_{<} = J_{>}$. Similarly for K. The proof obligations are thus:

(g) $J{\circ}R{\circ}K \subseteq R$,

(h) $R_{\prec} \circ J \circ R \circ K \circ R_{\succ} \;=\; R$.

(i) $(J{\circ}R{\circ}K)_{<} \circ R_{\prec} \circ (J{\circ}R{\circ}K)_{<} \;=\; (J{\circ}R{\circ}K)_{<}$,

(j) $(J{\circ}R{\circ}K)_{>} \circ R_{\succ} \circ (J{\circ}R{\circ}K)_{>} \;=\; (J{\circ}R{\circ}K)_{>}$,

Property (g) is an easy combination of (a) and (d). For (h) we have:

$$
\begin{array}{cl}
 & R_{\prec} \circ J \circ R \circ K \circ R_{\succ} \\
= & \quad \{ \quad \text{per domains: (29)} \quad \} \\
 & R_{\prec} \circ J \circ R_{\prec} \circ R \circ R_{\succ} \circ K \circ R_{\succ} \\
= & \quad \{ \quad \text{(b) and (f)} \quad \} \\
 & R_{\prec} \circ R \circ R_{\succ} \\
= & \quad \{ \quad \text{per domains: (29)} \quad \} \\
 & R \;\; .
\end{array}
$$

For (i), we have

$$
\begin{array}{cl}
 & (J{\circ}R{\circ}K)_{>} \circ R_{\succ} \circ (J{\circ}R{\circ}K)_{>} \\
= & \quad \{ \quad (J{\circ}R{\circ}K)_{>} \subseteq K_{>} = K, \\
 & \quad\quad\;\; \text{composition of coreflexives is intersection} \quad \} \\
 & (J{\circ}R{\circ}K)_{>} \circ K \circ R_{\succ} \circ K \circ (J{\circ}R{\circ}K)_{>} \\
= & \quad \{ \quad \text{(e)} \quad \} \\
 & (J{\circ}R{\circ}K)_{>} \circ K \circ (J{\circ}R{\circ}K)_{>} \\
= & \quad \{ \quad (J{\circ}R{\circ}K)_{>} \subseteq K_{>} = K \\
 & \quad\quad\;\; \text{composition of coreflexives is intersection} \quad \} \\
 & (J{\circ}R{\circ}K)_{>} \;\; .
\end{array}
$$

The proof is (j) is symmetrical. □

Theorem 76 shows how to construct an index of a relation R from indexes J and K of its left and right per domains. In combination with Lemma 53 and Corollary 54, the construction is unique. Specifically, the steps are, first to choose from each equivalence class of $R_{\prec}$ and each equivalence class of $R_{\succ}$ a single representative. The collection of such representatives defines the coreflexives J and K. Then the index is defined to be $J{\circ}R{\circ}K$.

6. Characterisations of Pers and Difunctions

This section is about characterising pers and difunctions in terms of functional relations. Although the characterisations are well known, they are not derivable in point-free relation algebra. We show that they are derivable using our axiom of choice.

6.1. Characterisation of Pers

A well-known property is that a relation R is a per iff

$$\langle \exists f \,:\, f \circ f^{\cup} \;=\; f{<} \,:\, R = f^{\cup} \circ f \rangle \;. \tag{77}$$

This property is said to be a *characteristic* property of pers. Perhaps surprisingly, it is *not* derivable in systems that axiomatise point-free relation algebra. Freyd and Ščedrov [Fv90, 1.281] call the functional f witnessing the existential quantification a "splitting[2]" of R. Typically, the existence of "splittings" is either postulated as an axiom (eg. Winter [Win04]) or by adding axioms formulating relations as a so-called "power allegory" [Fv90, 2.422], or by adding the so-called "all-or-nothing" axiom [Glü17]. (See Section 7.6 for discussion of "all or nothing".) See [BO23] for a comparison of the techniques used to establish (77) using these different axiom systems. Here we show that the existence of "splittings" is a consequence of our axiom of choice:

Theorem 78. If per P has a coreflexive index J , then

$$P \;=\; (J{\circ}P)^{\cup} \circ (J{\circ}P) \quad \wedge \quad J \;=\; (J{\circ}P) \circ (J{\circ}P)^{\cup} \;.$$

Thus, assuming the axiom of choice, for all relations R,

$$\mathsf{per}.R \;\equiv\; \langle \exists f \,:\, f \circ f^{\cup} \;=\; f{<} \,:\, R = f^{\cup} \circ f \rangle \;.$$

Proof: The proof is very straightforward. We have

$$\begin{array}{ll}
& (J{\circ}P)^{\cup} \circ (J{\circ}P) \\
= & \quad \{ \quad \text{distributivity} \quad \} \\
& P^{\cup} \circ J \circ J \circ P \\
= & \quad \{ \quad J \text{ is coreflexive, so } J{\circ}J = J;\ P = P^{\cup} \quad \} \\
& P{\circ}J{\circ}P \\
= & \quad \{ \quad J \text{ is an index of } P\text{, Definition 74(c)} \quad \} \\
& P
\end{array}$$

and

[2] Freyd and Ščedrov define a "splitting" in the more general context of a category rather than an allegory; the notion is applicable to "idempotents" which are also more general than pers.

$$
\begin{array}{ll}
 & (J{\circ}P) \circ (J{\circ}P)^{\cup} \\
= & \quad \{ \quad \text{distributivity} \quad \} \\
 & J \circ P \circ P^{\cup} \circ J \\
= & \quad \{ \quad P \text{ is a per, so by Lemma 32(ii), } P = P^{\cup} \circ P \quad \} \\
 & J{\circ}P{\circ}J \\
= & \quad \{ \quad J \text{ is an index of } P \text{, Definition 74(b)} \quad \} \\
 & J \ .
\end{array}
$$

This proves the first property. It also establishes that (assuming the axiom of choice), for all R,

$$\mathsf{per}.R \;\Rightarrow\; \langle \exists f \,:\, f \circ f^{\cup} \;=\; f{<} \,:\, R = f^{\cup} \circ f \rangle \ .$$

(The witness is $J{\circ}R$.) The converse is obvious: see [BO23] for details. The equivalence follows by mutual implication. □

A second so-called "characteristic" property is that a relation R is a difunctional iff

$$\langle \exists\, f{,}g \;\;:\;\; f \circ f^{\cup} \;=\; f{<} \;=\; g \circ g^{\cup} \;=\; g{<} \;\;:\;\; R = f^{\cup} \circ g \rangle \ .$$

Like the characteristic property of pers, it is not derivable in systems that axiomatise point-free relation algebra. It is, however, a corollary of the existence of "splittings" (and thus of Theorem 78), as shown by Winter [Win04].

6.2. Unicity of Characterisations

The characterisation of a per in the form $f^{\cup} \circ f$ where f is a functional relation is not unique. (There are typically many representatives one can choose for each equivalence class; so there are very many distinct indexes of a per.) The characterisation is sometimes described as being "essentially" unique or sometimes as unique "up to isomorphism". See our working document [BV22] for full details.

7. Enabling Pointwise Reasoning

In this section, our goal is to capture the notion that a relation is a set with elements pairs of points.

In traditional pointwise reasoning about relations, a basic assumption is that a type is a set that forms a complete, universally distributive lattice under the subset ordering; the type of a (binary) relation is a set of pairs. The set of relations of a given type thus forms a powerset of a set of pairs.

In Section 7.1, we recall a general theorem on the structure of powersets. Briefly, Theorem 81 states that a set is isomorphic to the powerset of its "atoms" iff it is "saturated". The section defines these concepts; the concepts then form the backbone of later sections where we specialise the theorem to relations.

