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A Theory of Conversion Relations for Prefixed Units of Measure

Baltasar Trancón y Widemann
Technische Hochschule Brandenburg
Brandenburg an der Havel, DE
trancon@th-brandenburg.de

Markus Lepper
semantics gGmbH
Berlin, DE

Abstract. Units of measure with prefixes and conversion rules are given a formal semantic model in terms of categorial group theory. Basic structures and both natural and contingent semantic operations are defined. Conversion rules are represented as a class of ternary relations with both group-like and category-like properties. A hierarchy of subclasses is explored, each satisfying stronger useful algebraic properties than the preceding, culminating in a direct efficient conversion-by-rewriting algorithm.

CCS Concepts: • *Theory of computation* → *Theory and algorithms for application domains*; • **Applied computing** → **Physical sciences and engineering**; • **Software and its engineering** → **Software notations and tools** → **Formal language definitions** → **Semantics**; • *Mathematics of computing* → *Discrete mathematics*;

1. Introduction

In the mathematics of science, *dimensions* and *units of measure* are used as static metadata for understanding and checking *quantity* relations [30]. Quantities are variables that may assume a value expressing a fact about a model, in terms of a formal multiplication of a *numerical value* with a unit of measure.[1] For example, $42\,195\ \mathrm{m}$ is a quantity value of *distance*, with numerical value $42\,195$ and

[1] In many fields of mathematics that deal with scalable entities, the former part would be called a *magnitude*; however, in the context of quantities, the term *magnitude* is applied to the pair. Hence we avoid that term for the sake of clarity.

unit of measure m. Quantities themselves have a complicated and controversial ontology, which shall not be discussed here.

Quantities and quantity values are quite distinct epistemological types, and the need to distinguish them properly is demonstrated beyond doubt by Partee's Paradox [12]: "The temperature is ninety" and "the temperature is rising" do not entail "ninety is rising." The former premise is a statement about (an instantaneous assignment of) a quantity value; note that the numerical value is explicit but the unit is implicit. The latter premise is a statement about a quantity that persists over time. The conclusion, being in type error, is not a meaningful statement about anything physical.

Clearly, neither half of the numerical value–unit pair is sufficient for interpretation: The numerical value carries the relative data, and the unit of measure the context of reference. A system of quantities assigns a dimension to each quantity, such that only quantities of identical dimension are *commensurable*, that is, may be added or compared (as apples with apples), and may furthermore be associated with preferred units of measurement. For example, the quantities of *molecular bond length* and *planetary perihelion* are both of dimension *length* and thus formally commensurable, but traditionally use quite different units of measure.

In scientific programming, traditional practice expresses only the numerical value part of quantity values, whereas units of measure, and their consistency among related quantities, are left implicit, just like in the quotation above. The evident potential for disastrous software errors inherent in that practice has been demonstrated, for instance, by the famous crash of the Mars Climate Orbiter probe, where *pounds of force* were confounded with *newtons* at a subsystem interface [23]. Practical tool solutions abound [16]. Theoretical foundations are rare, but essential for program correctness, specification and verification.

In the seminal work of Kennedy [10], the checking of stated or implied units of measure is cast formally as a *type inference* problem. The approach has been implemented successfully, for example in F# [11] or Haskell [6]. However, the underlying concept of units is strongly simplified, and falls far short of the traditional scientific practice, embodied in the International Systems of Units and Quantities (SI, ISQ [18], respectively).

A particular challenge is the concept of *convertible* units, where the scaling factors for equivalent transformation of numerical values relative to some other unit are fixed and known, and hence could be applied implicitly. In the Mars Climate Orbiter case, implicit multiplication by $4.448\,221\,615\,260\,5$ would have avoided the observed failure, and thus potentially saved the mission.

Furthermore, as a special case of convertibility of such practical importance that it comes with its own notation, there are *prefixes* that can be applied to any unit, and modify the scaling factor in a uniform way. For example, multiplication by 1000 serves to convert from seconds to milliseconds, and exactly in the same way from meters to millimeters.

1.1. A Survey of Problem Cases

Traditional concepts of units of measure also have their own curious, historically grown idiosyncrasies and restrictions. The following five problem cases shall serve to illustrate the need for more rigorous formalization, and also as challenges for the explanatory power of the model to be developed in the main matter of the present paper, to be recapitulated in Section 5.2.

1.1.1. Problem Case: Neutral Elements

The treatment of neutral elements of the algebras of dimensions and units in the standard literature exhibits several subtle inexactitudes and inconsistencies.

For the neutral element of the algebra of dimensions, the version of the ISQ that we are citing throughout this paper discusses the form

> "[...] $A^0 B^0 C^0 \cdots = 1$, where the symbol 1 denotes the corresponding dimension. Such a quantity is called a quantity of dimension one and is expressed by a number." [18, §5]

whereas the slightly later German version [5] replaces "dimension one" by "dimension number". [2] That this is not merely a liberty of translation, but a conscious choice is reflected by remarks to the effect that the terms "quantity of dimension one" and "dimensionless quantity" are deprecated[3], and that 1 as the expression of a dimension is not allowed[4].

A recent revision of [18] sums up the controversy with the more cautious, but rather indecisive statement:

> "[...] $A^0 B^0 C^0 \cdots = 1$, where the symbol 1 denotes the corresponding dimension. There is no agreement on how to refer to such quantities. They have been called dimensionless quantities (although this term should now be avoided), quantities with dimension one, quantities with dimension number, or quantities with the unit one. Such quantities are dimensionally simply numbers. To avoid confusion, it is helpful to use explicit units with these quantities where possible, e.g., m/m [...]" [19, §5]

Note the self-contradiction between "quantities with the unit one" and "use explicit units".

With regard to the neutral element of the algebra of units, the SI has a precise notion of its semantic status,

> "The unit one is the neutral element of any system of units—necessary and present automatically." [25, §2.3.3]

but still, somewhat contradictory, classifies it as derived by means of a syntactic criterion:

> "All other units [i.e. other than the seven base units], described as *derived units*, are constructed as products of powers of the base units." [25, §1.2].

The ISQ likewise classifies the neutral unit as not a base unit, but acknowledges that there are non-derived quantities measured in that unit, namely *counts* of entities. This is possible by virtue of the laxer, non-exclusive characterization of derived quantities (note the omission of "all"):

> "Other quantities, called derived quantities, are defined or expressed in terms of base quantities by means of equations." [18, §4.3]

[2] "Eine solche Größe wird als *Größe der Dimension Zahl* bezeichnet, und ihr Wert wird durch eine Zahl angegeben." [5, §5]
[3] "Die Benennungen 'Größe der Dimension Eins' und 'dimensionslose Größe' sind veraltet und sollten nicht mehr verwendet werden." [5, p. 3]
[4] "Die Zahl 1 ist keine Dimension. Hier sollte korrekterweise 'Dimension Zahl' stehen." [5, footnote N12]
"Die Zahl 1 ist keine Dimension. Ein Zeichen für die Dimension Zahl ist derzeit noch nicht festgelegt." [5, footnote N14]

The notation rules common to SI and ISQ state that the neutral unit has the symbol 1, but should not be written as such in a quantity value expression. It follows that, unlike any other unit symbol with a definition, it cannot take a prefix.

1.1.2. Problem Case: Equational Reasoning

The ISQ standard defines derived units by equations, equating each to some expression over other, more basic units [18, §6.5]. However, equals often cannot be substituted for equals, without making arbitrary choices, or violating semantic conventions or even basic syntax:

- Applying the prefix *micro-* ($\mathrm{\mu}$) to the SI "base" unit *kilogram* (kg) yields the syntactically illegal $\mathrm{\mu kg}$; the recommended, semantically equivalent alternative is the *milligram* (mg).
- A unit of *volume*, the *centilitre*, is defined as the prefix *centi-*, implying a multiplicator of 0.01, being applied to the unit *litre* (L), which in turn defined as the cubic *decimeter* (dm^3). The expanded expression $\mathrm{cL} = \mathrm{c}(\mathrm{dm}^3)$ is syntactically invalid, and there is no semantically equivalent expression in SI syntax proper; the equation $10^{-5} = 10^{3k}$ obviously has no integer solutions.
- The coherent SI unit of *power*, the *watt* (W) is defined, formally via several reduction steps involving other units, as $\mathrm{kg}{\cdot}\mathrm{m}^2{\cdot}\mathrm{s}^{-3}$. For the *hectowatt*, there are no less than 44 distinct semantically equivalent alternatives of prefix distribution, such as $\mathrm{fg}{\cdot}\mathrm{cm}^2{\cdot}\mathrm{ns}^{-3}$, or $\mathrm{dg}{\cdot}\mathrm{\mu m}^2{\cdot}\mathrm{\mu s}^{-3}$, or $\mathrm{Eg}{\cdot}\mathrm{cm}^2{\cdot}\mathrm{ks}^{-3}$, neither of which can be called obviously preferable.

Besides the apparently arbitrary restrictions of syntactic expressibility, there is the nontrivial problem of getting the circumventing semantic equivalence right; see Section 1.1.5 below.

1.1.3. Problem Case: Cancellation

The ISQ standard defines the units of *plane angle*, radians, and of *solid angle*, steradians, as

$$\mathrm{rad} = \mathrm{m}/\mathrm{m} \qquad\qquad \mathrm{sr} = \mathrm{m}^2/\mathrm{m}^2$$

respectively, but considers them different from each other, and from the unit 1 [18, §6.5]. Thus, the operation written as unit division appears cancellative in many, but not nearly all contexts.

1.1.4. Problem Case: Prefix Families

The SI prefix families are geometric sequences, but cannot be written as powers of a generator. For example, the General Conference on Weights and Measures (CGPM) has recently adopted the metric prefixes *ronna-* and *quetta-* with the values 10^{27} and 10^{30}, respectively, as well as their inverses *ronto-* and *quecto-* [20]. These are clearly intended to represent the ninth and tenth powers, respectively, of the generating prefix *kilo-*, but cannot be written as such: Defining equations of the form

$$\mathrm{R} = \mathrm{k}^9 \qquad\qquad \mathrm{r} = \mathrm{k}^{-9}$$
$$\mathrm{Q} = \mathrm{k}^{10} \qquad\qquad \mathrm{q} = \mathrm{k}^{-10}$$

are syntactically forbidden.

1.1.5. Problem Case: Units, Kinds and Dimensions

There is a tendency in both theory and practice to assert a distinguished reference unit per dimension. This has a number of consequences, some being clearly unintended.

It is common for tools to specify conversion factors between commensurable units in relation to the reference unit of that dimension, both in order to avoid inconsistencies and as a means of data compression [2]. But dimensions are too coarse to distinguish all semantically incommensurable quantities, leading to manifestly unsound conversion judgements. For example, *newton-meters* (of *torque*) and *joules* (of *work*) become interconvertible in this manner; the same holds for *becquerels* (of *radioactivity*) and *revolutions-per-minute* (of *angular velocity*)[5].

The ISQ admits a concept of *kind of quantity*, such that units of the same dimension must additionally be of the same kind in order for their values to be commensurable. Thus, the above problem could be prevented by noting that the respective quantities are of different kind.

However, the standard gives no guidelines how a particular choice of kinds and their assignment to quantities can be made, justified or predicated on a context. Instead, it states that “division into several kinds [...] is to some extent arbitrary”, and that quantities may be “generally considered” or “by convention not regarded” as being of the same kind [18, §3.2]. Furthermore, there is no indication whether and how kinds of quantities should relate to the units used for expressing their respective values: Can incommensurable quantities have values involving the same units, and if so, what are the appropriate algebraic rules?

Logically related, but even more worrying from the formal perspective is the ensuing confusion about the very identity of units. For example, the CGPM has recognized the medically important, even potentially life-saving, need to distinguish the units *gray* and *sievert* of *absorbed* and *equivalent/effective dose*, respectively, of ionizing radiation [21]. Nevertheless, the same resolution ends with the ominous statement “The sievert is equal to the joule per kilogram”, where the latter is the reductionistic definition of the *gray*. Whatever equality is invoked in this statement, it is not the usual mathematical one that satisfies the *indiscernibility of identicals*, seeing as that would utterly defeat the purpose of said resolution.

1.2. Research Programme

The current state of the theory of quantity values and units of measure is at best semi-formal. There is a collection of conventions, embodied explicitly in international standards and implicitly in scientific education, that prescribes the notations, pronounciations and calculations associated with quantity value expressions. What is lacking is an internal method, such as deduction from axioms, for justification or critique of these rules, in particular where they give rise to ambiguous, peculiar, awkward or outright contradictory usage.

Our present work is a proposal of how to improve the formal rigor of reasoning by mathematical remodeling. An interpretation of the domain of discourse is given that is *informed*, but not *governed*, by the traditional readings. For such a model to be *adequate*, it is required that the uncontroversial consequences of the theory are upheld. Where the model begs to differ from conventional wisdom, it can be studied on an objective formal basis, opening new and potentially fruitful avenues of criticism.

[5] The GNU command line tool `units`, as of version 2.21, happily suggests the conversion factor $30/\pi$.

The result of this research aspires not to produce only *some* model that satisfies the intended propositions in arbitrary ways. Rather, the goal is an *epistemological* model, where the reasons for the truth of some satisfied proposition can be analyzed. To this end, a clear line is carefully drawn between *necessary* and *contingent* properties of the model: The former are those that follow from the mathematical structure without regard to the meaning of particular entity symbols, and cannot be subverted. The latter are those that represent the actual semantic conventions of science, which "could be different but aren't" — i.e., which could be extended or modified without breaking the underlying logic.

1.3. Contributions

The present work, an extended revision of [27, 28], lays the foundation for a formal model of dimensions and units of measure that extends and complements the findings of [10]. We refine the mathematical structures outlined there, in order to accomodate unit conversions and prefixes. In addition, the proposed model rectifies irrelevant, historically accrued restrictions of traditional notation systems.

In particular, the main contributions of the present work are as follows:

- A formal mathematical model of dimensions and units with optional prefixes, separated into the generic structures shared by all unit systems (Section 3.1) and the properties to be interpreted contingently by each unit system (Section 3.2), together with a hierarchy of equivalence relations.

- A formal mathematical model of unit conversion rules as a class of ternary relations with a closure operator, that function both as groups and as categories (Section 4), together with a hierarchy of six subclasses, each satisfying stronger useful algebraic properties than the preceding, and in particular an efficient rewriting procedure for calculating conversion factors.

2. Prerequisites

We assume that the reader is familiar with basic concepts of abstract algebra and category theory. In particular, the proposed model hinges on the concepts of *abelian groups* and *adjoint functors*; see [1] for the categorial approach to the former and [13] for an introduction to the pervasive structural role of the latter.

Namely, the promised epistemological distinction shall be realized by modeling the necessary aspects of data and operations in terms of functors and natural transformations that arise from adjunctions. By contrast, contingent aspects are modeled as particular objects and non-natural morphisms.

2.1. Abelian Groups

The category $\mathbf{Ab}$ has abelian groups as objects. We write $\mathcal{U}(G)$ for the carrier set of a group G. (A group and its carrier set are not the same.) A homomorphism $f : G \to H$ is a map $f : \mathcal{U}(G) \to \mathcal{U}(H)$ that commutes with the group operations. The carrier-set operator $\mathcal{U}$, together with the identity operation on homomorphism maps, $\mathcal{U}(f) = f$, is the forgetful functor from the category $\mathbf{Ab}$ of abelian groups to the category $\mathbf{Set}$ of sets.

