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Normal Forms for Elements of *-Continuous Kleene Algebras Representing the Context-Free Languages

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Abstract. Within the tensor product $K \otimes_{\mathcal{R}} C_2'$ of any *-continuous Kleene algebra K with the polycyclic *-continuous Kleene algebra C_2' over two bracket pairs there is a copy of the fixed-point closure of K: the centralizer of C_2' in $K \otimes_{\mathcal{R}} C_2'$. Using an automata-theoretic representation of elements of $K \otimes_{\mathcal{R}} C_2'$ à la Kleene and with the aid of normal form theorems that restrict the occurrences of brackets on paths through the automata, we develop a foundation for a calculus of context-free expressions without variable binders. We also give some results on the bra-ket *-continuous Kleene algebra C_2, motivate the "completeness equation" that distinguishes C_2 from C_2', and show that C_2' validates a relativized form of this equation.

1. Introduction

A Kleene algebra $K = (K, +, \cdot, ^*, 0, 1)$ is *-continuous if

$$a \cdot c^* \cdot b = \sum \{\, a \cdot c^n \cdot b \mid n \in \mathbb{N} \,\}$$

for all $a, b, c \in K$, where $\sum$ is the least upper bound with respect to the natural partial order $\leq$ on K given by $a \leq b$ iff $a + b = b$. Well-known examples of *-continuous Kleene algebras are the algebras

$\mathcal{R}M = (\mathcal{R}M, +, \cdot, ^*, 0, 1)$ of regular or "rational" subsets of a monoid $M = (M, \cdot^M, 1^M)$, where $0 := \emptyset$, $1 := \{1^M\}$ and $+$ is union, $\cdot$ is elementwise product, and * is iteration or "monoid closure", i.e. for $A \in \mathcal{R}M$, A^* is the least $B \supseteq A$ that contains 1^M and is closed under $\cdot^M$.

We will make use of two other kinds of *-continuous Kleene algebras: quotients K/ρ of *-continuous Kleene algebras K under $\mathcal{R}$-congruences ρ on K, i.e. semiring congruences which make suprema of regular subsets congruent if their elements are congruent in a suitable sense, and tensor products $K \otimes_{\mathcal{R}} K'$ of *-continuous Kleene algebras K and K'.

Let Δ_m be a set of m pairs of "brackets", p_i, q_i, $i < m$, and $\mathcal{R}\Delta_m^*$ the *-continuous Kleene algebra of regular subsets of Δ_m^*. Hopkins [3] considers the $\mathcal{R}$-congruence ρ_m on $\mathcal{R}\Delta_m^*$ generated by the equation set

$$\{ p_i q_j = \delta_{i,j} \mid i, j < m \} \cup \{q_0 p_0 + \ldots + q_{m-1} p_{m-1} = 1\} \tag{1}$$

and the finer $\mathcal{R}$-congruence ρ_m' generated by the equations

$$\{ p_i q_j = \delta_{i,j} \mid i, j < m \}, \tag{2}$$

where $\delta_{i,j}$ is the Kronecker δ. The latter equations allow us to algebraically distinguish matching brackets, where $p_i q_j = 1$, from non-matching ones, where $p_i q_j = 0$.[1] These $\mathcal{R}$-congruences give rise to the *bra-ket* and the *polycyclic* *-continuous Kleene algebra $C_m = \mathcal{R}\Delta_m^*/\rho_m$ and $C_m' = \mathcal{R}\Delta_m^*/\rho_m'$, respectively. For $m > 2$, C_m can be coded in C_2 and C_m' in C_2', so we focus on the case $m = 2$.

Two *-continuous Kleene algebras K and C can be combined to a *tensor product* $K \otimes_{\mathcal{R}} C$ which, intuitively, is the smallest common *-continuous Kleene algebra extension of K and C in which elements of K commute with those of C.