One (of several) mechanisms for introducing pointwise reasoning within the framework of point-free relation algebra involves the introduction of the so-called "all-or-nothing rule" which was postulated as an axiom by Glück [Glü17]. This rule is combined with completeness and "extensionality" axioms which state that, for each type A, the coreflexives of type A form a complete, saturated lattice.

This was the approach taken in [BDGv22] where pointwise reasoning was used to formulate and prove properties of graphs. Theorem 102 establishes that the all-or-nothing rule is a consequence of our axiom of choice (Axiom 75: every per has an index). Together with the "extensionality" axiom, this enables the application of Theorem 81 to establish that the type $A{\sim}B$ is isomorphic to the powerset $2^{A\times B}$ (the set of subsets of the cartesian product $A{\times}B$). See Theorems 102 and 103 in Section 7.6.

Section 7.2 introduces "points" and states the extensionality axiom that we assume. A number of sections are then necessary in order to establish Theorem 103. Section 7.3 introduces "particles" and "pairs"; it is then shown that particles are points whilst section 7.4 shows that —assuming the axiom of choice— points are particles. (For this reason, the terminology "particle" is temporary.) Section 7.5 shows that proper atoms (of a given type) are "pairs". These are the ingredients for deriving the "all-or-nothing" rule in Section 7.6. Section 7.6 also shows that the point-free definition of a "pair" in Section 7.3 does correspond to what one normally understands to be a pair of points. The section concludes with Theorem 103.

7.1. Powersets

As mentioned above, this section defines "atoms" and "saturated" in the context of a partially ordered set. We then state a fundamental theorem relating these concepts to powersets.

The definition of an atom is the following.

Definition 79. (Atom and Atomicity)
Suppose $\mathcal{A}$ is a set partially ordered by the relation $\sqsubseteq$. Then, the element p is an *atom* iff

$$\langle \forall q \;::\; q\sqsubseteq p \;\equiv\; q{=}p \vee q{=}\perp\!\!\!\perp\rangle \ .$$

Note that $\perp\!\!\!\perp$ is an atom according to this definition. If p is an atom that is different from $\perp\!\!\!\perp$ we say that it is a *proper* atom. A lattice is said to be *atomic* if

$$\langle \forall q \;::\; q{\neq}\perp\!\!\!\perp \;\equiv\; \langle \exists a : \mathsf{atom}.a \wedge a{\neq}\perp\!\!\!\perp : a\sqsubseteq q\rangle\rangle \ .$$

In words, a lattice is atomic if every proper element includes a proper atom. □

The definition of saturated is as follows.

Definition 80. (Saturated)
A complete lattice (ordered by $\sqsubseteq$) is *saturated* iff

$$\langle \forall p \;::\; p \;=\; \langle \sqcup a : \mathsf{atom}.a \wedge a\sqsubseteq p : a\rangle\rangle \ .$$

□

The set of subsets of a type is a powerset iff the lattice is saturated, as formulated in the following theorem.

Theorem 81. Suppose $\mathcal{A}$ is a complete, universally distributive lattice. Then the following statements are equivalent.

(a) $\mathcal{A}$ is saturated,

(b) $\mathcal{A}$ is atomic and complemented,

(c) $\mathcal{A}$ is isomorphic to the powerset of its atoms. □

(See [ABH+92, Theorem 6.43] for the proof of Theorem 81.)

We use Theorem 81 in two ways. Firstly, for all types A, we simply postulate that the set of coreflexives of type A is isomorphic to a powerset under the $\subseteq$ ordering: the atoms are the "points" introduced in Section 7.2. Second, we use this postulate together with our axiom of choice to show that, for all types A and B, the type $A{\sim}B$ of (heterogeneous) relations is also isomorphic to a powerset under the $\subseteq$ ordering: the atoms are "pairs" introduced in Section 7.3. The proof that "pairs" are indeed atoms is the subject of Section 7.5. A prelude to this is Theorem 94, proved in Sections 7.3 and 7.4, which asserts that "points" are a special case of "pairs".

7.2. Points

We begin by postulating that each type A is a set of "points". We also postulate that the set of coreflexives of type A forms a complete, universally distributive lattice under the subset ordering. Finally, we postulate that the lattice is saturated. With Theorem 81 in mind, we define "points" to be the proper atoms of the lattice:

Definition 82. (Point)
A homogeneous relation a of type A is a *point* iff it has the following three properties.

(a) $a \neq \perp\!\!\!\perp$,

(b) $a \subseteq \mathbb{I}$, and

(c) $\langle \forall b : b \neq \perp\!\!\!\perp \wedge b \subseteq a : b = a \rangle$.

In words, a point is a proper, coreflexive atom. □

If A is a type, we use a, a' etc. to denote points of type A. Similarly for points of type B. Points represent elements of the appropriate type.

For points a and a' of the same type,

$$a = a' \;\vee\; a{\circ}a' = \perp\!\!\!\perp \;. \tag{83}$$

The proof is straightforward. Suppose a and a' are points. Then

$$
\begin{array}{ll}
 & a = a{\circ}a' \\
\Leftarrow & \quad \{ \quad a \text{ is an atom, Definition 79} \quad \} \\
 & a{\circ}a' \neq \perp\!\!\!\perp \;\wedge\; a{\circ}a' \subseteq a \\
\Leftarrow & \quad \{ \quad a' \subseteq \mathbb{I} \quad \} \\
 & a{\circ}a' \neq \perp\!\!\!\perp \;.
\end{array}
$$

Interchanging a and a',

$$a' = a \circ a' \Leftarrow a' \circ a \neq \bot\!\!\!\bot \ .$$

But, since composition of coreflexives is symmetric, $a \circ a' = a' \circ a$. We conclude that

$$a = a \circ a' = a' \quad \Leftarrow \quad a \circ a' \neq \bot\!\!\!\bot \ .$$

This is equivalent to (83).

In point-free relation algebra, subsets of a type are modelled by coreflexives of that type. In order to model the property that the coreflexives of a given type form a lattice that is isomorphic to the set of subsets of the type we need to add to our axiom system a *saturation* property, viz.:

Definition 84. (Saturation)
Suppose A is a type. The lattice of coreflexives of type A is said to be *saturated* iff

$$\langle \forall p \ :: \ p \subseteq \mathbb{I}_A \ \equiv \ p = \langle \cup a \ : \ \mathsf{point}.a \wedge a \subseteq p \ : \ a \rangle \rangle \ . \tag{85}$$

□

The axiom that we call "extensionality" is then:

Axiom 86. (Extensionality)
For each type A, the points of type A form a complete, universally distributive, saturated lattice under the subset ordering. □

Applying Theorem 81, a consequence of Axiom 86 is that the coreflexives of type A form a lattice that is isomorphic to the powerset 2^A. In this sense, the coreflexives in point-free relation algebra represent sets of points in traditional pointwise formulations of relation algebra.

We now want to show how to formulate the property that the set of relations of type $A{\sim}B$ is isomorphic to the powerset $2^{A \times B}$, i.e. relations in point-free relation algebra represent pairs (a, b) of points a and b of type A and B, respectively.

7.3. Pairs and Particles

We now turn our attention to the lattice of relations of a given type. We begin with a point-free definition of a "pair". In Subsection 7.6, we show that Definition 87 does indeed capture the notion of a "pair of points" whereby the points are the "particles" also introduced in the definition.

Definition 87. (Pair)
A relation Z is a *pair* iff it has the following properties:

(a) $Z \neq \bot\!\!\!\bot$,

(b) $Z = Z \circ \top\!\!\!\top \circ Z$,

(c) $Z< \ = \ Z \circ Z^{\cup}$,

(d) $Z> \ = \ Z^{\cup} \circ Z$.

We call a relation a *particle* if it is a pair and it is symmetric. □

In words, a pair Z is a non-empty rectangle (properties 87(a) and 87(b)) that is a bijection on its left domain and right domains (properties 87(c) and 87(d)).