Wherever a generic group variable G is mentioned, we write $\diamond$ for the group operation, e for the neutral element, and † for inversion. Alternatively, groups could be spelled out as formal tuples of the form $G = (\mathcal{U}(G), \diamond, e, ^\dagger)$, but there is no pressing need for that level of formality. Powers of group elements are written and understood as follows:

$$x^{(0)} = e \qquad x^{(1)} = x \qquad x^{(m+n)} = x^{(m)} \diamond x^{(n)} \qquad x^{(-n)} = (x^{(n)})^\dagger$$

In this and other places, the generic model structure makes use of the additive group of integers $\mathbb{Z}$. Contingent actual numbers in some model instance are from the set $\mathbb{Q}_+$ of positive rationals. For arithmetics only the multiplicative group Qm on that set is required. We refer to these numbers as *ratios*.

Kennedy [10] observed that the algebras of dimensions and units are essentially *free abelian groups*.

2.2. Free Abelian Groups

We write $\mathcal{A}(X)$ for the free abelian group over a set X of generators. Whereas the concept is specified only up to isomorphism in category theory, we conceive of a particular construction of the carrier set $\mathcal{U}(\mathcal{A}(X))$, namely the space $\mathrm{Zf}(X)$ of finitely supported integer-valued functions on X:

$$\mathrm{Zf}(X) = \{f : X \to \mathbb{Z} \mid \mathrm{supp}(f) \text{ finite}\} \qquad \mathrm{supp}(f) = \{x \in X \mid f(x) \neq 0\}$$

If X is finite, then the group $\mathcal{A}(X)$ is called *finitely generated*. Note that the finite-support constraint is redundant in this case.

For denoting a particular element of a free abelian group directly, it suffices to refer to the support. The finite partial maps of type $f : X \to \mathbb{Z}\setminus\{0\}$ are in one-to-one correspondence with group elements $f_{/0} \in \mathrm{Zf}(X)$:

$$f_{/0}(x) = \begin{cases} f(x) & \text{if defined} \\ 0 & \text{otherwise} \end{cases}$$

For example, consider the positive rational numbers $\mathbb{Q}_+$, which by the fundamental theorem of arithmetic are isomorphic to the free abelian group over the prime numbers $\mathbb{P}$. Namely, the map $^\star : \mathrm{Zf}(\mathbb{P}) \to \mathbb{Q}_+$ that sends every finitely supported integer-valued map on the primes to the corresponding product of prime powers,

$$f^\star = \prod_{p \in P} p^{f(p)}$$

is a bijection. Then we can express the fact that $2/9$ factors as $2 \cdot 3^{-2}$ concisely as follows:

$$\frac{2}{9} = \{2 \mapsto 1, 3 \mapsto -2\}_{/0}{}^\star$$

Figure 1. Laws of the Monad Zf

The operation that turns the set $\mathrm{Zf}(X)$ into the abelian group $\mathcal{A}(X)$ is realized by pointwise addition. However, since the standard interpretation of function values is *power exponents*, we shall use multiplicative notation, such that $(fg)(x) = f(x) + g(x)$, with neutral element $1 = \varnothing_{/0}$.

For the following definitions, it is useful to introduce the *Iverson bracket*:

$$[p] = \begin{cases} 1 & \text{if } p \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

For one, $\mathcal{A}$ is a functor

$$\mathcal{A}(f : X \to Y)(g \in \mathrm{Zf}(X))(y \in Y) = \sum_{x \in X} [f(x) = y] \cdot g(x)$$

that accumulates the contents of a group element g over the kernel of the map f. Furthermore, $\mathcal{A}$ is left adjoint to $\mathcal{U}$: There are two natural transformations

$$\delta : 1 \Rightarrow \mathcal{U}\mathcal{A} \qquad\qquad \varepsilon : \mathcal{A}\mathcal{U} \Rightarrow 1$$

in $\mathbf{Set}$ and $\mathbf{Ab}$, named the *unit*[6] and *counit* of the adjunction, respectively, such that:

$$\varepsilon\mathcal{A} \circ \mathcal{A}\delta = 1\mathcal{A} \qquad\qquad \mathcal{U}\varepsilon \circ \delta\mathcal{U} = 1\mathcal{U}$$

Their concise definitions are:

$$\begin{aligned} \delta_X(x) &= \{x \mapsto 1\}_{/0} \\ &= [x = \_] \end{aligned} \qquad\qquad \varepsilon_G(f \in \mathcal{A}\mathcal{U}(G)) = \mathop{\Diamond}_{x \in \mathcal{U}(G)} x^{(f(x))}$$

Note how the finite-support property of f ensures that the infinitary operator $\Diamond$ is well-defined.

From the adjoint situation follows that the composite functor $\mathrm{Zf} = \mathcal{U}\mathcal{A}$ is a monad on $\mathbf{Set}$, with *unit* δ and *multiplication* $\lambda = \mathcal{U}\varepsilon\mathcal{A}$:

$$\begin{aligned} &\delta : 1 \Rightarrow \mathrm{Zf} \\ &\lambda : \mathrm{Zf}^2 \Rightarrow \mathrm{Zf} \qquad\qquad \lambda_X(f)(x) = \sum_{g \in \mathrm{Zf}(X)} f(g) \cdot g(x) \end{aligned}$$

[6]Adjunction/monad units are unrelated to units of measure.

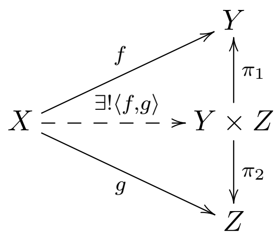

Figure 2. Cartesian Pair Operations

Monads and their operations are often best understood as abstract syntax: The elements of $\mathrm{Zf}(X)$ encode group words, up to necessary equivalence. The unit δ embeds the generators into the words as *literals*. The multiplication λ *flattens* two *nested* layers of words. The counit ε embodies the more general concept of interpreting words over a particular target group by "multiplying things out".

Lemma 2.1. Any element of a free abelian group can be written, uniquely up to permutation of factors, as a finite product of nonzeroth powers:

$$\{x_1 \mapsto z_1, \dots, x_n \mapsto z_n\}_{/0} = \prod_{i=1}^{n} \delta(x_i)^{z_i}$$

2.3. Pairs

Both the Cartesian product of two sets and the direct sum of two abelian groups are instances of the categorial binary product. All of their relevant structures and operations arise from a *diagonal functor*

$$\Delta_C : C \to C \times C \qquad \Delta_C(X) = (X, X) \qquad \Delta_C(f : X \to Y) = (f, f)$$

into the product category of some suitable category C with itself, and the fact that it has a right adjoint $(\_ \times \_)$. Namely, there are unit and counit natural transformations:

$$\psi : 1 \Rightarrow (\_ \times \_)\Delta \qquad \pi : \Delta(\_ \times \_) \Rightarrow 1$$

$$\psi_X(x) = (x, x) \qquad \pi_{(X,Y)} = (\pi_1, \pi_2) \quad \text{where} \quad \begin{array}{l} \pi_1(x, y) = x \\ \pi_2(x, y) = y \end{array}$$

In traditional notation, the counit components (projections) π_1 and π_2 are used separately, whereas the pairing of morphisms is written with angled brackets:

$$\langle f : X \to Y, g : X \to Z \rangle = (f \times g) \circ \psi_X : X \to (Y \times Z)$$

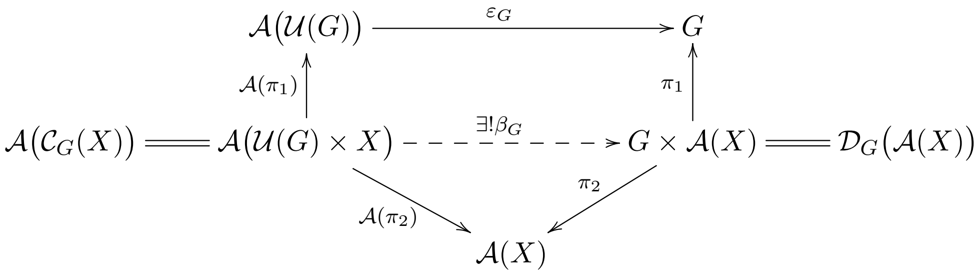

Figure 3. Construction of β

2.4. Pairing With an Abelian Group

Consider the functor that pairs elements of arbitrary sets X with elements of (the carrier of) a fixed abelian group G:

$$\mathcal{C}_G(X) = \mathcal{U}(G) \times X \qquad \mathcal{C}_G(f) = \mathrm{id}_{\mathcal{U}(G)} \times f$$

From the group structure of G (or actually any monoid), we obtain a simple monad, with unit η_G and multiplication μ_G:

$$\eta_G : 1 \Rightarrow \mathcal{C}_G \qquad \eta_{G,X}(x) = (e, x)$$
$$\mu_G : {\mathcal{C}_G}^2 \Rightarrow \mathcal{C}_G \qquad \mu_{G,X}\big(a, (b, x)\big) = (a \diamond b, x)$$

When the second component is changed to abelian groups as well, a corresponding functor on $\mathbf{Ab}$ is obtained, which creates the direct sum of abelian groups instead of the Cartesian product of carrier sets:

$$\mathcal{D}_G(H) = G \times H \qquad \mathcal{D}_G(f) = \mathrm{id}_G \times f$$

The two monads are related by the forgetful functor:

$$\mathcal{C}_G \mathcal{U} = \mathcal{U} \mathcal{D}_G$$

For the present discussion, there is no need to spell out particular adjunctions that give rise to these monads, nor their counits.

2.5. Monad Composition

There is a natural transformation in $\mathbf{Ab}$ that exchanges pairing and free abelian group construction:

$$\beta_G : \mathcal{A}\,\mathcal{C}_G \Rightarrow \mathcal{D}_G\,\mathcal{A} \qquad \beta_{G,X} = \big\langle \varepsilon_G \circ \mathcal{A}(\pi_1), \mathcal{A}(\pi_2) \big\rangle$$

Back down in $\mathbf{Set}$, this is a natural transformation

$$\mathcal{U}(\beta_G) = \beta_G : \mathrm{Zf}\,\mathcal{C}_G \Rightarrow \mathcal{C}_G\,\mathrm{Zf}$$

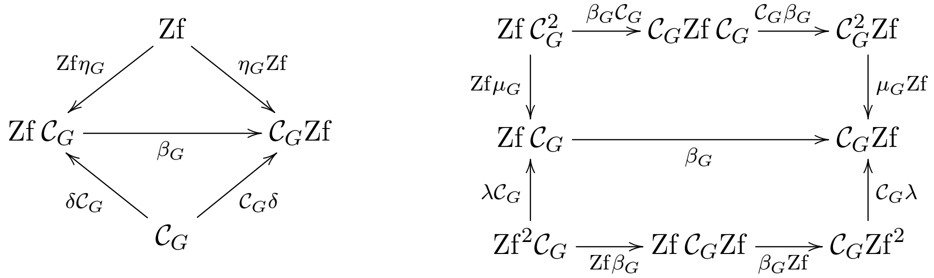

Figure 4. β is a Distributive Law

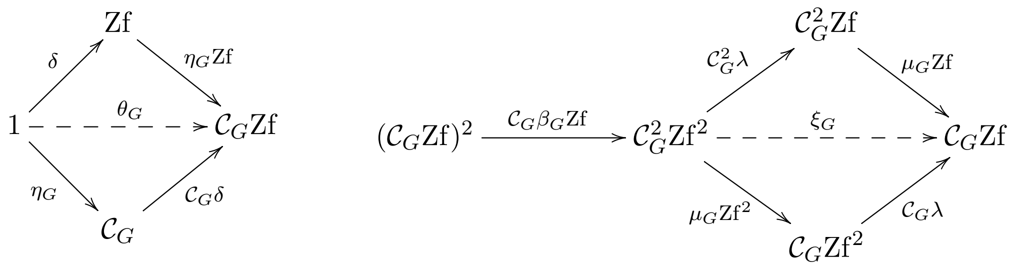

Figure 5. Construction of Composite Monad $\mathcal{C}_G\mathrm{Zf}$

which can be shown to be a distributive law between the monads $\mathcal{C}_G$ and Zf:

$$\beta_G \circ \delta\mathcal{C}_G = \mathcal{C}_G\delta \qquad \beta_G \circ \mathrm{Zf}\mu_G = \lambda\mathrm{Zf} \circ \mathcal{C}_G\beta_G \circ \beta_G\mathcal{C}_G$$
$$\beta_G \circ \mathrm{Zf}\eta = \eta\mathrm{Zf} \qquad \beta_G \circ \lambda\mathcal{C}_G = \mathcal{C}_G\mu_G \circ \beta_G\mathrm{Zf} \circ \mathrm{Zf}\beta_G$$

This turns the composite functor $\mathcal{C}_G\mathrm{Zf}$ into a monad, with unit θ_G and multiplication ξ_G:

$$\theta_G : 1 \Rightarrow \mathcal{C}_G\,\mathrm{Zf} \qquad \theta_G = \eta_G\delta$$
$$\xi_G : (\mathcal{C}_G\,\mathrm{Zf})^2 \Rightarrow \mathcal{C}_G\,\mathrm{Zf} \qquad \xi_G = \mu_G\lambda \circ \mathcal{C}_G\beta_G\mathrm{Zf}$$

2.6. Concrete Examples

The structures and operations introduced so far are all natural in the general categorial sense, but range from quite simple to rather nontrivial in complexity. To illustrate this, the following section gives concrete examples with actual elements.

In all of the following formulae, let $X = \{a, b, c, d, \dots\}$.