In unpublished work, the first author showed that for any *-continuous Kleene algebra K, the tensor product $K \otimes_{\mathcal{R}} C_2$ contains an isomorphic copy of the fixed-point closure of K. In particular, for finite alphabets X, each context-free set $L \subseteq X^*$ is represented in $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2$ as the value of a regular expression over the disjoint union $X \,\dot{\cup}\, \Delta_2$ of X and Δ_2. In fact, the *centralizer of* C_2 in $K \otimes_{\mathcal{R}} C_2$, i.e. the set of those elements of $K \otimes_{\mathcal{R}} C_2$ that commute with every element of C_2, consists of exactly the representations of context-free subsets of the multiplicative monoid of K. These results constitute a generalization of the Chomsky and Schützenberger representation theorem ([1], Proposition 2) in formal language theory, which says that any context-free set $L \subseteq X^*$ is the image $h(R \cap D)$ of a regular set $R \subseteq (X \cup \Delta)^*$ under a homomorphism $h : (X \cup \Delta)^* \to X^*$ that keeps elements of X fixed and "erases" symbols of Δ to 1. The generalization is shown in [11] with the simpler algebra $K \otimes_{\mathcal{R}} C_2'$ instead of $K \otimes_{\mathcal{R}} C_2$.

It is therefore of some interest to understand the structure of $K \otimes_{\mathcal{R}} C_2$ and $K \otimes_{\mathcal{R}} C_2'$. In this article, an extension of [6], we focus on $K \otimes_{\mathcal{R}} C_2'$, using ideas from and improvements of unpublished results on $K \otimes_{\mathcal{R}} C_2$ by the first author. Our main results are normal forms for elements of $K \otimes_{\mathcal{R}} C_2'$ that relate arbitrary elements to those of the centralizer of C_2'. We also present some results specific to C_2 and its matrix algebra. The rest of this article is structured as follows.

Section 2 recalls the definitions of *-continuous Kleene algebras (aka $\mathcal{R}$-dioids), bra-ket and polycyclic *-continuous Kleene algebras, and quotients and tensor products of *-continuous Kleene algebras. We then show a Kleene representation theorem, i.e. that each element φ of $K \otimes_{\mathcal{R}} C_2'$ is the value

[1] In $\mathcal{R}\Delta_m^*$, elements of Δ_m^* are interpreted by their singleton sets, 0 by the empty set.

$L(\mathcal{A}) = SA^*F$ of an automaton $\mathcal{A} = \langle S, A, F\rangle$, where $S \in \{0,1\}^{1\times n}$ resp. $F \in \{0,1\}^{n\times 1}$ code the set of initial resp. accepting states of the n states of $\mathcal{A}$ and $A \in Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$ is a transition matrix.

Section 3 refines the representation $\varphi = L(\mathcal{A})$ to a *normal form* where brackets on paths through the automaton $\mathcal{A}$ occur mostly in a balanced way. Section 3.1 identifies, in any Kleene algebra with elements u, x, v, the value $(u + x + v)^*$ with the value $(Nv)^*N(uN)^*$, provided the algebra has a least solution N of the inequation $y \geq (x + uyv)^*$ defining Dyck's language $D(x) \subseteq \{u, x, v\}^*$ with "bracket" pair u, v. We then show that for any *-continuous Kleene algebra K and $n \geq 1$, $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$ has such a solution N of $y \geq (UyV + X)^*$ for matrices U of 0's and opening brackets from C_2', X of elements of K, and V of 0's and closing brackets from C_2', and that entries of N belong to the centralizer of C_2' in $K \otimes_{\mathcal{R}} C_2'$.

Section 3.2 refines the representation $\varphi = L(\mathcal{A})$ to the sketched normal form: the transition matrix A can be split as $A = U + X + V$ into a matrix $X \in K^{n\times n}$ of transitions by elements of K, a matrix $U \in \{0, p_0, p_1\}^{n\times n}$ of transitions by 0 or opening brackets of C_2', and a matrix $V \in \{0, q_0, q_1\}^{n\times n}$ of transitions by 0 or closing brackets of C_2'. Then A^* can be normalized to $(NV)^*N(UN)^*$, where N is balanced in U and V and all other occurrences of closing brackets V are in front of all other occurrences of opening brackets U. We call $SA^*F = S(NV)^*N(UN)^*F$ the first normal form of φ. This result is a generalization of a normal form for elements of the polycyclic monoid $P_2'[X]$, the quotient of $(\Delta_2 \cup X \cup \{0\})^*$ by the monoid congruence generated by the bracket match- and mismatch equations, the equations for commuting brackets of Δ_2 with symbols of X, and the annihilator equations for 0. Namely, if $\Delta_2 = U \cup V$ is split into opening brackets U and closing brackets V, any $w \in (\Delta_2 \cup X \cup \{0\})^*$ is congruent to a normal form $\mathit{nf}(w) \in V^*X^*U^* \cup \{0\}$. (The centralizer of Δ_2 in $P_2'[X]$ is $X^* \cup \{0\}$, so the analogues of N are contracted in the factor X^*.)