(Definition 87 was introduced in [Voe99] but using the terminology "singleton" instead of "pair", and "singleton square" instead of "particle".)

Our goal is to prove that the points are exactly the particles. This section is about showing that a particle is a point. See Corollary 92.

One task is to show that particles are atoms. The more general property, which we need in later sections, is that pairs are atoms.

Lemma 88. A pair is an atom.

Proof:
Suppose Z is a pair and suppose Y is such that $Y \subseteq Z$. By the definition of atom, definition 79, we must show that $Y = \perp\!\!\!\perp \vee Y = Z$. Equivalently, assuming $Y \neq \perp\!\!\!\perp$, we must show that $Y = Z$. This is done as follows.

$$\begin{array}{ll}
 & Y \\
= & \{ \quad \text{assumption: } Y \subseteq Z. \text{ So, } Y{<} \subseteq Z{<} \text{ and } Y{>} \subseteq Z{>}\text{; domains: (18)} \quad \} \\
 & Z{<} \circ Y \circ Z{>} \\
= & \{ \quad Z \text{ is a pair, so } Z{<} = Z \circ Z^{\cup} = (Z \circ \top\!\!\!\top \circ Z) \circ Z^{\cup}\text{; similarly for } Z{>} \quad \} \\
 & Z \circ \top\!\!\!\top \circ Z \circ Z^{\cup} \circ Y \circ Z^{\cup} \circ Z \circ \top\!\!\!\top \circ Z \\
= & \{ \quad \text{domains: Theorem 19(a) and Theorem 19(b)} \quad \} \\
 & Z \circ \top\!\!\!\top \circ Z{<} \circ Y \circ Z{>} \circ \top\!\!\!\top \circ Z \\
= & \{ \quad Z{<} \circ Y \circ Z{>} = Y \text{ (see first step above)} \quad \} \\
 & Z \circ \top\!\!\!\top \circ Y \circ \top\!\!\!\top \circ Z \\
= & \{ \quad \text{assumption: } Y \neq \perp\!\!\!\perp\text{, cone rule} \quad \} \\
 & Z \circ \top\!\!\!\top \circ Z \\
= & \{ \quad Z \text{ is a pair} \quad \} \\
 & Z \;\; .
\end{array}$$

□

Since a particle is, by definition, a pair, we have:

Corollary 89. A particle is an atom. □

Lemma 90. A particle is coreflexive.

Proof: Suppose Z is square and a pair. Then

$$\begin{array}{ll}
 & Z \\
= & \{ \quad \text{assumption: } Z \text{ is a pair, so } Z = Z \circ \top\!\!\!\top \circ Z; \\
 & \quad\;\; [\; \top\!\!\!\top \circ Z \;=\; \top\!\!\!\top \circ Z{<} \circ Z \;=\; \top\!\!\!\top \circ Z^{\cup} \circ Z \;] \quad \}
\end{array}$$

$$
\begin{array}{ll}
 & Z \circ \mathbb{T} \circ Z^{\cup} \circ Z \\
= & \{ \quad \text{assumption: } Z \text{ is a square, so } Z = Z \circ \mathbb{T} \circ Z^{\cup} = Z^{\cup} \quad \} \\
 & Z^{\cup} \circ Z \\
= & \{ \quad \text{assumption: } Z \text{ is a pair, so } Z> = Z^{\cup} \circ Z \quad \} \\
 & Z> \ .
\end{array}
$$

That is, Z equals $Z>$ which is coreflexive. □

Corollary 91. (Particle)
A relation Z is a *particle* iff it has the following three properties.

(a) $Z \neq \perp\!\!\!\perp$,

(b) $Z \subseteq \mathbb{I}$, and

(c) $Z = Z \circ \mathbb{T} \circ Z$.

In words, a particle is a proper, coreflexive rectangle.

Proof: "Only-if" is the combination of the definition of a particle and Lemma 90. "If" is a straightforward consequence of the properties of domains and coreflexives. □

Corollary 92. A particle is a proper, coreflexive atom. That is, a particle is a point.

Proof: This is a combination of Lemmas 88 and 90. □

7.4. Points are Particles

We now prove the converse of Corollary 92. We use the assumption that every per has a coreflexive index: the axiom of choice (Axiom 75).

Lemma 93. Assuming Axiom 75, a point is a particle.

Proof:
Suppose that a is a point. Comparing the definition of a point, Definition 82, with the defining properties of a particle, Corollary 92, it suffices to prove that $a = a \circ \mathbb{T} \circ a$. Clearly $a \circ \mathbb{T} \circ a$ is a per. (The simple proof uses the fact that $a = a^{\cup}$, because a is coreflexive, and $\mathbb{T} \circ a \circ \mathbb{T} = \mathbb{T}$ because $a \neq \perp\!\!\!\perp$.) So, by the axiom of choice, $a \circ \mathbb{T} \circ a$ has an index J, say. We show that J is a particle and $J = a$.

To show that J is a particle, we must establish the three properties listed in Corollary 91 with the instantiation $Z := J$. Part (a) is proved as follows.

$$
\begin{array}{ll}
 & J = \perp\!\!\!\perp \\
\Rightarrow & \{ \quad \perp\!\!\!\perp \text{ is zero of composition} \quad \} \\
 & a \circ \mathbb{T} \circ a \circ J \circ a \circ \mathbb{T} \circ a = \perp\!\!\!\perp \\
= & \{ \quad J \text{ is an index of per } a \circ \mathbb{T} \circ a \text{, Definition 74(c)} \quad \} \\
 & a \circ \mathbb{T} \circ a = \perp\!\!\!\perp
\end{array}
$$

$$
\begin{array}{ll}
\Rightarrow & \{ \quad a\circ a\circ a \subseteq a\circ\mathbb{T}\circ a \text{ and } a\circ a\circ a = a \text{ (because } a\subseteq\mathbb{I}) \quad \} \\
& a\subseteq\perp\!\!\!\perp \\
= & \{ \quad [\, R\subseteq\perp\!\!\!\perp \;\equiv\; R=\perp\!\!\!\perp \,] \text{ with } R:=a \quad \} \\
& a=\perp\!\!\!\perp \\
= & \{ \quad \text{assumption: } a \text{ is proper, i.e. } a\neq\perp\!\!\!\perp \quad \} \\
& \mathsf{false} \;.
\end{array}
$$

We conclude that $J\neq\perp\!\!\!\perp$. The next step is to show that $J=a$.

$$
\begin{array}{ll}
& J=a \\
\Leftarrow & \{ \quad \text{assumption: } a \text{ is an atom} \quad \} \\
& J=\perp\!\!\!\perp \;\vee\; J\subseteq a \\
= & \{ \quad J\neq\perp\!\!\!\perp \text{ (see above)} \quad \} \\
& J\subseteq a \\
= & \{ \quad \text{assumption: } a\subseteq\mathbb{I}, \text{ so } a=(a\circ\mathbb{T}\circ a)< \quad \} \\
& J \subseteq (a\circ\mathbb{T}\circ a)< \\
= & \{ \quad \text{assumption: } J \text{ is an index of } a\circ\mathbb{T}\circ a \\
& \quad\;\; \text{Definition 74(a)} \quad \} \\
& \mathsf{true} \;.
\end{array}
$$

Property (b) of Corollary 91 immediately follows because a is coreflexive. We now show that $J=J\circ\mathbb{T}\circ J$.

$$
\begin{array}{ll}
& J\circ\mathbb{T}\circ J \\
= & \{ \quad J=a \text{ (proved above) and } a\subseteq\mathbb{I} \quad \} \\
& J\circ a\circ\mathbb{T}\circ a\circ J \\
= & \{ \quad \text{assumption: } J \text{ is an index of } a\circ\mathbb{T}\circ a \\
& \quad\;\; \text{Definition 74(c) with } P:=a\circ\mathbb{T}\circ a \quad \} \\
& J \;.
\end{array}
$$

We conclude that $J=a=J\circ\mathbb{T}\circ J$. Thus $a=a\circ\mathbb{T}\circ a$ as required. □

Combining Corollary 92 with Lemma 93, we conclude:

Theorem 94. Assuming Axiom 75, a relation is a point iff it is a particle. □

7.5. Proper Atoms are Pairs

The goal of this section is to show that a proper atom is a pair. Aiming to exploit the equivalence of points and particles, we begin with lemmas on the left and right domains of a proper atom.