Consider two elements $g_1, g_2 \in \mathrm{Zf}(X)$. Thanks to the finite map notation, their support is clearly visible:

$$g_1 = \{a \mapsto 2, b \mapsto -1, c \mapsto 1\}_{/0} \qquad g_2 = \{b \mapsto 2, c \mapsto -1, d \mapsto -2\}_{/0}$$
$$\mathrm{supp}(g_1) = \{a, b, c\} \qquad \mathrm{supp}(g_2) = \{b, c, d\}$$

The group operation acts by pointwise addition, zeroes vanish in the notation:

$$g_1 g_2 = \{a \mapsto 2, b \mapsto 1, d \mapsto -2\}_{/0}$$

There is no general relationship between $\mathrm{supp}(g_1)$, $\mathrm{supp}(g_2)$ and $\mathrm{supp}(g_1 g_2)$, because of cancellation. Now consider a map $f : X \to X$, such that:

$$f(a) = f(b) = a \qquad f(c) = f(d) = b$$

Mapping f over group elements yields

$$\mathcal{A}(f)(g_1) = \{a \mapsto 1, b \mapsto 1\}_{/0} \qquad \mathcal{A}(f)(g_2) = \{a \mapsto 2, b \mapsto -3\}_{/0} \qquad \mathcal{A}(f)(g_1 g_2) = \{a \mapsto 3, b \mapsto -2\}_{/0} = \mathcal{A}(f)(g_1)\,\mathcal{A}(f)(g_2)$$

where the coefficients of identified elements are lumped together. Note that $\mathcal{A}(f)(g)$ is very different from (the contravariant) $g \circ f$. Monad multiplication takes integer-valued maps of integer-valued maps, distributes coefficients, and combines the results:

$$\lambda_X\big(\{g_1 \mapsto -2, g_2 \mapsto -1\}_{/0}\big) = g_1^{-2} g_2^{-1} = \{a \mapsto -4, c \mapsto -1, d \mapsto 2\}_{/0}$$

The counit takes an integer-valued map of group elements, and evaluates it as a group expression:

$$\varepsilon_{\mathrm{Qm}}\big(\underbrace{\{2 \mapsto -3, 3 \mapsto 2, \tfrac{2}{5} \mapsto -1\}}_{q_1}\big) = 2^{-3}\, 3^2 \left(\tfrac{2}{5}\right)^{-1} = \frac{45}{16}$$

Note how the integer coefficients in $\mathrm{Zf}(X)$ function as exponents when the target group is multiplicative. The operations of the pairing functor $\mathcal{C}_G$ are straightforward:

$$\eta_{\mathrm{Qm},X}(a) = (1, a) \quad \mu_{\mathrm{Qm},X}\big(2, (3, b)\big) = (2 \cdot 3, b) = (6, b) \quad \mathcal{C}_{\mathrm{Qm}}(f)(5, c) = \big(5, f(c)\big) = (5, b)$$

By contrast, the distributive law β functions in a fairly complex way, splitting the left hand sides of a group element map:

$$\beta_{\mathrm{Qm},X}\big(\{(2, a) \mapsto -3, (3, b) \mapsto 2, (\tfrac{2}{5}, c) \mapsto -1\}\big) = \big(\varepsilon_{\mathrm{Qm}}(q_1), \underbrace{\{a \mapsto -3, b \mapsto 2, c \mapsto -1\}}_{g_3}\big)$$

The operations of the composite monad $\mathcal{C}_G\mathrm{Zf}$ just combine the above.

$$\theta_{\mathrm{Qm}}(c) = \big(1, \{c \mapsto 1\}_{/0}\big)$$

$$\begin{aligned} &\xi_{\mathrm{Qm},X}\Big(2, \big\{\big(3, \{a \mapsto 5\}_{/0}\big) \mapsto -2, \big(7, \{b \mapsto -1\}_{/0}\big) \mapsto 1\big\}_{/0}\Big) \\ &= \big(2 \cdot 3^{-2} \cdot 7^1, \{a \mapsto 5 \cdot -2, b \mapsto -1 \cdot 1\}_{/0}\big) \\ &= \big(\tfrac{14}{9}, \{a \mapsto -10, b \mapsto -1\}_{/0}\big) \end{aligned}$$

3. Data Structures

3.1. Generic Structures

This section defines the structures of a formal model of dimensions and units. These are *generic*: no interpretation of base symbols is presupposed. This is achieved by δ, λ, ε, π, ψ, η, μ, β, θ, ξ all being *natural transformations*: They are parametric families of maps that transform data between *shapes* specified by functors, in a way that is logically independent of the elementary *content* specified by the type argument.

Definition 3.1. (Dimension)
The *dimensions* Dm are the free abelian group over $\mathrm{Dm_b}$, a given finite set of *base dimensions*:

$$\mathrm{Dm} = \mathcal{A}(\mathrm{Dm_b})$$

Example 3.2. (SI Dimensions)
The SI/ISQ recognizes seven physical base dimensions, $\mathrm{Dm}_\mathrm{b}^\mathrm{SI} = \{\mathsf{L}, \mathsf{T}, \mathsf{M}, \mathsf{I}, \Theta, \mathsf{N}, \mathsf{J}\}$. From these, compound dimensions can be formed; for example, quantities of *thermodynamical entropy* are associated with $d_H = \{\mathsf{L} \mapsto 2, \mathsf{M} \mapsto 1, \mathsf{T} \mapsto -2, \Theta \mapsto -1\}_{/0}$. However, the actual physical interpretation of these symbols need not be considered at all for formal analysis.

Remark 3.3. We do not use the product-of-powers notation of Lemma 2.1 for concrete elements. Neither do we recommend that use for formal analysis, because it is prone to causing subtle misunderstandings, in particular the highly overloaded expression for the empty case, $1 = \varnothing_{/0}$. Confer Section 1.1.1.

Definition 3.4. (Root Unit)
The *root units* $\mathrm{Ut_r}$ are the free abelian group over $\mathrm{Ut_b}$, a given finite set of *base units*:

$$\mathrm{Ut_r} = \mathcal{A}(\mathrm{Ut_b})$$

Thus, root units have a structure analogous to dimensions, compare Definition 3.1. The reason that they are called "root units" and not just "units" is that prefixes are not yet accounted for; see Definition 3.7 below.

Example 3.5. (Base Units and the SI)
We use the qualifier *base* always in a strictly syntactic sense, namely to denote the atomic symbols over which compound expressions, or group words, can be formed. By contrast, the SI/ISQ attaches to it a pragmatic meaning, namely to denote entities that require no definition by reduction to other, more basic entities. While these contending concepts overlap to some degree, there are notable caveats and exceptions in both directions.

On the one hand, [18] distinguishes one base unit per base dimension. Six of these are also base units in the formal syntactic sense. The seventh however, for historical reasons, is not: The distinguished unit of *mass* is the *kilogram* (kg), which is syntactically a compound derived from the simpler unit *gram* (g); see Example 3.11 below.

$$\mathrm{Ut}_\mathrm{b}^{\mathrm{SI}_0} = \{\mathrm{m}, \mathrm{s}, \mathrm{kg}, \mathrm{A}, \mathrm{K}, \mathrm{mol}, \mathrm{cd}\}$$

On the other hand, SI units that have both a dedicated symbol and a definition are not considered base units in the pragmatic sense, but *derived units with special names*. For example, the *ohm* is is associated with both the symbol Ω and the defining term $\mathrm{kg}{\cdot}\mathrm{m}^2{\cdot}\mathrm{s}^{-3}{\cdot}\mathrm{A}^{-2}$.

The devil is in the details: Before the major revision of SI definitions by the 26th CGPM in 2018, the SI standard stated that "the ohm, symbol Ω, is uniquely defined by the relation $\Omega = \mathrm{kg}{\cdot}\mathrm{m}^2{\cdot}\mathrm{s}^{-3}{\cdot}\mathrm{A}^{-2}$" [24, §2.1.1], implying that the two expressions are distinct, relatable entities. By contrast, the statement has been dropped from the later version [25]. There, the pragmatic foundation is discussed thus:

> "[...] this distinction is, in principle, not needed, since all units, *base* as well as *derived units*, may be constructed directly from the defining constants. Nevertheless, the concept of base and derived units is maintained because it is useful and historically well established [...]" [25, §2.3]

Compatibility with the ISQ, where base units are in one-to-one correspondence with base quantities and hence base dimensions, is also cited. See Example 4.24 and Section 4.2.2 for a formal discussion of the traditional base–derived dichotomy in the SI/ISQ.

Since the explanatory power of our model depends essentially on a sharp distinction between syntax and semantics, and a full abstraction from pragmatics, we hold that Ω is a base unit, whereas $\mathrm{kg}{\cdot}\mathrm{m}^2{\cdot}\mathrm{s}^{-3}{\cdot}\mathrm{A}^{-2}$ is a (derived/compound) root unit, and that the two can and must be related explicitly.

Example 3.6. (Root Units and SI Coherence)
The concept of root units is not explicit in the SI/ISQ, but a construct of our remodeling. Again, it overlaps partially with a concept found there, namely *coherent units*. And again, the differences are mostly due to a distinction between syntax and semantics.

Coherence has two alternative characterizations (semiformal definitions), and standards disagree about their relative logical precedence; the SI gives the first as a definition and the second as a corollary [25, §2.3.4], whereas the ISQ has them in reverse [18, §6]:

In the first, semantic approach, defined units are considered coherent, if their definition does not imply a numerical factor other than one. See Definition 3.25 and Remark 4.2 below for two distinct semantic sources of numerical factors.

In the second, syntactic approach, defined units are be considered coherent if the form of their definition has the same shape as the underlying compound quantities or dimensions. This notion can be formalized: Define a function $u : \mathrm{Dm_b} \to \mathrm{Ut_b}$ that chooses a distinguished base unit per base dimension, such as in [18, §6.5.3]. Let $d \in \mathrm{Dm}$ be the dimension associated with a quantity q. Then the coherent (root) unit for q is $\mathcal{A}(u)(d)$.

Unfortunately the latter, seemingly syntactic definition is clearly meant only up to semantic equivalence in the SI/ISQ: For example, the unit symbol Ω is listed as coherent, but is clearly not of the proposed form, whereas the representation of its intended definition as a root unit,

$$\{\mathrm{kg} \mapsto 1, \mathrm{m} \mapsto 2, \mathrm{s} \mapsto -3, \mathrm{A} \mapsto -2\}_{/0}$$

matches the shape of the dimension of the quantity *electric resistance*

$$\{\mathsf{M} \mapsto 1, \mathsf{L} \mapsto 2, \mathsf{T} \mapsto -3, \mathsf{I} \mapsto -2\}_{/0}$$

perfectly, given $u(\mathrm{kg}) = \mathsf{M}$ etc.

In order to disentangle syntax and semantics in our model, we shall use the novel concept of root units for the syntactic characterization, which is narrower than traditional coherence, and remodel the semantic aspect later; see Definition 4.14 below.

Definition 3.7. (Prefix)
The *prefixes* Px are the free abelian group over $\mathrm{Px_b}$, a given finite set of *base prefixes*:

$$\mathrm{Px} = \mathcal{A}(\mathrm{Px_b})$$

Example 3.8. (SI Prefixes)
Three families of prefixes are recognized in combination with the SI units. The symbols come with associated numerical values, discussed in detail in Section 3.2 below. For now it suffices to simply name them:

$$\mathrm{Px_b^{SI}} = \left\{ \begin{array}{l} \mathrm{d, c, m, \mu, n, p, f, a, z, y, r, q,} \\ \mathrm{da, h, k, M, G, T, P, E, Z, Y, R, Q,} \\ \mathrm{ki, Mi, Gi, Ti, Pi, Ei, Zi, Yi} \end{array} \right\}$$

The symbol m occurs as both a base prefix and a base unit in the SI vocabulary. However, this causes ambiguity issues only for parsing traditional notations, where prefix and unit symbols are simply concatenated: For example, ms^{-1} could parse either as *meter per second* or as *per millisecond*; see [18, §7.2.2]. The formal semantics discussed here are not affected. See also Example 3.24 below for unambiguous usage.

Remark 3.9. Combinations of multiple base prefixes are forbidden in modern scientific notation. However, there are good reasons to deviate from that rule:

- The empty prefix is always allowed, and modeled adequately as $\varnothing_{/0}$.
- Some traditional usages of *double* prefixes are known, and double prefixes beyond *quetta-* have recently been resuggested [3].
- The inner logic of each SI prefix family is a geometric sequence, confer Section 1.1.4 and Example 3.26 below.
- Last but not least, composite prefixes arise virtually in the algebra of composite units. For example, semantic analysis of the *centilitre* discussed in Section 1.1.2 by reduction to the coherent SI unit of volume, the *cubic meter*, involves the effective composite prefix $\mathrm{c\,d^3}$. For a formalization of the procedure, see Definitions 3.19 and 3.20 below.

Thus, the generalization of prefixes to the free abelian group is a theoretical simplification and unification.

Definition 3.10. (Preunit)
The *preunits* $\mathrm{Ut_p}$ are the Cartesian pairs of a prefix, forgetting the group structure, and a base unit:

$$\mathrm{Ut_p} = \mathcal{C}_{\mathrm{Px}}(\mathrm{Ut_b})$$

Example 3.11. The SI unit *kilogram*, already discussed above, is represented formally as a preunit, $\mathrm{kg} = \big(\delta_{\mathrm{Px_b}}(\mathrm{k}), \mathrm{g}\big)$.

Definition 3.12. (Unit)
The *units* Ut are the free abelian group over the preunits:

$$\mathrm{Ut} = \mathcal{A}(\mathrm{Ut_p})$$

Remark 3.13. Unlike in the preceding constructions, the generator set $\mathrm{Ut_p}$ is generally infinite. Fortunately, this causes no problems for the remainder of the present work; see Theorem 4.36 below.

Definition 3.14. (Embedding of Base Units)
Since the structure Ut arises from the composition of two monadic functors, the composition of monad unit maps is bound to occur often in its use. We write just the bracket $\lfloor\_\rfloor : \mathrm{Ut_b} \to \mathrm{Ut}$ for the horizontal composition of unit transformations, i.e., the precise but verbose map:

$$(\delta\eta_{\mathrm{Px}})_{\mathrm{Ut_b}} = \delta_{\mathrm{Ut_p}} \circ \eta_{\mathrm{Px},\mathrm{Ut_b}} = \mathrm{Zf}(\eta_{\mathrm{Px},\mathrm{Ut_b}}) \circ \delta_{\mathrm{Ut_b}}$$

More concretely, this boils down to the construction:

$$\lfloor x \rfloor = \{(\varnothing_{/0}, x) \mapsto 1\}_{/0}$$

In other occurrences, the subscripts of δ and η may be omitted in applications, and can be inferred from the context.

Example 3.15. There are additional so-called *derived* units in the SI, which have a base unit symbol of their own, but whose semantics are defined by reduction to other (composite) units, for example the *newton*, $\lfloor \mathrm{N} \rfloor \equiv \{\mathrm{kg} \mapsto 1, \eta(\mathrm{m}) \mapsto 1, \eta(\mathrm{s}) \mapsto -2\}_{/0}$, where $\equiv$ is some semantic equivalence relation yet to be specified; see Definitions 3.23, 3.35, 4.14 and 4.22 below.

Definition 3.16. (Prefix/Root of a Unit)
Any unit can be decomposed into a prefix and a root unit, and the root units embedded back into the units, by means of natural group homomorphisms:

$$\begin{aligned}
\mathrm{pref} &: \mathrm{Ut} \to \mathrm{Px} & \mathrm{pref} &= \pi_1 \circ \beta_{\mathrm{Px},\mathrm{Ut_b}} = \varepsilon_{\mathrm{Px}} \circ \mathcal{A}(\pi_1) \\
\mathrm{root} &: \mathrm{Ut} \to \mathrm{Ut_r} & \mathrm{root} &= \pi_2 \circ \beta_{\mathrm{Px},\mathrm{Ut_b}} = \mathcal{A}(\pi_2) \\
\mathrm{unroot} &: \mathrm{Ut_r} \to \mathrm{Ut} & \mathrm{unroot} &= \mathcal{A}(\eta_{\mathrm{Px},\mathrm{Ut_b}}) \\
\mathrm{strip} &: \mathrm{Ut} \to \mathrm{Ut} & \mathrm{strip} &= \mathrm{unroot} \circ \mathrm{root}
\end{aligned}$$

Lemma 3.17. The function root is the left inverse of unroot. Their flipped composition, the function strip, is idempotent.