Section 3.3 proves a conjecture of [6]: if $\varphi = L(\mathcal{A})$ belongs to the centralizer of C_2' in $K \otimes_{\mathcal{R}} C_2'$, then the normal form $SA^*F = S(NV)^*N(UN)^*F$ can be simplified to $SA^*F = SNF$. We call this the reduced normal form. For this, we have to assume that K is non-trivial and has no zero divisors, which is satisfied e.g. when $K = \mathcal{R}M$ for a monoid M. A second normal form is given for a slightly more general transition matrix A than $U + X + V$, which is useful for the representation of products of context-free subsets. For the elements of the centralizer of C_2' in $K \otimes_{\mathcal{R}} C_2'$ only, a different characterization had been given in [11]. The normal form theorems presented here improve on this by showing how the elements of the centralizer of C_2', i.e. the representations of context-free subsets of K in $K \otimes_{\mathcal{R}} C_2'$, relate to the remaining elements of $K \otimes_{\mathcal{R}} C_2'$.

For a finite set X, the elements of $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ are named by regular expressions over $\Delta_2 \,\dot{\cup}\, X$, as mentioned above. A subset of those, called the context-free expressions over X, name the elements of the centralizer of C_2' in $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$, i.e. the representations of the context-free languages $L \subseteq X^*$. Section 4 provides a foundation of a calculus of context-free expressions by showing how to combine normal forms for elements of any $K \otimes_{\mathcal{R}} C_2'$ by regular operations.

Section 5 deals with the bra-ket *-continuous Kleene algebras C_m. Section 5.1 gives an interpretation of C_m in the algebra of binary relations on a countably infinite set, $Mat_{\omega,\omega}(\mathbb{B})$. We also show that C_m is isomorphic to $Mat_{m,m}(C_m)$ and $C_m \otimes_{\mathcal{R}} Mat_{m,m}(\mathbb{B})$, thereby excluding an interpretation by finite-dimensional matrices. Section 5.2 considers a natural interpretation of brackets as stack operations, where p_i pushes symbol $i \in \{1, \ldots, m-1\}$ to the stack and q_i pops i from the stack. Then

$q_i p_i$ tests if symbol i is on the stack top, while $q_0 p_0$ tests if the stack boundary 0 is on top, so that the equation $q_0 p_0 + \ldots + q_{m-1} p_{m-1} = 1$ distinguishing C_m from C_m' asserts a *completeness condition* for a stack with stack alphabet $\{1, \ldots, m-1\}$. For regular programs $r \in \mathit{RegExp}(\{q_0 p_0, p_1, \ldots, q_{m-1}\})$, the scope $p_0 \ldots q_0$ of $p_0 r q_0$ asserts that we start and end with an empty stack. Section 5.2 shows that the completeness equation of C_m in a sense already holds in C_m' in the scope of $p_0 \ldots q_0$.

Finally, the conclusion summarizes our results and indicates possible future extensions.

2. *-continuous Kleene algebras and $\mathcal{R}$-dioids

A *Kleene algebra*, as defined in [8], is an idempotent semiring or dioid $(K, +, \cdot, 0, 1)$ with a unary operation $^* : K \to K$ such that for all $a, b \in K$

$$a \cdot a^* + 1 \leq a^* \quad \wedge \quad \forall x(a \cdot x + b \leq x \to a^* \cdot b \leq x),$$
$$a^* \cdot a + 1 \leq a^* \quad \wedge \quad \forall x(x \cdot a + b \leq x \to b \cdot a^* \leq x),$$

where $\leq$ is the natural partial order on K given by $a \leq b$ iff $a + b = b$.

A Kleene algebra is *non-trivial* if $0 \neq 1$, and it *has zero-divisors* if there are non-zero elements a, b such that $a \cdot b = 0$. The boolean Kleene algebra $\mathbb{B} = (\{0, 1\}, +, \cdot, ^*, 0, 1)$ with boolean addition and multiplication and * given by $0^* = 1^* = 1$ is a subalgebra of any non-trivial Kleene algebra K.