Lemma 95. Suppose R is a proper atom. Then $R<$ and $R>$ are proper atoms[3].

[3] Note: strictly we should detail the lattice under consideration here. However, it is easy to show that a coreflexive being an atom in the lattice of coreflexives is equivalent to its being an atom in the lattice of relations. This justifies the omission.

Proof:
First, that $R{<}$ and $R{>}$ are both proper is immediate from (15).

To show that $R{<}$ is an atom we have to show that, for all p,

$$p \subseteq R{<} \;\wedge\; p \neq \perp\!\!\!\perp \;\;\equiv\;\; p = R{<} \;\;.$$

We do this by mutual implication. First, the follows-from:

$$
\begin{array}{ll}
 & p \subseteq R{<} \;\wedge\; p \neq \perp\!\!\!\perp \;\;\Leftarrow\;\; p = R{<} \\
= & \quad \{ \quad \text{predicate calculus} \quad \} \\
 & (p \subseteq R{<} \;\Leftarrow\; p = R{<}) \;\wedge\; (p \neq \perp\!\!\!\perp \;\Leftarrow\; p = R{<}) \\
\Leftarrow & \quad \{ \quad \text{left conjunct: anti-symmetry, right conjunct: Leibniz} \quad \} \\
 & \mathsf{true} \;\wedge\; R{<} \neq \perp\!\!\!\perp \\
\Leftarrow & \quad \{ \quad R{<} \text{ is proper (see above)} \quad \} \\
 & \mathsf{true} \;\;.
\end{array}
$$

Now we prove the converse. Assume $p \subseteq R{<}$ and $p \neq \perp\!\!\!\perp$. Then

$$
\begin{array}{ll}
 & p = R{<} \\
= & \quad \{ \quad \text{anti-symmetry and assumption: } p \subseteq R{<} \quad \} \\
 & R{<} \subseteq p \\
\Leftarrow & \quad \{ \quad \text{assumption: } p \subseteq R{<} \text{ and } R{<} \subseteq \mathbb{I}\text{, so } p = p{<};\ (p{\circ}R){<} \subseteq p{<} \quad \} \\
 & R{<} \;=\; (p{\circ}R){<} \\
\Leftarrow & \quad \{ \quad \text{Leibniz} \quad \} \\
 & R = p{\circ}R \\
= & \quad \{ \quad p{\circ}R \neq \perp\!\!\!\perp \text{ (see below for proof)} \\
 & \qquad\quad R \text{ is an atom, Definition 79 (appropriately instantiated)} \quad \} \\
 & p{\circ}R \subseteq R \\
= & \quad \{ \quad \text{assumption: } p \subseteq R{<} \text{ and } R{<} \subseteq \mathbb{I}\text{, monotonicity} \quad \} \\
 & \mathsf{true} \;\;.
\end{array}
$$

In order to verify the penultimate step in the above calculation, we show that $p{\circ}R = \perp\!\!\!\perp \Rightarrow \mathsf{false}$ under the assumption that $p \subseteq R{<}$ and $p \neq \perp\!\!\!\perp$.

$$
\begin{array}{ll}
 & p{\circ}R = \perp\!\!\!\perp \\
= & \quad \{ \quad \text{cone rule: (4)} \quad \} \\
 & \top\!\!\!\top{\circ}p{\circ}R{\circ}\top\!\!\!\top \;=\; \perp\!\!\!\perp \\
= & \quad \{ \quad \text{domains: Theorem 19(a)} \quad \} \\
 & \top\!\!\!\top \circ p \circ R{<} \circ \top\!\!\!\top \;=\; \perp\!\!\!\perp \\
\Rightarrow & \quad \{ \quad \text{assumption: } p \subseteq R{<}\text{, composition of coreflexives is intersection} \quad \} \\
 & \top\!\!\!\top{\circ}p{\circ}\top\!\!\!\top \;=\; \perp\!\!\!\perp \\
= & \quad \{ \quad \text{assumption: } p \neq \perp\!\!\!\perp\text{, cone rule: (4)} \quad \} \\
 & \mathsf{false} \;\;.
\end{array}
$$

$\Box$

Corollary 96. If R is a proper atom, $R{<}$ and $R{>}$ are particles.

Proof: By Lemma 95 and Definition 82 of a point, if R is a proper atom, $R{<}$ and $R{>}$ are points. Thus, by Lemma 93, $R{<}$ and $R{>}$ are particles. □

We now aim to verify properties 87(b), (c) and (d) of a pair, with Z instantiated to proper atom R. Property 87(b) is the following lemma.

Lemma 97. A proper atom is a rectangle.

Proof: Suppose R is a proper atom. Then

$$
\begin{array}{ll}
 & R\circ\mathbb{T}\circ R \\
= & \quad\{\qquad \text{domains: Theorem 19(a)} \quad\} \\
 & R{<}\circ\mathbb{T}\circ R{>} \\
= & \quad\{\qquad R\neq\perp\!\!\!\perp,\ \text{cone rule: (4)} \quad\} \\
 & R{<}\circ\mathbb{T}\circ R\circ\mathbb{T}\circ R{>} \\
= & \quad\{\qquad \text{domains: (14)} \quad\} \\
 & R{<}\circ\mathbb{T}\circ R{<}\circ R\circ R{>}\circ\mathbb{T}\circ R{>} \\
= & \quad\{\qquad \text{by Corollary 96, } R{<} \text{ and } R{>} \text{ are particles;} \\
 & \qquad\qquad \text{Corollary 91(c) with } Z:=R{<} \text{ and } Z:=R{>} \quad\} \\
 & R{<}\circ R\circ R{>} \\
= & \quad\{\qquad \text{domains: (14)} \quad\} \\
 & R\ \ .
\end{array}
$$

That is, $R\circ\mathbb{T}\circ R = R$. Thus, by definition, R is a rectangle. □

We now have all the ingredients for our goal.

Lemma 98. Suppose R is a proper atom. Then, assuming Axiom 75, R is a pair.

Proof:
Suppose R is a proper atom. We have to verify properties 87(b), (c) and (d) (with $Z:=R$) of a pair.

Property 87(b) is Lemma 97. Properties 87(c) and (d) assert that R is a bijection. To prove this, let J be an index of R. (This is where Axiom 75 is assumed.) Then

$$
\begin{array}{ll}
 & J=R \\
= & \quad\{\qquad R \text{ is an atom} \quad\} \\
 & J\neq\perp\!\!\!\perp\ \wedge\ J\subseteq R \\
= & \quad\{\qquad J \text{ is an index of } R\text{, Definition 49} \quad\} \\
 & \mathtt{true}\ \ .
\end{array}
$$

That is, $J=R$. But R is a rectangle and thus a difunction. So, applying Lemma 70, J —and thus R— is a bijection, as required. □

To conclude this section and Sections 7.3 and 7.4, we have:

Theorem 99. Assuming Axiom 75, for all types A and B, and all relations R of type $A{\sim}B$, R is a proper atom iff R is a pair.