Proof:

$$\mathrm{root} \circ \mathrm{unroot} = \mathcal{A}(\pi_2) \circ \mathcal{A}(\eta_{\mathrm{Px},\mathrm{Ut_b}}) = \mathcal{A}(\pi_2 \circ \eta_{\mathrm{Px},\mathrm{Ut_b}}) = \mathcal{A}(\mathrm{id}_{\mathrm{Ut_b}}) = \mathrm{id}_{\mathrm{Ut_r}}$$

$$\mathrm{strip} \circ \mathrm{strip} = \mathrm{unroot} \circ \underbrace{\mathrm{root} \circ \mathrm{unroot}}_{\mathrm{id}_{\mathrm{Ut_r}}} \circ \mathrm{root} = \mathrm{unroot} \circ \mathrm{root} = \mathrm{strip}$$

□

Definition 3.18. (Root Equivalence)
Two units are called *root equivalent*, written $\simeq_\mathrm{r}$, iff their roots coincide:

$$u \simeq_\mathrm{r} v \iff \mathrm{root}(u) = \mathrm{root}(v)$$

The units as defined above are a faithful semantic model of traditional notations, confer [25]. However, an algebraically more well-behaved structure can be derived by transposing prefixes and free abelian group construction:

Definition 3.19. (Normalized Unit)
The *normalized units* $\mathrm{Ut_n}$ are the direct sum of a prefix and a root unit:

$$\mathrm{Ut_n} = \mathcal{D}_{\mathrm{Px}}(\mathrm{Ut_r})$$

Definition 3.20. (Normalization)
The normalized units are derived naturally from the units proper, in terms of the natural group homomorphism β:

$$\mathrm{norm} : \mathrm{Ut} \to \mathrm{Ut_n} \qquad\qquad \mathrm{norm} = \beta_{\mathrm{Px},\mathrm{Ut_b}} = \langle \mathrm{pref}, \mathrm{root} \rangle$$

The composite monad structure of $\mathcal{U}(\mathrm{Ut_n}) = \mathcal{C}_{\mathrm{Px}}\mathrm{Zf}(\mathrm{Ut_b})$ comes with a multiplication $\xi_{\mathrm{Px},\mathrm{Ut_b}}$, and thus provides exactly the compositionality lacking in the traditional model Ut: In a normalized unit, other normalized units can be substituted for base units and just “multiplied out”. In Ut, this is formally impossible:

Theorem 3.21. The group $\mathrm{Ut} = \mathrm{Zf}\mathcal{C}_{\mathrm{Px}}(\mathrm{Ut_b})$ cannot be given a composite monadic structure in the same way as $\mathrm{Ut_n}$; except for degenerate cases, there is no distributive law $\mathcal{C}_G\,\mathrm{Zf} \Rightarrow \mathrm{Zf}\,\mathcal{C}_G$ that is also a group homomorphism.

Proof:
A distributive law $\tau_G : \mathcal{C}_G\,\mathrm{Zf} \Rightarrow \mathrm{Zf}\,\mathcal{C}_G$ of the monad Zf over $\mathcal{C}_G$ is s special case of a distributive law of the monad Zf over the endofunctor $\mathcal{U}(G) \times -$. The latter is exactly a *strength* for the monad Zf with respect to the Cartesian product in $\mathbf{Set}$. Such monad strengths are known to exist uniquely [15, §3.1]. They take the form

$$\lambda(x, y).\, T\big(\lambda\, z.\, (x, z)\big)(y)$$

for any $\mathbf{Set}$-monad T.

Proceed by contradiction. Let G be nontrivial. Pick some arbitrary $a \in \mathcal{U}(G)$ with $a \neq e$ and hence $a^2 \neq a$, and some $x \in X$. Now let $p = \big(a, \delta_X(x)\big)$, and hence $p^2 = \big(a^2, \delta_X(x)^2\big)$. Then we get, on the one hand,

$$\tau_G(p) = \delta_{\mathcal{U}(G) \times X}(a, x)$$

and hence

$$\tau_G(p)^2 = \delta_{\mathcal{U}(G)\times X}(a, x)^2$$

but on the other hand:

$$\tau_G(p^2) = \delta_{\mathcal{U}(G)\times X}(a^2, x)^2$$

If τ_G were a group homomorphism, then the latter two had to coincide; but that is not the case by Lemma 2.1. □

Remark 3.22. Normalization is generally not injective, since the actual distribution of partial prefixes is also "multiplied out", and forgotten. For example, *micrometer-per-microsecond* cancels to *meter-per-second*:

$$\mathrm{norm}\big(\{(\delta(\mu), \mathrm{m}) \mapsto 1, (\delta(\mu), \mathrm{s}) \mapsto -1\}_{/0}\big) = \mathrm{norm}\big(\{\eta(\mathrm{m}) \mapsto 1, \eta(\mathrm{s}) \mapsto -1\}_{/0}\big) = \theta(\mathrm{m})\,\theta(\mathrm{s})^{-1}$$

Definition 3.23. (Normal Equivalence)
Two units are called *normally equivalent*, written $\simeq_{\mathrm{n}}$, iff their normalizations coincide:

$$u \simeq_{\mathrm{n}} v \iff \mathrm{norm}(u) = \mathrm{norm}(v)$$

Example 3.24. (Precipitation)
Normalization is not part of the traditional notation of units. However, it has the beneficial property that entangled, redundant prefixes and base units are cancelled out orthogonally:

Consider a meteorological unit p for *amount of precipitation*, denoting *litre-per-square-meter* (L/m^2), where a *litre* (L) is defined as the third power of a *decimeter* (dm), which parses as a simple preunit analogous to kg; that is formally:

$$p = \{(\delta(\mathrm{d}), \mathrm{m}) \mapsto 3, \eta(\mathrm{m}) \mapsto -2\}_{/0}$$

By normalization, a root unit factor of $\delta(\mathrm{m})^2$ is cancelled out:

$$\mathrm{norm}(p) = \big(\delta(\mathrm{d})^3, \delta(\mathrm{m})\big)$$

Thus we find that p is normally equivalent to a *deci-deci-deci-meter*, but not to a *millimeter*;

$$p \simeq_{\mathrm{n}} \delta\big((\delta(\mathrm{d})^3, \mathrm{m})\big) \qquad\qquad p \not\simeq_{\mathrm{n}} \delta\big((\delta(\mathrm{m}), \mathrm{m})\big)$$

In the semantic structures presented so far, the base symbols are free; they stand only for themselves, operated on exclusively by natural transformations, and do not carry any attributes for comparison. As the last example has shown, the resulting notions of semantic equivalence may be narrower than intended. The following section introduces one such attribute each for prefixes and units, and explores the semantic consequences. Note that any actual assignment of attribute values is *contingent*; it could well be different in another possible world, i.e., system of units, whereas all of the preceding reasoning is logically *necessary*.

3.2. Specific Attributes

Definition 3.25. (Base Prefix Value)
Every base prefix shall be assigned a ratio as its numerical value.

$$\mathrm{val_b} : \mathrm{Px_b} \to \mathbb{Q}_+$$

Example 3.26. (SI Prefix Values)
The three families of SI prefixes are defined numerically as negative and positive (mostly triple) powers of ten, and positive (tenfold) powers of two, respectively: [7]

$$\mathrm{val_b^{SI}} = \left\{ \begin{array}{llllll} \mathrm{d} \mapsto 10^{-1}, & \mathrm{c} \mapsto 10^{-2}, & \mathrm{m} \mapsto 10^{-3}, & \mu \mapsto 10^{-6}, & \mathrm{n} \mapsto 10^{-9}, & \dots \\ \mathrm{da} \mapsto 10^{+1}, & \mathrm{h} \mapsto 10^{+2}, & \mathrm{k} \mapsto 10^{+3}, & \mathrm{M} \mapsto 10^{+6}, & \mathrm{G} \mapsto 10^{+9}, & \dots \\ \mathrm{ki} \mapsto 2^{+10}, & \mathrm{Mi} \mapsto 2^{+20}, & \mathrm{Gi} \mapsto 2^{+30}, & \dots & & \end{array} \right\}$$

etc.

Definition 3.27. (Base Unit Dimension)
Every base unit shall be assigned a (possibly compound) dimension.

$$\mathrm{dim_b} : \mathrm{Ut_b} \to \mathcal{U}(\mathrm{Dm})$$

Example 3.28. (SI Unit Dimensions)
The SI base units correspond to the SI base dimensions in the respective order presented above, that is

$$\mathrm{dim_b^{SI}}(\mathrm{m}) = \delta(\mathsf{L}) \qquad\qquad \mathrm{dim_b^{SI}}(\mathrm{s}) = \delta(\mathsf{T})$$

etc., whereas derived SI units may have more complex dimensions:

$$\mathrm{dim_b^{SI}}(\mathrm{N}) = \{\mathsf{L} \mapsto 1, \mathsf{T} \mapsto -2, \mathsf{M} \mapsto 1\}_{/0}$$

Definition 3.29. (Prefix Value)
Prefix value assignment lifts naturally to all concerned structures:

$$\begin{aligned} \mathrm{val} &: \mathrm{Px} \to \mathrm{Qm} & \mathrm{val} &= \varepsilon_{\mathrm{Qm}} \circ \mathcal{A}(\mathrm{val_b}) \\ \mathrm{pval_p} &: \mathrm{Ut_p} \to \mathbb{Q}_+ & \mathrm{pval_p} &= \mathrm{val} \circ \pi_1 \\ \mathrm{pval} &: \mathrm{Ut} \to \mathrm{Qm} & \mathrm{pval} &= \varepsilon_{\mathrm{Qm}} \circ \mathcal{A}(\mathrm{pval_p}) \\ \mathrm{pval_n} &: \mathrm{Ut_n} \to \mathrm{Qm} & \mathrm{pval_n} &= \mathrm{val} \circ \pi_1 \end{aligned}$$

[7] It is scientific standard to assign context-free numerical values to prefixes; some traditional notations do not follow the practice. For example, consider the popular (ab)use of *kilobyte* for 2^{10} bytes, which has given rise to the binary family for proper distinction.

Definition 3.30. (Unit Dimension)
Dimension assignment lifts naturally to all concerned structures:

$$\begin{aligned}
\mathrm{dim_r} &: \mathrm{Ut_r} \to \mathrm{Dm} & \mathrm{dim_r} &= \varepsilon_{\mathrm{Dm}} \circ \mathcal{A}(\mathrm{dim_b}) \\
\mathrm{dim_p} &: \mathrm{Ut_p} \to \mathcal{U}(\mathrm{Dm}) & \mathrm{dim_p} &= \mathrm{dim_b} \circ \pi_2 \\
\mathrm{dim} &: \mathrm{Ut} \to \mathrm{Dm} & \mathrm{dim} &= \varepsilon_{\mathrm{Dm}} \circ \mathcal{A}(\mathrm{dim_p}) \\
\mathrm{dim_n} &: \mathrm{Ut_n} \to \mathrm{Dm} & \mathrm{dim_n} &= \mathrm{dim_r} \circ \pi_2
\end{aligned}$$

Example 3.31. (Density)
Consider a unit of *mass density*, $\mathrm{kg/cm^3}$, that is formally

$$q = \big\{(\delta(\mathrm{k}), \mathrm{g}) \mapsto 1, (\delta(c), \mathrm{m}) \mapsto -3\big\}_{/0}$$

In the SI interpretation, it follows that

$$\mathrm{pval}(q) = 10^3 \cdot (10^{-2})^{-3} = 10^9 \qquad \mathrm{dim}(q) = \{\mathsf{L} \mapsto -3, \mathsf{M} \mapsto 1\}_{/0}$$

Definition 3.32. (Codimensionality)
Two (base, pre-, ...) units are called *codimensional*, written $\sim_\Box$, with the appropriate initial substituted for $\Box$, iff their assigned dimensions coincide:

$$u \sim_\Box v \iff \mathrm{dim}_\Box(u) = \mathrm{dim}_\Box(v)$$

Definition 3.33. (Evaluated Unit)
The *evaluated units* $\mathrm{Ut_e}$ are the direct sum of a ratio and a root unit:

$$\mathrm{Ut_e} = \mathcal{D}_{\mathrm{Qm}}(\mathrm{Ut_r})$$

Definition 3.34. (Evaluation)
Evaluation separates prefix value and root unit:

$$\begin{aligned}
\mathrm{eval} &: \mathrm{Ut} \to \mathrm{Ut_e} & \mathrm{eval} &= \langle \mathrm{pval}, \mathrm{root} \rangle = \mathrm{eval_n} \circ \mathrm{norm} \\
\mathrm{eval_n} &: \mathrm{Ut_n} \to \mathrm{Ut_e} & \mathrm{eval_n} &= \mathrm{val} \times \mathrm{id}_{\mathrm{Ut_r}}
\end{aligned}$$

Definition 3.35. (Numerical Equivalence)
Two units are called *numerically equivalent*, written $\simeq_\mathrm{e}$, iff their evaluations coincide:

$$u \simeq_\mathrm{e} v \iff \mathrm{eval}(u) = \mathrm{eval}(v)$$

Theorem 3.36. Normalization equivalence entails numerical equivalence, which entails root equivalence, which entails codimensionality:

$$u \simeq_\mathrm{n} v \implies u \simeq_\mathrm{e} v \implies u \simeq_\mathrm{r} \implies u \sim v$$

Proof:
The relations arise from maps by successive composition:

1. The relation $\simeq_{\mathrm{n}}$ is the kernel of the map norm.
2. The relation $\simeq_{\mathrm{e}}$ is the kernel of the map $\mathrm{eval} = (\mathrm{val} \times \mathrm{id}_{\mathrm{Ut_r}}) \circ \mathrm{norm}$.
3. The relation $\simeq_{\mathrm{r}}$ is the kernel of the map $\mathrm{root} = \pi_2 \circ \mathrm{eval}$.
4. The relation $\sim$ is the kernel of the map $\mathrm{dim} = \mathrm{dim_r} \circ \mathrm{root}$.

Thus the kernel relations increase with each step. □

Numerical equivalence is coarser than normalization equivalence, because it uses an additional source of information: Whereas the latter depends only on the universal group structures of prefixes and units, the former takes contingent value assignments ($\mathrm{val_b}$) into account.

Example 3.37. (Precipitation revisited)
Continuing Example 3.24, we find that, given the SI interpretation of prefix values, the precipitation unit p is indeed numerically equivalent to the *millimeter*:

$$\mathrm{eval}^{\mathrm{SI}}(p) = \bigl(10^{-3}, \delta(\mathrm{m})\bigr) = \mathrm{eval}^{\mathrm{SI}}\Bigl(\delta\bigl(\bigl(\delta(\mathrm{m}), \mathrm{m}\bigr)\bigr)\Bigr) \implies p \simeq_{\mathrm{e}} \delta\bigl(\bigl(\delta(\mathrm{m}), \mathrm{m}\bigr)\bigr)$$

So far, three points on the syntax–semantics axis have been discussed, namely units proper as the formal model of traditional syntax, and normalized and evaluated units as two steps of semantic interpretation, giving rise to successively coarser equivalence relations. Evaluated units are the coarsest interpretation that is arguably adequate in general practice. For the sake of critical argument, a further semantic interpretation step shall be considered. That one fits the logical framework as well, see Figure 6 below, but is too coarse for sound reasoning in most cases.