A Kleene algebra $K = (K, +, \cdot, ^*, 0, 1)$ is **-continuous* if

$$a \cdot c^* \cdot b = \sum \{\, a \cdot c^n \cdot b \mid n \in \mathbb{N} \,\}$$

for all $a, b, c \in K$, where $\sum$ is the least upper bound with respect to the natural partial order. Well-known *-continuous Kleene algebras are the algebras $\mathcal{R}M = (\mathcal{R}M, +, \cdot, ^*, 0, 1)$ of regular subsets of monoids $M = (M, \cdot^M, 1^M)$, where $0 := \emptyset$, $1 := \{1^M\}$ and for $A, B \in \mathcal{R}M$,

$$A + B = A \cup B, \qquad A \cdot B = \{\, a \cdot^M b \mid a \in A, b \in B \,\},$$
$$A^* = \bigcup \{\, A^n \mid n \in \mathbb{N} \,\} \qquad \text{with } A^0 = 1, A^{n+1} = A \cdot A^n.$$

If K is a dioid $(K, +^K, \cdot^K, 0^K, 1^K)$ or a Kleene algebra, by $\mathcal{R}K$ we mean the Kleene algebra $\mathcal{R}M$ of its multiplicative monoid $M = (K, \cdot^K, 1^K)$.

An $\mathcal{R}$*-dioid* is a dioid $K = (K, +^K, \cdot^K, 0^K, 1^K)$ where each $A \in \mathcal{R}K$ has a least upper bound $\sum A \in K$, i.e. $\sum$ is $\mathcal{R}$*-complete*, and where $\sum(AB) = (\sum A)(\sum B)$ for all $A, B \in \mathcal{R}K$, i.e. $\sum$ is $\mathcal{R}$*-distributive*. An $\mathcal{R}$*-morphism* is a dioid morphism that preserves least upper bounds of regular sets.

Any $\mathcal{R}$-dioid K can be expanded to a *-continuous Kleene algebra by putting $c^* := \sum\{c\}^*$ for $c \in K$. Conversely, the dioid reduct of a *-continuous Kleene algebra K is an $\mathcal{R}$-dioid, since, by induction, every regular set C has a least upper bound $\sum C \in K$ satisfying $a \cdot (\sum C) \cdot b = \sum(aCb)$, which implies the $\mathcal{R}$-distributivity property $\sum(AB) = (\sum A)(\sum B)$ for $A, B \in \mathcal{R}K$ (see [3]).

The *-continuous Kleene algebras, with Kleene algebra homomorphisms (semiring homomorphisms that preserve *), form a category. It is isomorphic to the category $\mathbb{D}\mathcal{R}$ of $\mathcal{R}$-dioids and $\mathcal{R}$-morphisms, cf. [7, 3, 5], and a subcategory of the category $\mathbb{D}$ of dioids and dioid morphisms. There is an adjunction $(\mathcal{R}, \widehat{\mathcal{R}}, \eta, \epsilon)$ between the category $\mathbb{M}$ of monoids and the category $\mathbb{D}\mathcal{R}$, where $\widehat{\mathcal{R}}$ is the forgetful functor, the unit η is given by $\eta_M : M \to \mathcal{R}M$ with $\eta_M(m) = \{m\}$ and the counit ϵ by $\epsilon_K : \mathcal{R}K \to K$ with $\epsilon_K(A) = \sum A$, for monoids M and $\mathcal{R}$-dioids K, cf. Theorem 16 of [4].

The $\mathcal{R}$-dioids of the form $\mathcal{R}M$ with monoid M form the Kleisli subcategory of $\mathbb{D}\mathcal{R}$. The cases of most immediate interest are the algebras $\mathcal{R}X^*$ associated with regular expressions and regular languages over an alphabet X, and $\mathcal{R}\,(X^* \times Y^*)$ of rational relations and rational transductions with alphabets X and Y, respectively, of inputs and outputs.