Proof: This is a combination of Lemmas 88 and 98. □

7.6. Pairs of Points and the All-or-Nothing Rule

The final step is to show that we can derive the "all-or-nothing" rule. Throughout this section we assume Axiom 75.

Lemma 100. If Z is a pair then $Z{<}$ and $Z{>}$ are particles.

Proof:
Suppose Z is a pair. We begin by showing that its left and right domains are also pairs.

Properties 87(a), (c) and (d) —with $Z := Z{<}$ and $Z := Z{>}$— are properties of the domain operators. This leaves 87(b). For the instance $Z := Z{<}$, we have:

$$
\begin{array}{ll}
 & Z{<} \circ \mathbb{T} \circ Z{<} \\
= & \quad \{ \quad \text{domains: Theorem 19(a) and (b)} \quad \} \\
 & Z \circ \mathbb{T} \circ Z \circ Z^{\cup} \\
= & \quad \{ \quad \text{assumption: } Z \text{ is a pair, so } Z \circ \mathbb{T} \circ Z = Z \quad \} \\
 & Z \circ Z^{\cup} \\
= & \quad \{ \quad \text{assumption: } Z \text{ is a pair, so } Z \circ Z^{\cup} = Z{<} \quad \} \\
 & Z{<} \ .
\end{array}
$$

The proof that $Z{>}$ is a pair is symmetrical.

It now follows immediately that $Z{<}$ and $Z{>}$ are squares: a square is a symmetric rectangle, and both are rectangles (see above); also, both are coreflexives, and coreflexives are symmetric. □

The following theorem is [Voe99, Lemma 4.41(d)].

Theorem 101. For all Z,

$$\mathsf{pair}.Z \ \equiv \ \langle \exists\, a,b : \mathsf{point}.a \wedge \mathsf{point}.b : Z = a \circ \mathbb{T} \circ b \rangle \ .$$

Proof: By mutual implication. First,

$$
\begin{array}{ll}
 & \mathsf{pair}.Z \\
\Rightarrow & \quad \{ \quad \text{Lemma 100;} \\
 & \quad\quad \text{Definition 87(b) and } [\ Z \circ \mathbb{T} \circ Z \ = \ Z{<} \circ \mathbb{T} \circ Z{>} \ ] \quad \} \\
 & \mathsf{particle}\,.\,Z{<} \ \wedge \ \mathsf{particle}\,.\,Z{>} \ \wedge \ Z = Z{<} \circ \mathbb{T} \circ Z{>} \\
\Rightarrow & \quad \{ \quad \text{Corollary 92} \quad \} \\
 & \mathsf{point}\,.\,Z{<} \ \wedge \ \mathsf{point}\,.\,Z{>} \ \wedge \ Z = Z{<} \circ \mathbb{T} \circ Z{>} \\
\Rightarrow & \quad \{ \quad a,b := Z{<}\,,Z{>} \quad \} \\
 & \langle \exists\, a,b : \mathsf{point}.a \wedge \mathsf{point}.b : Z = a \circ \mathbb{T} \circ b \rangle \ .
\end{array}
$$

For the converse, assume that a and b are points. We have to prove that $a{\circ}\top\!\!\top{\circ}b$ is a pair. Applying Definition 87, this means checking four properties:

(a) $a{\circ}\top\!\!\top{\circ}b \neq \perp\!\!\perp$,

(b) $a{\circ}\top\!\!\top{\circ}b = a{\circ}\top\!\!\top{\circ}b{\circ}\top\!\!\top{\circ}a{\circ}\top\!\!\top{\circ}b$,

(c) $(a{\circ}\top\!\!\top{\circ}b)\!< \;=\; (a{\circ}\top\!\!\top{\circ}b)\circ(a{\circ}\top\!\!\top{\circ}b)^{\cup}$,

(d) $(a{\circ}\top\!\!\top{\circ}b)\!> \;=\; (a{\circ}\top\!\!\top{\circ}b)^{\cup}\circ(a{\circ}\top\!\!\top{\circ}b)$.

Properties (a) and (b) are instances of the cone rule together with the assumption that a and b are proper. We prove (c) as follows.

$$
\begin{array}{ll}
 & (a{\circ}\top\!\!\top{\circ}b)\circ(a{\circ}\top\!\!\top{\circ}b)^{\cup} \\
= & \quad\{ \quad \text{converse} \quad \} \\
 & a\circ\top\!\!\top\circ b\circ b^{\cup}\circ\top\!\!\top\circ a \\
= & \quad\{ \quad \text{assumption: } b \text{ is a point, cone rule: (4)} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}a \\
= & \quad\{ \quad \text{assumption: } a \text{ is a point; so, by Corollary 93, } a \text{ is a pair;} \\
 & \qquad\quad \text{Definition 87(b) with } Z:=a \quad \} \\
 & a \\
= & \quad\{ \quad a{\circ}\top\!\!\top{\circ}b \text{ is a non-empty rectangle} \quad \} \\
 & (a{\circ}\top\!\!\top{\circ}b)\!< \;\; .
\end{array}
$$

Property (d) is proved symmetrically. □

We conclude with the theorem that Glück's "all-or-nothing" axiom [Glü17] is a consequence of our axiom of choice.

Theorem 102. (All or Nothing)

$$\langle\forall\, a,b,R \;:\; \mathsf{point}.a \wedge \mathsf{point}.b \;:\; a{\circ}R{\circ}b = \perp\!\!\perp \,\vee\, a{\circ}R{\circ}b = a{\circ}\top\!\!\top{\circ}b\rangle \;\; .$$

Proof:
Suppose a and b are points. By Theorem 101, $a{\circ}\top\!\!\top{\circ}b$ is a pair. So, by Lemma 88, $a{\circ}\top\!\!\top{\circ}b$ is an atom. Applying the definition of atomic, we have, for all R,

$$
\begin{array}{ll}
 & \mathsf{true} \\
= & \quad\{ \quad \text{monotonicity, } R \subseteq \top\!\!\top \quad \} \\
 & a{\circ}R{\circ}b \subseteq a{\circ}\top\!\!\top{\circ}b \\
= & \quad\{ \quad a{\circ}\top\!\!\top{\circ}b \text{ is an atom, Definition 79} \quad \} \\
 & a{\circ}R{\circ}b = \perp\!\!\perp \,\vee\, a{\circ}R{\circ}b = a{\circ}\top\!\!\top{\circ}b \;\; .
\end{array}
$$

□

The significance of the all-or-nothing rule is that, together with Theorem 81, it follows that the lattice of relations of type $A{\sim}B$ is isomorphic to the powerset $2^{A\times B}$.

Theorem 103. Suppose, for types A and B, the lattices of coreflexives of types A and B are both extensional (i.e. complete, universally distributive and saturated). Then the lattice of relations of type $A{\sim}B$ is saturated; the atoms are elements of the form $a{\circ}\mathbb{T}{\circ}b$ where a and b are atoms of the poset of coreflexives (of types A and B, respectively). It follows that, if the lattice of relations of type $A{\sim}B$ is complete and universally distributive, it is isomorphic to the powerset of the set of elements of the form $a{\circ}\mathbb{T}{\circ}b$ where a and b are points of types A and B, respectively.