Definition 3.38. (Abstract Unit)
The *abstract units* $\mathrm{Ut_a}$ are the direct sum of a ratio and a dimension:

$$\mathrm{Ut_a} = \mathcal{D}_{\mathrm{Qm}}(\mathrm{Dm})$$

Definition 3.39. (Abstraction)
Abstraction separates prefix value and dimension assignments:

$$\begin{aligned}
&\mathrm{abs} : \mathrm{Ut} \to \mathrm{Ut_a} && \mathrm{abs} = \langle \mathrm{pval}, \mathrm{dim} \rangle = \mathrm{abs_n} \circ \mathrm{norm} \\
&\mathrm{abs_n} : \mathrm{Ut_n} \to \mathrm{Ut_a} && \mathrm{abs_n} = \langle \mathrm{pval_n}, \mathrm{dim_n} \rangle = \mathrm{val} \times \mathrm{dim_r} = \mathrm{abs_e} \circ \mathrm{eval_n} \\
&\mathrm{abs_e} : \mathrm{Ut_e} \to \mathrm{Ut_a} && \mathrm{abs_e} = \mathcal{D}_{\mathrm{Qm}}(\mathrm{dim_r})
\end{aligned}$$

Abstract units arise in the remodeling of theoretical works that assert one canonical unit per dimension [2, 10, e.g.]; see also Section 5.2.5 below.

Example 3.40. (Pecipitation Abstracted)
The precipitation unit p abstracts to

$$\mathrm{abs}^{\mathrm{SI}}(p) = \bigl(\mathrm{pval}^{\mathrm{SI}}(p), \mathrm{dim}^{\mathrm{SI}}(p)\bigr) = (10^{-3}, \mathsf{L})$$

which means “one thousandth of the coherent SI unit of length”.

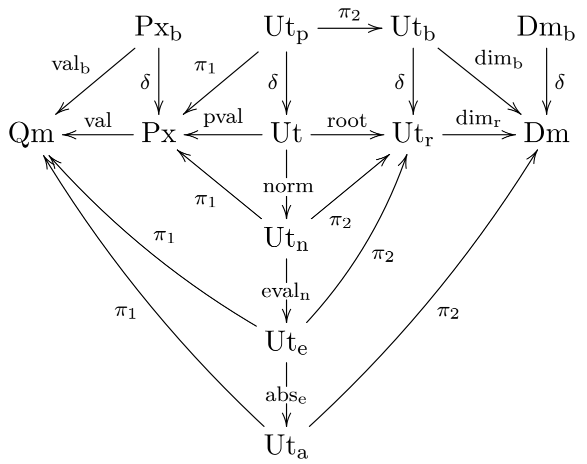

Figure 6. Overview of Model Structures

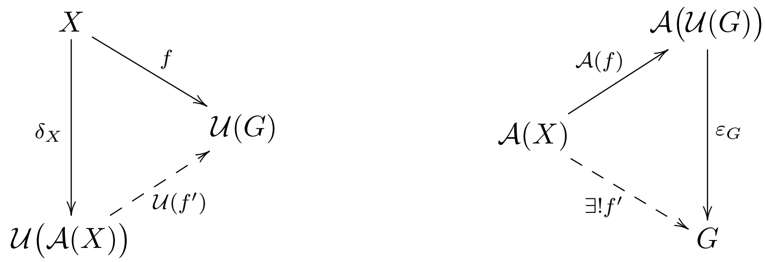

Figure 7. Correspondence Between $\mathbf{Set}$ and $\mathbf{Ab}$ by Adjunction

3.3. The Big Picture

Figure 6 depicts all mathematical structures defined in the preceding sections and their relating operations in a single diagram. The diagramm commutes, such that, for every pair of objects, all paths from one to the other denote the same function, namely *the* model operation of that type signature.

The depiction abuses notation mildly for the sake of conciseness; the top row and the rest of the diagram live in categories $\mathbf{Set}$ and $\mathbf{Ab}$, respectively. However, the perspective shift is easily explained: The adjunction of functors $\mathcal{A} \dashv \mathcal{U}$ ensures that, for any map f that takes values from the carrier of a group G, there exists a unique homomorphic extension f' to the free abelian group over the domain, i.e., the corresponding diagrams commute simultaneously in $\mathbf{Set}$ and $\mathbf{Ab}$, respectively; see Figure 7.

All horizontal arrows in the second row of Figure 6 arise from this principle (Definitions 3.29, 3.30). The syntactic model Ut is in the center, its semantics projections extend to the left and right. The two independent sources of contingent data are in the top left and right corner, respectively; all other operations arise from adjoint functors and their natural transformations. Extending downward from the center is a sequence of componentwise semantic interpretations. Named operations that are just compositions of others are not shown.

4. Unit Conversion Relations

In the following sections, we deal with ternary relations of a particular type, namely the carrier of a direct sum of three groups:

$$\mathrm{Conv} = \mathrm{Ut} \times \mathrm{Qm} \times \mathrm{Ut} \qquad\qquad \mathcal{U}(\mathrm{Conv}) = \mathrm{Zf}(\mathrm{Ut_p}) \times \mathbb{Q}_+ \times \mathrm{Zf}(\mathrm{Ut_p})$$

The middle component is called a *conversion ratio*. We shall write $u \xrightarrow{r}_R v$ for $(u, r, v) \in R$, a notation that alludes to categorial diagrams; see Theorem 4.8 below for justification. As with diagrams, the subscript R shall be omitted where it is obvious from the context, and adjacent triples $u \xrightarrow{r} v, v \xrightarrow{s} w$ shortened to $u \xrightarrow{r} v \xrightarrow{s} w$.

4.1. Unit Conversions and Closures

Definition 4.1. (Unit Conversion)
A relation $C \subseteq \mathcal{U}(\mathrm{Conv})$ is called a *(unit) conversion*, iff it is dimensionally consistent and functional in its arguments of unit type:

$$u \xrightarrow{r}_C v \implies u \sim v \tag{1a}$$

$$u \xrightarrow{r}_C v \wedge u \xrightarrow{r'}_C v \implies r = r' \tag{1b}$$

Remark 4.2. A judgement $u \xrightarrow{r}_C v$ means "One u is r vs," and hence introduces an algebraic rewriting rule for quantity values that converts (values measured in) u to v, since multiplication with the conversion factor r is taken as associative:

$$x\,u = x\,(r\,v) = (xr)\,v \tag{2}$$

Example 4.3. (UK Units)
Standards in the United Kingdom define the regional customary units *pound* (lb) and *pint* (pt) with the following conversion rules:

$$\lfloor \mathrm{lb} \rfloor \xrightarrow{453.592\,37} \lfloor \mathrm{g} \rfloor \qquad\qquad \lfloor \mathrm{pt} \rfloor \xrightarrow{568.261\,25} \delta\big((\delta(\mathrm{c}), \mathrm{m})\big)^3$$

Definition 4.4. (Conversion Closure)
Let $C \subseteq \mathcal{U}(\mathrm{Conv})$ be a unit conversion. Its *(conversion) closure* is the smallest relation C^* with $C \subseteq C^* \subseteq \mathcal{U}(\mathrm{Conv})$ such that the following axioms hold:

$$u \xrightarrow{r}_{C^*} v \wedge u' \xrightarrow{r'}_{C^*} v' \implies uu' \xrightarrow{rr'}_{C^*} vv' \tag{3a}$$

$$u \xrightarrow{r}_{C^*} v \implies u^{-1} \xrightarrow{r^{-1}}_{C^*} v^{-1} \tag{3b}$$

$$u \xrightarrow{\mathrm{pval}(u)}_{C^*} \mathrm{strip}(u) \xrightarrow{\mathrm{pval}(u)^{-1}}_{C^*} u \tag{3c}$$

Remark 4.5. The closure of a unit conversion may fail to be a unit conversion itself, since contradictory factors can arise from closure axioms, violating (1b). For example, take both $u \xrightarrow{2}_C v$ and $u^{-1} \xrightarrow{3}_C v^{-1}$, but clearly $2 \neq 3^{-1}$.

Remark 4.6. Closure ensures operational *completeness* of the reasoning, namely that all potential rewriting rules for composite units implied by rules for their constituents are actually available. For example, if we know how to convert from *furlongs* to *meters*, and from *seconds* to *fortnights*, then we can deduce how to convert from *furlongs-per-fortnight* to *millimeters-per-second*.

Conversion closures have a rich algebraic structure:

Theorem 4.7. The closure of a unit conversion forms a subgroup of Conv.

Proof:
Axioms (3a) and (3b) state directly that a unit conversion closure is closed under the direct sum group operation and inversion. The closure is nonempty by virtue of (3c); at least the triple $\varnothing_{/0} \xrightarrow{1} \varnothing_{/0}$ is always contained. Any subset of a group with these three properties is a subgroup. □

Theorem 4.8. The closure of a unit conversion forms an invertible category or groupoid, the categorial generalization both of a group and of an equivalence relation, with unit objects and conversion factor morphisms:

$$u \xrightarrow{1}_{C^*} u \tag{4a}$$

$$u \xrightarrow{r}_{C^*} v \xrightarrow{s}_{C^*} w \implies u \xrightarrow{rs}_{C^*} w \tag{4b}$$

$$u \xrightarrow{r}_{C^*} v \implies v \xrightarrow{r^{-1}}_{C^*} u \tag{4c}$$

Proof:

- For (4a): By (3c) we have $u \xrightarrow{\mathrm{pval}(u)}_{C^*} \mathrm{strip}(u)$ and $\mathrm{strip}(u) \xrightarrow{\mathrm{pval}(u)^{-1}}_{C^*} u$. Multiplication via (3a) yields:

  $$u\,\mathrm{strip}(u) \xrightarrow{1}_{C^*} u\,\mathrm{strip}(u)$$

  By substitution of $\mathrm{strip}(u)$ for u in (3c), we also obtain $\mathrm{strip}(u) \xrightarrow{\mathrm{pval}(\mathrm{strip}(u))}_{C^*} \mathrm{strip}(\mathrm{strip}(u))$, which simplifies via Lemma 3.17 to $\mathrm{strip}(u) \xrightarrow{1}_{C^*} \mathrm{strip}(u)$, and inversion via (3b) yields:

  $$\mathrm{strip}(u)^{-1} \xrightarrow{1}_{C^*} \mathrm{strip}(u)^{-1}$$

  Multiplication of both via (3a) concludes:

  $$u \xrightarrow{1}_{C^*} u$$

- For (4b) assume $u \xrightarrow{r}_{C^*} v$ and $v \xrightarrow{s}_{C^*} w$. Multiplication via (3a) yields:

$$uv \xrightarrow{rs}_{C^*} vw$$

By substitution of v^{-1} for u in (4a), we also obtain $v^{-1} \xrightarrow{1}_{C^*} v^{-1}$. Multiplication of both via (3a) concludes:

$$u \xrightarrow{rs}_{C^*} w$$

- For (4c) assume $u \xrightarrow{r}_{C^*} v$. Inversion via (3b) yields:

$$u^{-1} \xrightarrow{r^{-1}}_{C^*} v^{-1}$$

By (4a), used both directly and substituting v for u, we also obtain $u^{-1} \xrightarrow{1}_{C^*} u^{-1}$ and $v^{-1} \xrightarrow{1}_{C^*} v^{-1}$, respectively. Multiplication of all three via (3a) concludes:

$$v \xrightarrow{r^{-1}}_{C^*} u$$

□

Remark 4.9. The closure of a unit conversion is again a conversion iff it is *thin* as a category. Such relations shall take center stage in the next section.

Theorem 4.10. Conversions can be decomposed and partitioned by dimension:

$$\mathrm{Ut}_{(d)} = \{u \in \mathrm{Ut} \mid \dim(u) = d\} \subseteq \mathcal{U}(\mathrm{Ut}) \qquad C = \bigcup_{d \in \mathrm{Dm}} C_{(d)}$$
$$C_{(d)} = C \cap (\mathrm{Ut}_{(d)} \times \mathbb{Q}_+ \times \mathrm{Ut}_{(d)})$$

Proof: Follows directly from axiom (1a). □

Remark 4.11. A non-converting theory of units, such as [10], is characterized by being partitioned into *trivial* subgroups. This demonstrates that the theory being presented here is a complementary extension to previous work.

Remark 4.12. Closure cannot be performed on the partitions individually, mostly because of axiom (3a) that allows for multiplication of units with orthogonal dimensions. Thus it is generally the case that:

$$C^* \neq \bigcup_{d \in \mathrm{Dm}} {C_{(d)}}^*$$

Definition 4.13. (Unit Convertibility)
Two units are called *convertible*, with respect to a unit conversion C, written $\propto_C$, iff they are related by some factor:

$$u \propto_C v \iff u \xrightarrow{\exists r}_C v$$

Definition 4.14. (Unit Coherence)
Two units are called *coherent*, with respect to a unit conversion C, written $\cong_C$, iff they are related by the factor one:

$$u \cong_C v \iff u \xrightarrow{1}_C v$$

Theorem 4.15. Coherence entails convertibility, which entails codimensionality:

$$u \cong_C v \implies u \propto_C v \implies u \sim v$$

Proof: Follows directly from the definitions and axiom (1a). □

Definition 4.16. (Conversion Coherence)
By extension, a unit conversion C is called *coherent* iff all units that are convertible with respect to C are coherent:

$$u \cong_C v \iff u \propto_C v$$

Remark 4.17. The notion of coherence given here is a formal remodeling of the one used by the SI; recall Example 3.6. We take the semantic characterization, namely a conversion factor of one, as the actual definition. The pseudo-syntactic characterization shall be recovered later as a result for *regular* conversions, that are algebraically complete but otherwise sparse; see Theorem 4.35.

Example 4.18. (SI Unit Coherence)
The conversion relation of the seven canonical base units of the SI is trivially coherent in the more general sense, since they are pairwise inconvertible. In addition, the SI recognizes 22 derived units with coherent conversion rules, see Section 4.2.2. By contrast, many traditional units, such as the *hour* (h), are convertible but not coherently so: $\mathrm{h} \xrightarrow{3600} \mathrm{s}$.

4.2. Defining Unit Conversions and Rewriting

Interesting subclasses of conversions arise as the closures of syntactically restricted generators.