Proof:
By Theorems 101 and 99, $a{\circ}\mathbb{T}{\circ}b$ is an atom. It suffices to prove that the lattice of relations of type $A{\sim}B$ is saturated. This is easy: for all R of type $A{\sim}B$,

$$
\begin{array}{ll}
 & R \\
= & \{ \quad \mathbb{I} \text{ is unit of composition,} \\
 & \quad\quad \text{lattices of coreflexives of types } A \text{ and } B \text{ are extensional} \quad \} \\
 & \langle \cup a : \mathsf{point}.a : a\rangle \circ R \circ \langle \cup b : \mathsf{point}.b : b\rangle \\
= & \{ \quad \text{distributivity of composition over } \cup \quad \} \\
 & \langle \cup\, a,b \,:\, \mathsf{point}.a \wedge \mathsf{point}.b \,:\, a{\circ}R{\circ}b\rangle \\
= & \{ \quad \text{all-or-nothing rule: Theorem 102, } \perp\!\!\!\perp \text{ is zero of supremum} \quad \} \\
 & \langle \cup\, a,b \;:\; \mathsf{point}.a \wedge \mathsf{point}.b \wedge a{\circ}R{\circ}b \neq \perp\!\!\!\perp \;:\; a{\circ}\mathbb{T}{\circ}b\rangle \ .
\end{array}
$$

That the lattice of relations is a powerset follows from Theorem 81. By Theorem 101, every pair is a relation of the form $a{\circ}\mathbb{T}{\circ}b$; also, by Lemma 88, $a{\circ}\mathbb{T}{\circ}b$ is an atom. □

Summarising Theorem 103, the *saturation* property is that

$$\langle \forall R \ :: \ R = \langle \cup\, a,b \,:\, a{\circ}\mathbb{T}{\circ}b \subseteq R \,:\, a{\circ}\mathbb{T}{\circ}b\rangle\rangle \ . \tag{104}$$

Combining Theorem 103 with Theorem 81, we get the *irreducibility* property: if $\mathcal{R}$ is a function with range relations of type $A{\sim}B$ and source K, then, for all points a and b of appropriate type,

$$a{\circ}\mathbb{T}{\circ}b \ \subseteq\ \cup\mathcal{R} \ \equiv\ \langle \exists k \,:\, k{\in}K \,:\, a{\circ}\mathbb{T}{\circ}b \subseteq \mathcal{R}.k\rangle \ . \tag{105}$$

Property (104) formalises the interpretation of the property $a{\circ}\mathbb{T}{\circ}b \subseteq R$ as the property $(a,b){\in}R$ in standard set-theoretic accounts of relation algebra.

Theorem 103 assumes that the lattices of coreflexives (of appropriate type) are extensional. Conversely, if we assume that, for all types A and B, the lattice of relations of type $A{\sim}B$ is extensional then so is the lattice of coreflexives of type A, for all A. This is Theorem 106. (The proof of Theorem 106 can be found in the companion document [BV22].)

Theorem 106. Suppose, for all types A and B, the lattice of relations of type $A{\sim}B$ is extensional, whereby the atoms are elements of the form $a{\circ}\mathbb{T}{\circ}b$ where a and b are atoms of the poset of coreflexives (of types A and B, respectively). Then, for all A, the lattice of coreflexives of type A is extensional. □

Combining Theorems 103 and 106, we get:

Corollary 107. Suppose, for all types A and B, the lattice of relations of type $A{\sim}B$ is complete and universally distributive. Then for all types A and B, the lattice of relations of type $A{\sim}B$ is extensional iff for all types A, the lattice of coreflexives of type A is extensional. □

Although the saturation property allows us to identify atoms of the form $a{\circ}\mathbb{T}{\circ}b$ with elements (a,b) of the set $A{\times}B$, it does not establish that the operators of relation algebra (converse, composition etc.) correspond to their standard set-theoretic interpretations. This is straightforward. For example, for composition we have, for all R and S,

$$
\begin{array}{ll}
 & R{\circ}S \\
= & \quad \{ \quad \text{saturation: (104)} \quad \} \\
 & \langle \cup\, a,b : a{\circ}\mathbb{T}{\circ}b \subseteq R : a{\circ}\mathbb{T}{\circ}b\rangle \circ \langle \cup\, b',c : b'{\circ}\mathbb{T}{\circ}c \subseteq S : b'{\circ}\mathbb{T}{\circ}c\rangle \\
= & \quad \{ \quad \text{distributivity} \quad \} \\
 & \langle \cup\, a,b,b',c \;:\; a{\circ}\mathbb{T}{\circ}b \subseteq R \wedge b'{\circ}\mathbb{T}{\circ}c \subseteq S \;:\; a{\circ}\mathbb{T}{\circ}b{\circ}b'{\circ}\mathbb{T}{\circ}c\rangle \\
= & \quad \{ \quad b \text{ and } b' \text{ are points, so } b{\circ}b' \neq \perp\!\!\!\perp \equiv b' = b \\
 & \qquad\quad \text{case analysis on } b' = b \;\vee\; b' \neq b\text{, one-point rule} \quad \} \\
 & \langle \cup\, a,b,c \;:\; a{\circ}\mathbb{T}{\circ}b \subseteq R \;\wedge\; b{\circ}\mathbb{T}{\circ}c \subseteq S \;:\; a{\circ}\mathbb{T}{\circ}b{\circ}b{\circ}\mathbb{T}{\circ}c\rangle \\
= & \quad \{ \quad b \text{ ranges over points, so } b{\circ}b = b \neq \perp\!\!\!\perp\text{, cone rule: (4)} \quad \} \\
 & \langle \cup\, a,b,c \;:\; a{\circ}\mathbb{T}{\circ}b \subseteq R \;\wedge\; b{\circ}\mathbb{T}{\circ}c \subseteq S \;:\; a{\circ}\mathbb{T}{\circ}c\rangle \\
= & \quad \{ \quad \text{range disjunction} \quad \} \\
 & \langle \cup\, a,c \;:\; \langle \exists b :: a{\circ}\mathbb{T}{\circ}b \subseteq R \wedge b{\circ}\mathbb{T}{\circ}c \subseteq S\rangle \;:\; a{\circ}\mathbb{T}{\circ}c\rangle \;\; .
\end{array}
$$

Comparing the first and last lines of this calculation (and interpreting $a{\circ}\mathbb{T}{\circ}b \subseteq R$ as $(a,b){\in}R$ and $b{\circ}\mathbb{T}{\circ}c \subseteq S$ as $(b,c){\in}S$) we recognise the standard set-theoretic definition of $R{\circ}S$.

The important step to note in the above calculation is the use of the distributivity of composition over union. The validity of such universal distributivity — both from the left and from the right— is a consequence of the Galois connections (5) and (6) defining factors. A similar step needed in the calculation for converse relies on the fact that converse is the upper and lower adjoint of itself.

We conclude this section with a brief comparison of extensionality as formulated here with the notion of extensionality formulated by Voermans [Voe99].

Voermans [Voe99, Section 4.5] postulated that the lattice of binary relations of a given type is saturated by relations of the form $X{\circ}\mathbb{T}{\circ}Y$ where X and Y are *particles*. Relations of this form are then shown to model pairs (x,y) in standard set-theoretic presentations of relation algebra. Here, we have postulated that each type A forms a lattice that is saturated by *points*: see Axiom 86; this postulate is combined with our axiom of choice: all pers have an index. Then pairs in standard set-theoretic presentations of relation algebra are modelled by relations of the form $a{\circ}\mathbb{T}{\circ}b$, where a and b are points. Because particles are points (Corollary 92), the saturation property postulated by Voermans is formally stronger than Axiom 86. As a consequence, it becomes slightly harder to establish that, for example, the composition of two relations does indeed correspond to the set-theoretic notion of composition. (See [Voe99, Section 4.5] for details of what is involved.) More importantly, the combination of Axioms 75 and 86 facilitates a better separation of concerns: Axiom 75 provides a powerful extension of point-free reasoning, whilst Axiom 86 fills the gap where pointwise reasoning is unavoidable.

8. Independence of the Axioms

In this paper, we have considered various additions to the basic axioms of point-free relation algebra: the cone rule, our axiom of choice, extensionality and the all-or-nothing rule. The question arises to what extent these additional rules are independent of one another.

Elementary examples of point-free relation algebra give some insight. In all the examples, there is just one, unnamed, type. The 1-element algebra has just one element: in this algebra $\perp\!\!\!\perp = \mathbb{I} = \mathbb{T}$. The 2-element algebra has two elements: in this algebra $\perp\!\!\!\perp \neq \mathbb{I} = \mathbb{T}$. There is a 3-element algebra that has distinct elements $\perp\!\!\!\perp$, $\mathbb{I}$ and $\mathbb{T}$. (There is also a 3-element algebra in which $\mathbb{I} = \mathbb{T}$.)

The cone rule is independent of the other rules since, given any two algebras, one can define their product in such a way that the cone rule is not satisfied but the validity of other rules is preserved. See [Voe99, section 3.4.3] for more details.

The 1-element algebra is the algebra of concrete relations on the empty set and the 2-element algebra is the algebra of concrete relations on a set with exactly one element. Both these algebras satisfy all four additional axioms.

The 1-element and 2-element algebras have the property that $\mathbb{I} = \mathbb{T}$. So, for any R,

$$\mathbb{T}\circ R\circ\mathbb{T} = \mathbb{I}\circ R\circ\mathbb{I} = R.$$

In such algebras, our axiom of choice is valid (because every element is an index of itself); extensionality is also valid but, if there are more than two elements, the cone rule is not valid. (If the algebra is complemented, it is a Boolean algebra.)

The 3-element algebra does not satisfy our axiom of choice since $\mathbb{T}$ does not have an index. Nor does it satisfy the all-or-nothing rule. It does, however, satisfy the cone rule as well as being extensional.

A slightly more complicated example is needed to show that the all-or-nothing rule does not imply our axiom of choice. Consider the algebra with a single point a in addition to the three constants $\perp\!\!\!\perp$, $\mathbb{I}$ and $\mathbb{T}$. That is, $\perp\!\!\!\perp \subseteq a \subseteq \mathbb{I} \subseteq \mathbb{T}$. Add the requirement that $a\circ\mathbb{T} = a = \mathbb{T}\circ a$. Then $a = a\circ\mathbb{T}\circ a$ and the all-or-nothing rule is valid. However, $\mathbb{T}$ does not have an index: the only two possibilities are a and $\mathbb{I}$, and it is easily checked that neither satisfies the requirements.

The above example does not satisfy extensionality (because there is exactly one point different from $\mathbb{I}$); nor does it satisfy the cone rule (because $\mathbb{T}\circ a\circ\mathbb{T} = a$). Adding the cone rule as a requirement has the implication that $a\circ\mathbb{T}$, a and $\mathbb{T}\circ a$ must be different — with the knock-on effect that several additional elements must also be added to the algebra. The resulting algebra, which was constructed by Jules Desharnais [private communication] using Mace4 [McC10], is the minimal algebra with elements a and E, in addition to the constants $\perp\!\!\!\perp$, $\mathbb{I}$ and $\mathbb{T}$, such that

$$\perp\!\!\!\perp \subseteq a \subseteq \mathbb{I} \subseteq E \quad \wedge \quad a\circ E = a = E\circ a \ \ ,$$

and the cone rule and the all-or-nothing rule are both satisfied. The lattice structure of the algebra is shown in Fig. 3.

Note that the original 4-element algebra forms the "stem" of the algebra — $\mathbb{T}$ has been renamed E. Adding the cone rule causes the algebra to "blossom out" into a much larger structure. The algebra does not satisfy our axiom of choice, because E does not have an index.

Figure 3. All-or-Nothing and Cone Rule but not Axiom of Choice

The combination of the all-or-nothing rule, the cone rule and extensionality implies that the lattice of relations is isomorphic to the powerset of the set of elements of the form $a{\circ}\mathbb{T}{\circ}b$ where a and b are points [BDGv22, Theorem 57]. (We have already seen two examples of the theorem: the 1-element and the 2-element algebras.) In turn, this implies that, in standard set theory, our axiom of choice is valid. In constructive set theory, the axiom might be deemed to be inadmissible — in the same way that the universal axiom of choice might be deemed to be inadmissible. (Inadmissible is different from invalid. The law of the excluded middle is inadmissible in constructive logic but is not invalid; in constructive logic, particular instances of excluded middle must be proven by exhibiting witnesses, but current wisdom is that a counterexample to excluded middle will never be found.)

The final question we ask is whether, in point-free relation algebra, the universal axiom of choice is implied by our axiom of choice. Formally, the universal axiom of choice is that, for any relation R, there is a relation f such that $f \subseteq R$, $f \circ f^{\cup} \subseteq \mathbb{I}$ and $f{>} = R{>}$. (In words, f is a functional approximation to R with the same right domain as R; for given point b, $f{\circ}b$ "chooses" a point a in the left domain of R such that a and b are related by R.)

The 3-element algebra satisfies the universal axiom of choice but not our axiom of choice. So the universal axiom of choice does not imply our axiom of choice. It is an open question whether or not our axiom of choice implies the universal axiom of choice.

9. Conclusion

Point-free relation algebra has been developed over many, many years (beginning in the 19th century) and is generally regarded as a much better basis for the development of the theory of relations than pointwise reasoning. However, for practical applications, pointwise reasoning is at times unavoidable. For example, path-finding algorithms on graphs must ultimately be expressed in terms of the nodes and edges of the graph (the points and elements of the relation defined by the graph). Good practice is to develop such algorithms in a stepwise fashion, beginning with point-free reasoning (typically using regular algebra) and delaying the introduction of points until absolutely necessary.

It is common practice to represent an equivalence relation by choosing a specific element of each equivalence class. For example, the class of integers modulo 3 is commonly represented by the set of three elements 0, 1 and 2. The characterisation of an equivalence relation by a representative function is not derivable in point-free relation algebra since there is a constructive element in the choice of representatives. Extensions to point-free relation algebra, such as the postulate that relations form a so-called "power allegory" [Fv90, 2.4], are intended to enable pointwise reasoning but nevertheless fail to properly capture the use of representatives. Our axiom of choice (Axiom 75) together with our point-free formulation of the notion of an index of a relation does capture the use of representatives. The strength of the axiom together with the fact that an index of a relation has the same type as the relation makes the notion of an index —in our view— very attractive and useful. Moreover, its combination with the extensionality axiom (Axiom 86) permits the derivation of Glück's "all-or-nothing" axiom [Glü17]. In this way, point-free reasoning has been strengthened whilst also facilitating pointwise reasoning when unavoidable.

It might be argued that our axiom of choice is too strong. On the contrary, we would argue that it corresponds much better to standard practice. For example, the computation of the strongly connected components of a graph involves computing a representative node for each component. (Tarjan [Tar72] and Sharir [Sha81] call the representative of a strongly connected component of a graph the "root" of the component; Cormen, Leiserson and Rivest [CLR90, p.490] call it the "forefather" of the component.) A suggestion for future work is to exploit our notion of an index in order to reformulate —much more succinctly— the properties of depth-first search that underlie its effectiveness in such computations.

Our focus in this paper has been on documenting the properties of indexes and the consequences for axiom systems enabling pointwise reasoning. The original motivation for this work was, however, quite different. Seventy years ago, in a series of publications [Rig48, Rig50, Rig51], Jacques Riguet introduced the notions of a "relation difonctionelle", the "différence" of a relation and "relations de Ferrers". In view of possible practical implications, particularly in respect of relational databases, our goal was to bring Riguet's work up to date, making it more accessible to modern audiences. In the process, we began to realise that substantial improvements could be made by introducing the notions of "core" and "index" of a relation, drawing inspiration from Voerman's [Voe99] notion of the (left- and right-) per domains of a relation. Results on cores and indexes relevant to Riguet's work, in particular practical applications of his notion of the "différence" of a relation (which we rename the "diagonal" of a relation), are documented in [BV23]. Further insight into the nature of cores and indices is documented in [VDB] where we introduce the "thins" ordering on relations; among the theorems we prove is that an index of a relation is minimal with respect to the thins ordering. Proofs we have omitted here are documented in the companion working document [BV22].