Definition 4.19. (Defining Conversion)
A unit conversion C is called *defining* iff it is basic and functional in its first component:

$$u \xrightarrow{r}_C v \implies \exists\, u_0.\, u = \lfloor u_0 \rfloor \tag{5a}$$

$$u \xrightarrow{r}_C v \wedge u \xrightarrow{r'}_C v' \implies v = v' \tag{5b}$$

Definition 4.20. (Definition Expansion)
Every defining unit conversion C gives rise to a totalized *expansion* function:

$$\mathrm{xpd}(C) : \mathrm{Ut}_\mathrm{b} \to \mathcal{C}_\mathrm{Qm}(\mathrm{Ut}) \qquad\qquad \mathrm{xpd}(C)(u_0) = \begin{cases} (r, v) & \text{if } \lfloor u_0 \rfloor \xrightarrow{r}_C v \\ \eta(\lfloor u_0 \rfloor) & \text{if no match} \end{cases}$$

This in turn gives rise to a mapping of base to evaluated units, and ultimately to an iterable rewriting operation on evaluated units:

$$\mathrm{rwr_b}(C) : \mathrm{Ut_b} \to \mathrm{Ut_e} \qquad \mathrm{rwr_b}(C) = \mu_{\mathrm{Qm},\mathrm{Ut_r}} \circ \mathcal{C}_{\mathrm{Qm}}(\mathrm{eval}) \circ \mathrm{xpd}(C)$$
$$\mathrm{rwr_e}(C) : \mathrm{Ut_e} \to \mathrm{Ut_e} \qquad \mathrm{rwr_e}(C) = \xi_{\mathrm{Qm},\mathrm{Ut_b}} \circ \mathcal{C}_{\mathrm{Qm}}\mathrm{Zf}\big(\mathrm{rwr_b}(C)\big)$$

Definition 4.21. (Dependency Order)
Let C be a defining unit conversion. The relation $(>_C) \subseteq \mathcal{U}(\mathrm{Ut_b})^2$ is the smallest transitive relation such that:

$$\lfloor u_0 \rfloor \propto_C v \wedge v_0 \in \mathrm{supp}\big(\mathrm{root}(v)\big) \implies u_0 >_C v_0$$

We say that, with respect to C, u_0 depends on v_0.

Definition 4.22. (Well-Defining Conversion)
A defining unit conversion C is called *well-defining* iff its dependency order $>_C$ is well-founded. Since $\mathrm{Ut_b}$ is finite, this is already the case if $>_C$ is antireflexive.

Theorem 4.23. Let C be a well-defining conversion. The iteration of $\mathrm{rwr_e}(C)$ has a fixed point, which is reached after a number N_C of steps that is bounded by the number of distinct base units, and independent of the input unit:

$$\lim_{n\to\infty} \mathrm{rwr_e}(C)^n = \mathrm{rwr_e}(C)^{N_C \le |\mathrm{Ut_b}|}$$

The proof requires some auxiliary machinery, and is given in Section 4.2.1 below.

Example 4.24. (SI Derived Units)
Traditional systems of units, including the SI, follow a well-defining approach: Starting from an irreducible set of base units, additional "derived" base units are added in a stratified way, by defining them as convertible to (expressions over) preexisting ones. The SI ontology is rather complex; the details are discussed in Section 4.2.2 below.

4.2.1. Rewriting Termination

Definition 4.25. (Unit Depth)
Let C be a well-defining conversion. The *domain* of C is the set of base units occurring in its first component.

$$\mathrm{dom}(C) = \big\{u_0 \mid \lfloor u_0 \rfloor \xrightarrow{\exists r}_C \exists v\big\}$$

The *depth* of a (base/root/evaluated) unit is defined by well-founded recursion as follows:

$$d_{\mathrm{b}}^C : \mathrm{Ut_b} \to \mathbb{N} \qquad d_{\mathrm{b}}^C(u_0) = \begin{cases} 1 + \sup\{d_{\mathrm{b}}^C(v_0) \mid u_0 >_C v_0\} & u_0 \in \mathrm{dom}(C) \\ 0 & \text{otherwise} \end{cases}$$
$$d_{\mathrm{r}}^C : \mathrm{Ut_r} \to \mathbb{N} \qquad d_{\mathrm{r}}^C(u) = \sup\{d_{\mathrm{b}}^C(u_0) \mid u_0 \in \mathrm{supp}(u)\}$$
$$d_{\mathrm{e}}^C : \mathrm{Ut_e} \to \mathbb{N} \qquad d_{\mathrm{e}}^C = d_{\mathrm{r}}^C \circ \pi_2$$

Note that $\sup \varnothing = 0$, such that a base unit u_0 with $\lfloor u_0 \rfloor \xrightarrow{r}_C \varnothing_{/0}$ (for example, the *dozen* with $r = 12$) has depth 1, not 0.

Lemma 4.26. Base unit depth is monotonic with respect to dependency.

$$u_0 >_C v_0 \implies d_\mathrm{b}^C(u_0) > d_\mathrm{b}^C(v_0)$$

Proof: Follows directly from the definition. □

Lemma 4.27. Evaluated unit depth is linearly bounded.

$$d_\mathrm{e}^C(u) \leq |\mathrm{Ut_b}|$$

Proof: Follows from $>_C$ being transitive and antireflexive, hence cycle-free. □

Lemma 4.28. Evaluated unit depth is a termination function for rewriting.

$$d_\mathrm{e}^C(u) = 0 \implies \mathrm{rwr_e}(C)(u) = u$$
$$d_\mathrm{e}^C(u) > 0 \implies d_\mathrm{e}^C(u) > d_\mathrm{e}^C\big(\mathrm{rwr_e}(C)(u)\big)$$

Proof:

1. Assume that $d_\mathrm{e}^C(u) = 0$. Expand $\pi_2(u) = \delta(u_1)^{z_1} \cdots \delta(u_n)^{z_n}$ by Lemma 2.1. A longish but straightforward calculation, mostly with natural transformations, yields $\mathrm{rwr_e}(C)(u) = u$.
2. Assume that $n = d_\mathrm{e}(u) > 0$. Then $M_u^C = \{v_0 \in \mathrm{supp}(\pi_2(u)) \mid d_\mathrm{b}^C(v_0) = n - 1\}$ are the base units occuring in u that have maximal depth. Each of these is replaced in $\mathrm{rwr_e}(C)(u)$ by base units of strictly lesser depth. Thus $d_\mathrm{e}^C(\mathrm{rwr_e}(C)(u)) \leq n - 1$.

□

Proof: (Theorem 4.23)
The depth function d_e^C provides a bound on the number of iterations of $\mathrm{rwr_e}(C)$ required to fix a particular element. Namely, from Lemma 4.28 it follows that:

$$n \geq d_\mathrm{e}^C(u) \implies {h_C}^n(u) = u$$

By maxing over all possible base unit depths we obtain a global bound.

$$N_C = \max\{d_\mathrm{b}^C(u_0) \mid u_0 \in \mathrm{Ut_b}\} \qquad\qquad d_\mathrm{e}^C(u) \leq N_C$$

It follows that

$$n \geq N_C \implies \mathrm{rwr_e}(C)^n(u) = u$$

and, by Lemma 4.27

$$n \geq |\mathrm{Ut_b}| \implies {h_C}^n(u) = u$$

□

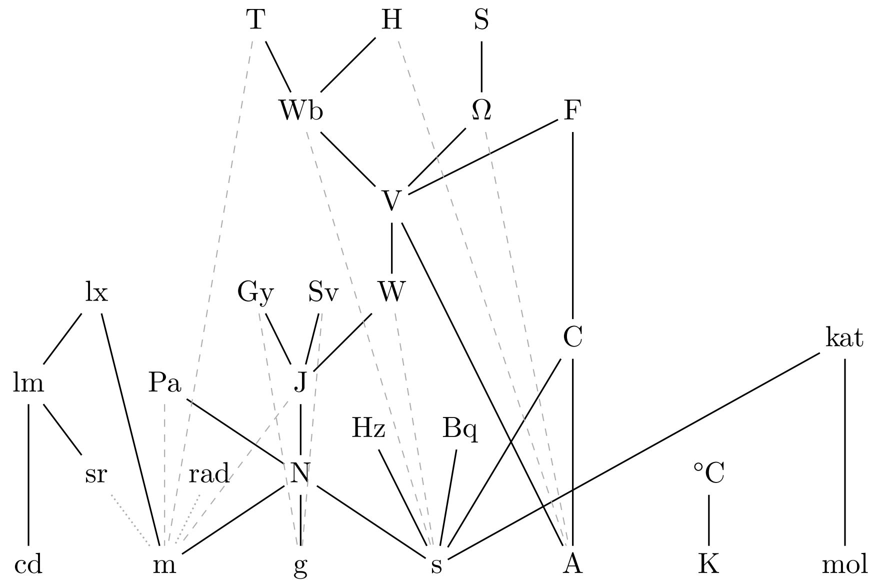

Figure 8. Hasse diagram ($>_{\mathrm{SI}}$) of coherent SI base units. Light dashed lines indicate dependencies that are explicit in the definition, but omitted in the transitive reduct. Dotted lines indicate pathological cases.

4.2.2. Extended Example: SI Derived Units

This section gives an overview of the named coherent SI units, namely the seven irreducible base units and 22 "derived units with special names". In terms of the model presented here, of course all 29 are base units, and being irreducible is synonymous with being $>_C$-minimal.

Table 1 lists the standardized names and reductionistic definitions of all "derived coherent SI units with special names" [25]. The definitions are usually given as equations, which is quite problematic: Two terms being equated, without the possibility to substitute one for the other, violates the principle of the indiscernibility of identicals. For easy reference, all tabulated definitions have been normalized, and then expanded in terms of exponents of the relevant base prefix (k) and the seven irreducible base units.

In the model presented here, the table forms a well-defining coherent conversion. Figure 8 depicts the corresponding dependency relation $>_C$ in terms of a Hasse diagram. Dotted lines indicate pathological cases to be discussed in Section 5.2.3. Compare also [26].

Table 1. SI derived units with special names [18]

Symbol	**Definition**	**Normalization Exponents**							
		pref	root						
		k	m	g	s	A	K	mol	cd
rad	m/m *	0	0	0	0	0	0	0	0
sr	m^2/m^2 *	0	0	0	0	0	0	0	0
Hz	s^{-1}	0	0	0	−1	0	0	0	0
N	$kg{\cdot}m/s^2$	+1	+1	+1	−2	0	0	0	0
Pa	N/m^2	+1	−1	+1	−2	0	0	0	0
J	N·m	+1	+2	+1	−2	0	0	0	0
W	J/s	+1	+2	+1	−3	0	0	0	0
C	A·s	0	0	0	+1	+1	0	0	0
V	W/A	+1	+2	+1	−3	−1	0	0	0
F	C/V	−1	−2	−1	+4	+2	0	0	0
Ω	V/A	+1	+2	+1	−3	−2	0	0	0
S	Ω^{-1}	−1	−2	−1	+3	+2	0	0	0
Wb	V·s	+1	+2	+1	−2	−1	0	0	0
T	Wb/m^2	+1	0	+1	−2	−1	0	0	0
H	Wb/A	+1	+2	+1	−2	−2	0	0	0
°C	K	0	0	0	0	0	+1	0	0
lm	cd·sr	0	0	0	0	0	0	0	+1
lx	lm/m^2	0	−2	0	0	0	0	0	+1
Bq	s^{-1}	0	0	0	−1	0	0	0	0
Gy	J/kg	0	+2	0	−2	0	0	0	0
Sv	J/kg	0	+2	0	−2	0	0	0	0
kat	mol/s	0	0	0	−1	0	0	+1	0

4.3. The Conversion Hierarchy

Definition 4.29. (Conversion Hierarchy)
A conversion is called ...

1. *consistent* iff its closure is again a conversion;
2. *closed* iff it is its own closure;
3. *finitely generated* iff it is the closure of a finite conversion;
4. *defined* iff it is the closure of a defining conversion;
5. *well-defined* iff it is the closure of a well-defining conversion;
6. *regular* iff it is the closure of an empty conversion.

Theorem 4.30. Each property in the conversion hierarchy entails the preceding.

Proof:

1. *Closed entails consistent.* – If $C = C^*$, and C is a conversion, then evidently so is C^*.
2. *Finitely generated entails closed.* – The closure operator is idempotent. Thus, if $C = B^*$, where B is irrelevantly finite, then also $B^* = (B^*)^* = C^*$.
3. *Defined entails finitely generated.* – If $C = B^*$ and B is defining, then the cardinality of B is bounded by the cardinality of $\mathrm{Ut_b}$, which is finite.
4. *Well-defined entails defined.* – If $C = B^*$ and B is well-defining, then B is also defining.
5. *Regular entails well-defined.* – If $C = B^*$ and $B = \varnothing$, then B is vacuously well-defining.

□

Theorem 4.31. Consistency is non-local; namely the following three statements are equivalent in the closure of a conversion C:

a. contradictory factors exist for some pair of units;

b. contradictory factors exist for all convertible pairs of units;

c. $\varnothing_{/0} \xrightarrow{\neq 1}_{C^*} \varnothing_{/0}$.

Proof:
By circular implication. Assume (a), i.e., both $u \xrightarrow{r_1}_{C^*} v$ and $u \xrightarrow{r_2}_{C^*} v$ with $r_1 \neq r_2$. Multiplication via (3a) of the former with the inverse by (3b) of the latter yields:

$$\varnothing_{/0} \xrightarrow{r_1 r_2^{-1}}_{C^*} \varnothing_{/0}$$

with $s = r_1 r_2^{-1} \neq 1$, which is (c). Now let $u' \propto_{C^*} v'$ be any convertible pair, i.e.

$$u' \xrightarrow{\exists t_1}_{C^*} v'$$

Multiplication via (3a) with the preceding yields:

$$u' \xrightarrow{st_1}_{C^*} v'$$

with $st_1 \neq t_1$, which is (b). This in turn implies (a) by nonemptiness. □

Theorem 4.32. For closed conversions, convertibility and coherence are group congruence relations.

Proof:
That $\propto_C$ and $\cong_C$ are equivalences follows directly from Theorem 4.8, by forgetting the morphisms. Any equivalence that is also a subgroup of the direct sum (Theorem 4.7) is a congruence. □

Theorem 4.33. For closed conversions, the resulting category is strict dagger compact.

Proof:
A dagger compact category combines several structures. Firstly, it is symmetric monoidal, i.e., there is a bifunctor $\otimes$ and an object I such that:

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \qquad I \otimes A \cong A \cong A \otimes I \qquad A \otimes B \cong B \otimes A$$

In the general case, all isomorphisms $\cong$ are named natural transformations subject to additional coherence conditions, but in the strict case they are identities, and hence no extra laws are required. Here, the abelian group structures of Ut, Qm provide the operations:

$$u \otimes v = uv \qquad r \otimes s = rs \qquad I = \varnothing_{/0}$$

Secondly, there is a contravariant involutive functor † that is compatible with the above:

$$f : A \to B \implies f^\dagger : B \to A \qquad (f^\dagger)^\dagger = f \qquad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$$

Here, this is given by the inverse in Qm:

$$r^\dagger = r^{-1}$$

Lastly, the category is compact closed, i.e., endowed with a dual $A^\star$ of each object A, and natural transformations (not necessarily isomorphisms):

$$A \otimes A^\star \Rightarrow I \Rightarrow A^\star \otimes A$$

In the general case, these are again subject to more coherence conditions, but in the strict case they are simply identities. Here, the dual is given by the inverse in Ut:

$$u^\star = u^{-1}$$

□

Theorem 4.34. In a closed conversion C, any two units are convertible if they are root equivalent, and coherent if they are numerically equivalent.

$$u \simeq_{\mathrm{r}} v \implies u \xrightarrow{\mathrm{pval}(u)\,\mathrm{pval}(v)^{-1}}_C v \implies u \propto_C v \qquad\qquad u \simeq_{\mathrm{e}} v \implies u \cong_C v$$

Proof:
Let C be a closed conversion. Assume $u \simeq_{\mathrm{r}} v$. Unfolding the definition of $(\simeq_{\mathrm{r}})$ and applying unroot to both sides yields:

$$\mathrm{strip}(u) = \mathrm{strip}(v)$$

The two halves of (3c) for u and v, respectively, yield:

$$u \xrightarrow{\mathrm{pval}(u)}_C \mathrm{strip}(u) \qquad\qquad \mathrm{strip}(v) \xrightarrow{\mathrm{pval}(v)^{-1}}_C v$$

Transitivity via (4b) concludes

$$u \xrightarrow{\mathrm{pval}(u)\,\mathrm{pval}(v)^{-1}}_C v$$

which also proves convertibility, $u \propto_C v$.