Appendix. Notational Conventions The predence rules we assume are that unary operators take precedence over all binary operators; composition takes precedence over union and intersection, which have equal precedence; next comes the subset ordering and finally the logical operators, which follow the usual precedence rules except that disjunction and conjuction have the same precedence. We endeavour to add spacing to suggest the precedence.

Our calculations consist of a sequence of terms, successive terms being separated by a relation symbol and a "hint" indicating why the relation holds between the terms. The relations between terms are always ordering relations and, typically but not always, the conclusion of the calculation is the relation between the initial and final terms (which is implicitly deduced by transitivity properties of the connecting relations). We endeavour to use equality steps whenever possible, even in the case that the final conclusion is an ordering. (This is because an equality step is safe in the sense that no loss of information is incurred when making the step. Calculations that reach a dead end often do so because an ordering is introduced that is too strong.) Sometimes an equality is established by exploiting the anti-symmetry of an ordering relation. In such cases, the initial and final terms are identical and the conclusion is that all terms in the calculation are equal. Sometimes we announce the intention to exploit anti-symmetry by the use of the word "mutual" (as in, for example, "mutual implication" or "mutual inclusion").

The first rule of logic is the rule sometimes known as "substitution of equals for equals": two things are equal exactly when one can be substituted for the other in any context. The rule is generally attributed to Gottfried Wilhelm Leibniz. The use of the rule is often implicit in our calculations but occasionally we refer to it by the hint "Leibniz". The centrality of the rule is reinforced by the ubiquitous overloading of Robert Recorde's "$=$" symbol to denote equality. So the symbol is used for equality of numbers, sets, functions, etc. In some circumstances, however, the overloading can be confusing. A simple example is that the boolean $x=0$ is equal to the boolean $x{+}1=1$, whatever the value of x. However, it could be confusing to overload Robert Recorde's equality symbol to express their equality as in

$$(x=0) \;=\; (x{+}1=1) \;\;.$$

For this reason, it is convenient to introduce a special symbol "$\equiv$" which we sometimes use to denote the equality of booleans; this symbol is given a lower precedence than Robert Recorde's symbol, as exemplified by

$$x=0 \;\equiv\; x{+}1=1 \;\;.$$

A more important reason for introducing an additional symbol is that boolean equality is associative. In a continued equality $a=b=c$ (whatever the type of a, b and c), the transitivity of equality is implicitly assumed. For booleans a, b and c, $a\equiv b\equiv c$ has a different meaning: its value is determined by evaluating $(a=b)=c$ or $a=(b=c)$; both yield the same result. We don't exploit the associativity of boolean equality in this paper. We do exploit its transitivity as well as Leibniz's rule frequently in our calculations. Consequently, we most commonly use Robert Recorde's symbol to denote the equality of boolean expressions (as well as expressions of other types).

Acknowledgement Many thanks to Jules Desharnais for supplying models of the different axiom systems. We take full responsibility for any errors we may have introduced. Thanks also to the referees for their helpful comments.

References

[ABH+92] C.J. Aarts, R.C. Backhouse, P. Hoogendijk, T.S. Voermans, and J. van der Woude. A relational theory of datatypes. Available at ResearchGate and `http://www.cs.nott.ac.uk/~psarb2/papers`, September 1992.

[Bac06] Roland Backhouse. Regular algebra applied to language problems. *Journal of Logic and Algebraic Programming*, 66:71–111, 2006.

[Bac21] Roland Backhouse. Difunctions and block-ordered relations. Working document. Available at ResearchGate and `www.cs.nott.ac.uk/~psarb2/MPC/DifunBlockOrderedAndStaircase.pdf`, 2021.

[BDGv22] Roland Backhouse, Henk Doornbos, Roland Glück, and Jaap van der Woude. Components and acyclicity of graphs. An exercise in combining precision with concision. *Journal of Logical and Algebraic Methods in Programming*, 124(100730), January 2022.

[BH93] R.C. Backhouse and P. Hoogendijk. Elements of a relational theory of datatypes. In B. Möller, H.A. Partsch, and S.A. Schuman, editors, *Formal Program Development. Proc. IFIP TC2/WG 2.1 State of the Art Seminar, Rio de Janeiro, Jan. 1992*, volume 755 of *LNCS*, pages 7–42. Springer-Verlag, 1993.

[BO23] Roland Backhouse and José N. Oliveira. On difunctions. *Journal of Logical and Algebraic Methods in Programming*, 134, June 2023.

[BV22] Roland Backhouse and Ed Voermans. The index and core of a relation. With application to the axiomatics of relation algebra and block-ordered relations. Available at ResearchGate, April 2022.

[BV23] Roland Backhouse and Ed Voermans. Diagonals and block-ordered relations. Submitted for publication. Available at ResearchGate and `http://arXiv.org/abs/2401.17130`, September 2023.

[CLR90] Thomas H. Cormen, Charles E. Leiserson, and Ronald L. Rivest. *Introduction to Algorithms*. MIT Electrical Engineering and Computer Science Series, MIT Press, 1990.

[Con71] J.H. Conway. *Regular Algebra and Finite Machines*. Chapman and Hall, London, 1971.

[Fv90] P.J. Freyd and A. Ščedrov. *Categories, Allegories*. North-Holland, 1990.

[Glü17] Roland Glück. Algebraic investigation of connected components. In P. Höfner, D. Pous, and G. Struth, editors, *Relational and Algebraic Methods in Computer Science – 16th International Conference, RAMiCS 2017*, volume 10226 of *Lecture Notes in Computer Science*, pages 109–126. Springer, May 15–18 2017.

[McC10] W. McCune. Prover9 and Mace4. `http://www.cs.unm.edu/~mccune/mace4/`, 2005–2010.

[Oli18] José N. Oliveira. Programming from metaphorisms. *JLAMP*, 94:15–44, January 2018.

[Rig48] J. Riguet. Relations binaire, fermetures, correspondances de Galois. *Bulletin de la Société Mathématiques de France*, 76:114–155, 1948.

[Rig50] J. Riguet. Quelques propriétés des relations difonctionelles. *C.R.Acad.Sci. Paris*, 230:1999–2000, 1950.

[Rig51] J. Riguet. Les relations de Ferrers. *Compte Rendus des Séances hebdomadaires de l'Acadamie des Sciences (Paris)*, 232:1729–1730, 1951.

[Sha81] M. Sharir. A strong-connectivity algorithm and its application in data flow analysis. *Computers and Mathematics with Applications*, 7(1):67–72, 1981.

[Tar72] Robert Endre Tarjan. Depth first search and linear graph algorithms. *SIAM J. Computing*, pages 146–160, 1972.

[VDB] Ed Voermans, Jules Desharnais, and Roland Backhouse. The thins ordering on relations. Submitted for publication. Available at ResearchGate and `http://arXiv.org/abs/2401.14888`.

[Voe99] Ed (Theodorus Sebastiaan) Voermans. *Inductive Datatypes with Laws and Subtyping. A Relational Model*. Technische Universiteit Eindhoven, Department of Mathematics and Computer Science, 1999. Ph.D. thesis. DOI:10.6100/IR51.

[Win04] Michael Winter. Decomposing relations into orderings. In B. Möller R. Berghammer and G. Struth, editors, *Proc. of the Internat. Workshop RelMiCS '7 and 2nd Internat. Workshop on Applications of Kleene Algebra, in combination with a workshop of the COST Action 274: TARSKI. Revised Selected Papers*, volume 3051 of *LNCS*, pages 261–272. Springer-Verlag, 2004.