Now assume $u \simeq_{\mathrm{e}} v$. From this follows all of the above, and additionally:

$$\mathrm{pval}(u) = \mathrm{pval}(v)$$

Hence the conversion ratio simplifies

$$u \xrightarrow{1}_C v$$

which proves coherence, $u \cong_C v$. □

Theorem 4.35. In a regular conversion C, in addition to Theorem 4.34, any two units are convertible if *and only if* they are root equivalent, and coherent if *and only if* they are numerically equivalent:

$$u \simeq_{\mathrm{r}} v \iff u \propto_C v \qquad\qquad u \simeq_{\mathrm{e}} v \iff u \cong_C v$$

Proof:
Since any regular conversion is closed, it suffices to prove the converses of Theorem 4.34.

1. All closure axioms preserve root equivalence: If their preconditions are instantiated with root equivalent unit pairs, then so are their conclusions. It follows that a regular conversion can only relate root equivalent unit pairs:

$$u \propto_C v \implies u \simeq_{\mathrm{r}} v$$

2. Assume $u \cong_C v$. Then we have the above, and additionally

$$u \xrightarrow{1}_C v$$

This, together with Theorem 4.34 and (1b), yields:

$$\mathrm{pval}(u)\,\mathrm{pval}(v)^{-1} = 1$$

Thus we find both

$$\mathrm{pval}(u) = \mathrm{pval}(v) \qquad\qquad \mathrm{root}(u) = \mathrm{root}(v)$$

and combine components

$$\mathrm{eval}(u) = \mathrm{eval}(v)$$

which proves numerically equivalence, $u \simeq_{\mathrm{e}} v$.

□

Most of the entailments of Theorem 4.30 are generally proper, but one is not:

Theorem 4.36. All closed conversions are finitely generated.

Proof:
Let C be a closed conversion. The component group $\mathrm{Ut} = \mathcal{A}(\mathrm{Ut_p})$ is not finitely generated as a group, so neither is Conv; thus the result is not immediate.

Since closure is monotonic, any closed conversion has the corresponding regular conversion as a (normal) subgroup. Thus it suffices to show that the remaining quotient group is finitely generated.

By Theorem 4.35 we have:

$$u \simeq_{\mathrm{r}} u' \wedge v \simeq_{\mathrm{r}} v' \implies \bigl(u \propto_C v \iff u' \propto_C v'\bigr)$$

Pick a representative pair from each equivalence class, the obvious candidate being the stripped one. Let B be the relation such that:

$$u \xrightarrow{r}_C v \iff \mathrm{strip}(u) \xrightarrow{\mathrm{pval}(u)^{-1}\,\mathrm{pval}(v)}_B \mathrm{strip}(v)$$

By reasoning in analogy to Theorem 4.34, we can show that B is a conversion and a subgroup of Conv, and that $B^* = C$.

Thus it suffices to show that B is finitely generated as a group. Since the map root is bijective between stripped and root units, B is isomorphic to a subgroup of the binary direct sum $\mathrm{Ut_r} \times \mathrm{Ut_r}$, namely by forgetting the conversion factor. But $\mathrm{Ut_r} \times \mathrm{Ut_r}$ is a finitely generated free abelian group — in the wide sense, namely naturally isomorphic to $\mathcal{A}(\mathrm{Ut_b} \times \mathrm{Ut_b})$ — whose subgroups are all free and finitely generated. □

Remark 4.37. Being finitely generated as a closed conversion in this sense is entailed by being finitely generated as an abelian group in the sense of Theorem 4.7, but *not* vice versa; the former could be "larger" due to axiom (3c), which adds infinitely many elements.

Theorem 4.38. Every well-defining conversion C is consistent, and hence gives rise to a well-defined closure C^*.

Proof:
By well-founded induction. Let C be a well-defining conversion. Its finitely many element triples can be sorted w.r.t. $>_C$ in ascending topological order of the left hand sides. Now consider an incremental construction of C^*, starting from the empty set, and alternatingly adding the next element of C and applying the closure operator. It can be shown that no such step introduces contradictory conversion factors. □

4.4. Well-Defining Unit Conversions and Algorithmic Considerations

Remark 4.39. It follows from Theorem 4.38 that a defining but non-well-defining conversion can only be inconsistent due to a circular dependency.

Remark 4.40. The convertibility problem for closed conversions can encode the *word problem* for quotients of unit groups, thus there is potential danger of undecidability. We conjecture that, by Theorem 4.36, closed conversions are *residually finite*, such that a known decision algorithm [22] could be used in principle.

For well-defined conversions, a more direct and efficient algorithm exists.

Definition 4.41. (Exhaustive Rewriting)
Given a well-defining conversion C, units can be evaluated and exhaustively rewritten:

$$\mathrm{rwr}^*(C) : \mathrm{Ut} \to \mathrm{Ut_e} \qquad\qquad \mathrm{rwr}^*(C) = \mathrm{rwr_e}(C)^{N_C} \circ \mathrm{eval}$$

Each step requires a linearly bounded number of group operations. The kernel of this map can be upgraded to a ternary relation $C^\sharp \subseteq \mathcal{U}(\mathrm{Conv})$:

$$C^\sharp = \left\{ (u, rs^{-1}, v) \in \mathcal{U}(\mathrm{Conv}) \middle| \begin{matrix} \mathrm{rwr}^*(C)(u) = (r, u') \\ \mathrm{rwr}^*(C)(v) = (s, v') \end{matrix} \wedge u' = v' \right\}$$

Theorem 4.42. Exhaustive rewriting solves the well-defined conversion problem; namely, let C be a well-defining conversion, then:

$$C^* = C^\sharp$$

Proof: By mutual inclusion.

1. Define a relation $(\rightsquigarrow_{C^*}) \subseteq \mathrm{Ut} \times \mathrm{Ut_e}$:

   $$u \rightsquigarrow_{C^*} (r, u') \iff u \xrightarrow{r}_{C^*} \mathrm{unroot}(u')$$

   Show by induction

   $$\begin{aligned} & u \rightsquigarrow_{C^*} \mathrm{eval}(u) \\ u \rightsquigarrow_{C^*} v \implies & u \rightsquigarrow_{C^*} \mathrm{rwr_e}(C)(v) \end{aligned}$$

   that it contains the graph of the exhaustive rewriting map:

   $$u \rightsquigarrow_{C^*} \mathrm{rwr}^*(C)(u)$$

   Now assume $(u, rs^{-1}, v) \in C^\sharp$, i.e.:

   $$\begin{aligned} \mathrm{rwr}^*(C)(u) &= (r, u') \\ \mathrm{rwr}^*(C)(v) &= (s, v') \end{aligned} \qquad\qquad u' = v'$$

Combining this with the preceding property, both directly for u and reversed via (4c) for v, we get:

$$u \xrightarrow{r}_{C^*} \mathrm{unroot}(u') = \mathrm{unroot}(v') \xrightarrow{s^{-1}}_{C^*} v$$

Transitivity via (4b) concludes

$$u \xrightarrow{rs^{-1}}_{C^*} v$$

such that $C^\sharp \subseteq C^*$.

2. For the converse, show that $C^\sharp$ is a closure of C, by virtue of obeying the axioms,

$$\begin{aligned} u \xrightarrow{r}_{C} v &\implies u \xrightarrow{r}_{C^\sharp} v \\ u \xrightarrow{r}_{C^\sharp} v \wedge u' \xrightarrow{r'}_{C^\sharp} v' &\implies uu' \xrightarrow{rr'}_{C^\sharp} vv' \\ u \xrightarrow{r}_{C^\sharp} v &\implies u^{-1} \xrightarrow{r^{-1}}_{C^\sharp} v^{-1} \\ u \xrightarrow{\mathrm{pval}(u)}_{C^\sharp} \mathrm{strip}(u) &\xrightarrow{\mathrm{pval}(u)^{-1}}_{C^\sharp} u \end{aligned}$$

which requires rewriting to be exhaustive. Among such relations, C^* is minimal.

□

This theorem ensures that the following decision algorithm is correct.

Definition 4.43. (Conversion Algorithm)

1. Given two units u and v, rewrite them independently and exhaustively to evaluated units (r, u') and (s, v'), respectively.

2. Compare the root units u' and v':

   (a) If they are identical, then the conversion factor is rs^{-1}.

   (b) Otherwise, there is no conversion factor.

Example 4.44. (Mars Climate Orbiter)

The subsystems of the Orbiter attempted to communicate using different units of *linear momentum*, namely *pound-force-seconds* $u = \lfloor \mathrm{lbf} \rfloor \lfloor \mathrm{s} \rfloor$ vs. *newton-seconds* $v = \lfloor \mathrm{N} \rfloor \lfloor \mathrm{s} \rfloor$. The relevant base units are well-defined w.r.t the irreducible SI units as $\lfloor \mathrm{lbf} \rfloor \xrightarrow{1} \lfloor \mathrm{lb} \rfloor \lfloor g_\mathrm{n} \rfloor$ and $\lfloor \mathrm{N} \rfloor \xrightarrow{1} \delta(\mathrm{kg}) \lfloor \mathrm{m} \rfloor \lfloor \mathrm{s} \rfloor^{-2}$, with the auxiliary units *pound* $\lfloor \mathrm{lb} \rfloor \xrightarrow{a} \lfloor \mathrm{g} \rfloor$ (see Example 4.3) and *norm gravity* $\lfloor g_\mathrm{n} \rfloor \xrightarrow{b} \lfloor \mathrm{m} \rfloor \lfloor \mathrm{s} \rfloor^{-2}$, and the conversion factors $a = 453.59237$ and $b = 9.80665$. Exhaustive rewriting yields $\mathrm{rwr}^*(C)(u) = (ab, \lfloor \mathrm{g} \rfloor \lfloor \mathrm{m} \rfloor \lfloor \mathrm{s} \rfloor^{-1})$ and $\mathrm{rwr}^*(C)(v) = \big(1000, \lfloor \mathrm{g} \rfloor \lfloor \mathrm{m} \rfloor \lfloor \mathrm{s} \rfloor^{-1}\big)$, and hence $u \xrightarrow{ab/1000} v$. This conclusion reconstructs the factor given in the introduction.

By virtue of Theorems 4.8 and 4.33, the same reasoning can be presented in diagrammatic instead of textual form; see Figure 9.

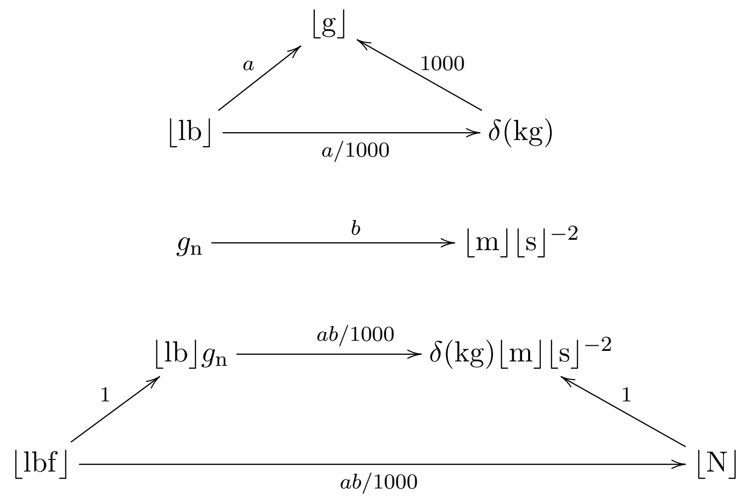

Figure 9. Calculation of the Mars Climate Orbiter Correction Factor

5. Conclusion

We have presented a group-theoretic formal model of units of measure, to our knowledge the first one that supports prefixes and arbitrary conversion factors. The model is based on two composeable monadic structures, and hence has the compositionality required for reasoning with substitution and for denotational semantics. The model is epistemologically stratified, and distinguishes cleanly between necessary properties and natural operations on the one hand, and contingent properties and definable interpretations on the other, leading to a hierarchy of semantic equivalences.

The semantic data structures of the proposed model deviate from historical practice in small, but important details:

- Prefixes apply to compound units as a whole, rather than to individual base units.
- Combinations of prefixes are allowed.

We have characterized unit conversion rules by a class of ternary relations equipped with a closure operator, that function both as a group congruence and as a category, together with a six-tiered hierarchy of subclasses, each satisfying stronger useful algebraic properties than the preceding. This model reconstructs and justifies the reductionistic approach of traditional scientific unit systems.

5.1. Related Work

[9] has found existing related work then to be generally lacking in both formal rigor and universality. With regard to the research programme outlined by [10], matters have improved only partially; see [16] for a recent critical survey. Formally rigorous and expressive type systems have been proposed

for various contexts, such as C [7], UML/OCL [14], generic (but instantiated for Java) [32]. However, in that line of research, neither flexibly convertible units in general nor prefixes in particular are supported.

An object-oriented solution with convertible units is described by [2]. Their approach, like Kennedy's, is syntactic; abelian groups are added as an ad-hoc language extension with "abelian classes" and a normalization procedure. Prefixes are not recognized. Unit conversion is defined always in relation to a fixed reference unit per dimension. This implies that all codimensional units are convertible, which is manifestly unsound; see Section 1.1.5. Nevertheless, the approach has been reiterated in later work such as [4]. By contrast, [8] had already proposed a more flexible relational approach to unit conversion, with methods of matrix calculus for checking consistency.

5.2. Problem Cases Revisited

5.2.1. Problem Case: Neutral Elements

The statement "The number 1 is not a dimension" (translated from [5]) is a rather meaningless outcome of the ambiguity of informal language. The *expression* 1, taken as the multiplicative notation for the neutral element $\varnothing_{/0}$ of the free abelian group $\mathrm{Dm} = \mathcal{A}(\mathrm{Dm_b})$, is of course a perfectly valid dimension; in this sense the above statement is false. 1 is however not a base dimension symbol, nor is the dimension that it represents the product of any base dimensions; in that much narrower sense the above statement is true. The misunderstanding is manifest in the attribute "dimensionless", which in the domain of quantity values has no meaning other than "of dimension 1".

The contradictory case of "1 is a derived unit" is more subtle, but no less problematic. The expression 1, taken as the multiplicative notation for the neutral element $\varnothing_{/0}$ of the free abelian group $\mathrm{Ut} = \mathcal{A}(\mathrm{Ut_b})$ is a valid unit. As above, it is not a product of any base units. As recognized in [25], it is a necessary inhabitant of the group-based model, not subject to explicit definition. Since the standard notions of being derived do not distinguish between syntax and semantics, they cannot distinguish between *compound* and *defined* entities: The former depend syntactically on the base units they are made of. The latter depend only semantically on their compound definition. The unit 1 is of the former kind, but there is no base unit that it depends on; as such it is derived literally from nothing.

Furthermore, although the unit proper $1 = \varnothing_{/0} \in \mathrm{Ut}$ may not be able to take a prefix $p \in \mathrm{Px_b}$, the normalized unit $\mathrm{norm}(1) = (\varnothing_{/0}, \varnothing_{/0}) \in \mathrm{Ut_n}$ surely can, the result being $(\delta(p), \varnothing_{/0})$.

5.2.2. Problem Case: Equational Reasoning

The traditional syntactic structure of prefixed units, in particular the interacting design decisions that prefixes are simple and bind stronger than products, is inherently non-compositional; it cannot be fixed to allow the substitution of complex expressions for atomic symbols in a straightforward and consistent way.

The model presented above side-steps the problem by considering a pair of structures: A faithful model of traditional syntax, Ut is juxtaposed with a slightly coarser abstract syntax, $\mathrm{Ut_n}$, the two being related precisely by the natural distributive law $\mathrm{norm} = \beta_{\mathrm{Px},\mathrm{Ut_b}}$. The latter model is a monad, which entails a number of well-understood virtues regarding the use as a compositional data structure

[29, e.g.]; monad unit and multiplication give a natural notion of compositional substitution that turns reductionistic defining equations into a fully effective algebraic rewriting system.

- The SI base unit kg parses as

$$u = \delta_{\mathrm{Ut_p}}((\delta_{\mathrm{Px_b}}(\mathrm{k}), \mathrm{g})) \in \mathrm{Ut}$$

  and normalizes to

$$u_{\mathrm{n}} = \mathrm{norm}(u) = \big(\delta_{\mathrm{Px_b}}(\mathrm{k}), \delta_{\mathrm{Ut_b}}(\mathrm{g})\big) \in \mathrm{Ut_n} = \mathcal{C}_{\mathrm{Px}}(\mathrm{Ut_r})$$

  Attaching a new prefix $\upmu$ is effected non-invasively by forming an intermediate two-level expression

$$e = \big(\delta_{\mathrm{Px_b}}(\upmu), u_{\mathrm{n}}\big) \in \mathcal{C}^2_{\mathrm{Px}}(\mathrm{Ut_r})$$

  and flattening it naturally by means of the multiplication of the monad $\mathcal{C}_{\mathrm{Px}}$:

$$v_{\mathrm{n}} = \mu_{\mathrm{Px,Ut_n}}(e) = \big(\delta_{\mathrm{Px_b}}(\upmu)\delta_{\mathrm{Px_b}}(\mathrm{k}), \delta_{\mathrm{Ut_b}}(\mathrm{g})\big) \in \mathrm{Ut_n}$$

  The compound prefix *micro-kilo* is distinct from, but has the same value as the simple prefix *milli*.

$$u_{\mathrm{n}} \not\simeq_{\mathrm{n}} v_{\mathrm{n}} \qquad\qquad u_{\mathrm{n}} \simeq_{\mathrm{e}} v_{\mathrm{n}}$$

  For the purpose of correct semantic calculation, it is of no importance whether compound prefix notation is allowed or forbidden syntactically; the issue arises only when the result of a calculation needs to be printed in a specific format.

- For dealing with centilitres, consider a well-defining conversion that just defines the litre as in Example 3.24:

$$\lfloor \mathrm{L} \rfloor \xrightarrow{1}_C \delta((\delta(\mathrm{d}), \mathrm{m}))^3$$

  Calculation as above would get stuck with the symbolically irreducible compound prefix *centi-deci-deci-deci-*. Conversion, on the hand, can work with numerical values. Since the conversion

C^* in question is well-defined, it suffices to consider the results of exhaustive rewriting:

$$\begin{aligned}
\mathrm{rwr}^*(C)(\delta((\delta(c), L))) &= \mathrm{rwr_e}(C)(\mathrm{eval}(\delta((\delta(c), L)))) \\
&= \xi(\mathcal{C}_{\mathrm{Qm}}\mathrm{Zf}(\mathrm{rwr_b}(C))(\mathrm{eval}(\delta((\delta(\mathrm{c}), L))))) \\
&= \xi(\mathcal{C}_{\mathrm{Qm}}\mathrm{Zf}(\mathrm{rwr_b}(C))((\mathrm{pval}(\delta(\mathrm{c})), \delta(\mathrm{L})))) \\
&= \xi(\mathcal{C}_{\mathrm{Qm}}\mathrm{Zf}(\mathrm{rwr_b}(C))((\mathrm{val}(\mathrm{c}), \delta(\mathrm{L})))) \\
&= \xi((\mathrm{val}(\mathrm{c}), \mathrm{Zf}(\mathrm{rwr_b}(C))(\delta(\mathrm{L})))) \\
&= \xi((\mathrm{val}(\mathrm{c}), \delta(\mathrm{rwr_b}(C)(\mathrm{L})))) \\
&= \xi((\mathrm{val}(\mathrm{c}), \delta(\mu(\mathcal{C}_{\mathrm{Qm}}(\mathrm{eval})(\mathrm{xpd}(C)(\mathrm{L})))))) \\
&= \xi((\mathrm{val}(\mathrm{c}), \delta(\mu(\mathcal{C}_{\mathrm{Qm}}(\mathrm{eval})((1, \delta((\delta(\mathrm{d}), \mathrm{m}))^3)))))) \\
&= \xi((\mathrm{val}(\mathrm{c}), \delta(\mu((1, \mathrm{eval}(\delta((\delta(\mathrm{d}), \mathrm{m}))^3)))))) \\
&= \xi((\mathrm{val}(\mathrm{c}), \delta(\mu((1, (\mathrm{pval}(\delta(\mathrm{d})^3), \delta(\mathrm{m})^3)))))) \\
&= \xi((\mathrm{val}(\mathrm{c}), \delta(\mu((1, (\mathrm{val}(\mathrm{d})^3, \delta(\mathrm{m})^3)))))) \\
&= \xi((\mathrm{val}(\mathrm{c}), \delta((1 \cdot \mathrm{val}(\mathrm{d})^3, \delta(\mathrm{m})^3)))) \\
&= \cdots \\
&= \mu((\mathrm{val}(\mathrm{c}), (1 \cdot \mathrm{val}(\mathrm{d})^3, \delta(\mathrm{m})^3))) \\
&= (\mathrm{val}(\mathrm{c}) \cdot 1 \cdot \mathrm{val}(\mathrm{d})^3, \delta(\mathrm{m})^3) \\
&= (10^{-5}, \delta(\mathrm{m})^3)
\end{aligned}$$

Thus, the judgement that a centilitre is 10^{-5} cubic meters arises as the product of the relevant prefix values and the conversion ratio for the litre. Note that the natural transformations employed in the calculation serve no other purpose but to transport contingent data to the right places.

- The defining conversion for the *watt* can be seen in Table 1. The semantically equivalent notations for the *hectowatt* are exactly the coherence (equivalence) class of $\delta((\delta(\mathrm{h}), \delta(\mathrm{W})))$. Among this set, 44 elements obey traditional syntax rules, while infinitely many more involve multiple base prefixes on the same base unit.

5.2.3. Problem Case: Cancellation

Units of measure are entities of calculation. As such, their usefulness depends crucially on their algebraic behavior. Making exceptions in the algebra of units in order to encode properties of quantities seems like putting the cart before the horse. Thus the distinction

$$\mathrm{rad} = \mathrm{m}/\mathrm{m} \neq 1 \neq \mathrm{sr} = \mathrm{m}^2/\mathrm{m}^2$$

makes no algebraic sense, yet is the expression of a valid distinction at some level, which our model should be able to accomodate.

We propose that the adequate level for making the distinction is quantities: Quantities have both algebraic and *taxonomical* (sub- and supertype) structure, with the interaction being largely unstudied. Quantities may be associated with a default unit to express their values in.

The quantity *plane angle* is defined as the quotient of the quantities *arc length* and *circle radius*. Both are subquantities of *length*, and hence inherit the default unit m. The quantity *solid angle* is defined as the quotient of the quantities *covered surface area* and *sphere radius* squared. The former inherits the default unit m^2 from its superquantity *area*, the latter inherits the squared default unit of *length*.

Thus, the equation $\mathrm{rad} = \mathrm{m}/\mathrm{m}$ is an attempt to encode the genealogy of the quantity *plane angle* into the definition of its default unit, in terms of the default units of its precursor quantities. The message is irreconcilably at odds with an algebraic reading, since the equation $\mathrm{m}/\mathrm{m} = 1$ is a necessary property of the group-based model.

In a model with fine-grained conversion relations, the problem is easily side-stepped: rad as a unit of plane angle does not require any reductionistic definition at all, let alone one that equates it unintendedly with other units. Rather, rad is codimensional with, but can be left unconvertible to, both sr and 1, in the well-defined conversion relation of choice.

In fact, the possibility to have a partition of multiple interconvertible components within a set of codimensional units behaves like a subtype system. We leave the intriguing question whether this feature models relevant issues of quantity ontology, in particular whether it is a suitable complement for the concept of quantity kinds, to future research.

5.2.4. Problem Case: Prefix Families

The group algebra of prefixes can express the semantics of geometric families of base prefixes in a formally precise manner within the model: Statements of interest correspond directly to relationships among mathematical objects. For example, the statement "*mega* means *kilo* squared" is embodied in the fact that the pair $(\delta(\mathrm{M}), \delta(\mathrm{k})^2) \in \mathrm{Px}^2$ is contained in the kernel of the model operation val.

5.2.5. Problem Case: Units and Dimensions

The approach that prefers one canonical unit per dimension is quite prevalent in the foundational literature [10, 2], motivated by both the desire to keep things simple and to manage conversion information in a concise and consistent way. It is, however, prone to overdoing convertibility. The following quotation demonstrates this by jumping to unwarranted conclusions twice [2, boldface added]:

> To convert between measurements in different units of the same dimension, we must specify conversion factors between **various** units of that dimension. A natural place to keep this information is in the definition of a unit: each unit specifies how to convert measurements in that unit to measurements in **any other** defined unit (for the same dimension).
>
> Although the number of such conversion factors is quadratic in the number of units, it is not necessary to maintain so many factors explicitly: if we can convert between measurements in units A and B, and between measurements in units B and C, then we

can convert between measurements in A and C via B. **Thus**, it is sufficient to include in the definition of every unit a single conversion factor to a *primary unit* of that dimension, and convert between any two **commensurable** units via their common primary unit.

In the model presented here, there is no sufficient reason for all codimensional units to be convertible; the latter equivalence relation can be seen as a proper refinement of the former. The placement of conversion data is described in an object-oriented fashion in [2]. Here, by contrast, we distinguish a (well-)defining conversion and its closure.

The statement "The sievert is equal to the joule per kilogram [i.e., the gray]" from [21] is likely a category mistake. It is obviously false in a literal reading: Three quantity value assignments that speak about the same irradiation event may assign very different numerical values to absorbed dose (in *gray*), equivalent and effective dose (both in *sievert*), since the relevant medical calculation models crucially involve context-dependent scaling factors both greater and less than one.

We conjecture that the actually intended message is threefold:

- The quantities of *absorbed*, *equivalent* and *effective dose of ionizing radiation* are specializations of the more general parent quantity *dose of ionizing radiation*. They shall be regarded as being of different kind.
- Absorbed dose inherits the default unit *gray*, reductionistically defined as *joule per kilogram*, from its parent quantity; equivalent and effective dose do not.
- The practice to notate the calculation models with unitless scaling factors is acknowledged, although a coefficient with a unit of *sievert per gray* would improve the formal rigor.

5.3. Future Work

5.3.1. Number Theory of Conversion Ratios

Conversion ratios have been defined axiomatically as positive rational numbers. The reason is an application of Occam's Razor: The positive rationals are the simplest multiplicative group, and no need for other numbers is evident from the examples discussed so far. However, there are some elephants in the room.

Of course, the conversion between full turns (or rational fractions thereof) and radians of planar angle famously features the irrational factor π. Fortunately, transcendental numbers such as π are nicely orthogonal to rationals, such that π itself can be considered a unit of dimension $\varnothing_{/0}$, and treated purely symbolically.

Another application of irrational factors arises in the treatment of logarithms. Since logarithms to different bases are merely linearly scaled copies of the same basic function shape, it is possible to consider a *single*, generic function $\log$ that makes no mention of the base. The result can then be expressed in logarithmic units such as *bit*, *nat* or *decibel*, which specify bases 2, e or $10^{1/10}$, respectively.

As usually in number theory, algebraic irrationals are more complicated, because their powers may overlap with the rationals in complicated ways. An illuminating example is provided by the "DIN" families of paper sizes [31]. The "A" family defines a sequence of length units, $a_0, \ldots, a_{11}$, such that paper sizes arise as $A_n = a_n \times a_{n+1}$. Ideally, perfect self-similarity would be achieved by the sequence being geometric, with a factor of $a_n/a_{n+1} = \sqrt{2}$. The largest paper size in the family, A_0, covers one square meter. These facts can be specified unambiguously by a conversion relation

$$a_0 a_1 \xrightarrow{1} \mathrm{m}^2 \qquad\qquad a_n^2 \xrightarrow{2} a_{n+1}^2$$

but not by a well-defined one. Consequently, the ensuing conversion rules are not complete with respect to the algebra of $\sqrt{2}$. The standard [31] uses a different approach to avoiding irrationals: The lengths are specified as integer multiples of a millimeter, with margins of tolerance such that the exact geometric solution is a valid "approximation".

5.3.2. Affine Conversion

Some traditional units of measure have the property that a numerical value of zero may have different meanings. The most well-known example are the units of *temperature*, namely *degrees Celsius* and *Farenheit*, in relation to the corresponding SI unit *Kelvin*. Their formal behavior can be accommodated in the present model, but with some severe limitations.

The first step is to note that in equation 2 in Remark 4.2, the ratio r is not just a number, but also a linear map acting from the right on numerical values. Thus, the middle component of conversion relations could be generalized to invertible linear maps, and subsequently to invertible *affine* maps, for example:

$$^\circ C \xrightarrow{\lambda\, x.\, \frac{9}{5}x+32} {}^\circ F$$

However, the concept of conversion closure fails for this generalization. Affine maps are not multiplicative: For two affine maps f and g, there is in general no affine map fg such that $fg(xy) = f(x)g(y)$.

Thus the convertibility of affine units does not survive being put into context. In particular, the product of any two quantity values measured in such units is already nonsensical.

5.3.3. Syntax

Perhaps surprisingly, the syntax of quantity values is, in actual computing practice, just as controversial and poorly understood as the semantics.

[18, 25] give some informal notational rules that document the standard practice within the SI/ISQ, but not an actual grammar, let alone proof of non-ambiguity. By contrast, the programming language F# supports an ad-hoc syntax that is not compatible with any standard. The example

```
55.0<miles/hour>
```

given in [17] features angled brackets instead of the standard invisible multiplication operator, and a plural form, which is explicitly ruled out by [18, 25].

A systematic study of standard-conforming syntax rules for quantity values, and of their interaction with common rules for arithmetic expressions, would be a useful basis for the design and implementation of ergonomic and clear unit-aware programming language extensions.

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