2.1. The polycyclic $\mathcal{R}$-dioids

We will make use of two kinds of $\mathcal{R}$-dioids which do not belong to the Kleisli subcategory, but are quotients of the regular sets $\mathcal{R}\Delta^*$ by suitable $\mathcal{R}$-congruence relations ρ on $\mathcal{R}\Delta^*$, where Δ is an alphabet of "bracket" pairs. In this section, we introduce the polycyclic $\mathcal{R}$-dioids C_m' over an alphabet Δ_m of m bracket pairs; the bra-ket $\mathcal{R}$-dioids C_m over Δ_m are deferred to Section 5.2.

Let ρ be a dioid congruence on an $\mathcal{R}$-dioid D. The set D/ρ of congruence classes is a dioid under the operations defined by $(d/\rho)(d'/\rho) := (dd')/\rho$, $1 := 1/\rho$, $d/\rho + d'/\rho := (d + d')/\rho$, $0 := 0/\rho$. Let $\leq$ be the partial order on D/ρ derived from $+$. For $U \subseteq D$, put $U/\rho := \{\, d/\rho \mid d \in U \,\}$ and

$$(U/\rho)^{\downarrow} = \{\, e/\rho \mid e/\rho \leq d/\rho \text{ for some } d \in U, e \in D \,\}.$$

An *$\mathcal{R}$-congruence* on D is a dioid-congruence ρ on D such that for all $U, U' \in \mathcal{R}D$, if $(U/\rho)^{\downarrow} = (U'/\rho)^{\downarrow}$, then $(\sum U)/\rho = (\sum U')/\rho$. It is easy to see that the kernel of an $\mathcal{R}$-morphism is an $\mathcal{R}$-congruence.

Proposition 2.1. (Proposition 1 of [5])
If D is an $\mathcal{R}$-dioid and ρ an $\mathcal{R}$-congruence on D, then D/ρ is an $\mathcal{R}$-dioid. For every $R \subseteq D \times D$ there is a least $\mathcal{R}$-congruence $\rho \supseteq R$ on D.

Let $\Delta_m = P_m \,\dot{\cup}\, Q_m$ be a set of m "opening brackets" $P_m = \{\, p_i \mid 0 \leq i < m \,\}$ and m "closing brackets" $Q_m = \{\, q_i \mid 0 \leq i < m \,\}$, with $P_m \cap Q_m = \emptyset$. The *polycyclic $\mathcal{R}$-dioid C_m'* is the quotient $C_m' = \mathcal{R}\Delta_m^*/\rho$ of $\mathcal{R}\Delta_m^*$ by the $\mathcal{R}$-congruence ρ generated by the relations

$$\{\, p_i q_j = \delta_{i,j} \mid i, j < m \,\}. \tag{3}$$

These equations allow us to algebraically distinguish matching brackets, where $p_i q_j = 1$, from non-matching ones, where $p_i q_j = 0$. The *polycyclic monoid P_m'* of m generators is the quotient of $(\Delta_m \,\dot{\cup}\, \{0\})^*$ by the monoid congruence σ_m generated by

$$\{\, p_i q_j = \delta_{i,j} \mid i, j < m \,\} \cup \{\, x0 = 0 \mid x \in \Delta_m \,\dot{\cup}\, \{0\} \,\} \cup \{\, 0x = 0 \mid x \in \Delta_m \,\}.$$

Each element $w \in (\Delta_m \,\dot{\cup}\, \{0\})^*$ has a *normal form nf*$(w) \in Q_m^* P_m^* \cup \{0\}$, obtained by using the equations to shorten w, that represents $w/\sigma_m \in P_m'$. Hence,

$$P_m' \simeq (Q_m^* P_m^* \cup \{0\}, \cdot, 1) \quad \text{with } v \cdot w = \mathit{nf}(vw).$$

The polycyclic $\mathcal{R}$-dioid C_m' can be understood as the regular sets of strings in normal form:

Proposition 2.2. (Proposition 9 of [11])
Let ν be the least $\mathcal{R}$-congruence on $\mathcal{R}P_m'$ that identifies $\{0\}$ with the empty set. Then $C_m' \simeq \mathcal{R}P_m'/\nu$ via the mapping defined by $A/\rho \mapsto \{\, nf(w) \mid w \in A\,\}/\nu$ for $A \in \mathcal{R}\Delta_m^*$. Each element A/ρ of C_m' is uniquely represented by a subset of $Q_m^* P_m^*$, namely $\{\, nf(w) \mid w \in A\,\} \setminus \{0\}$.

The normal form can be extended from P_m' to monoid extensions $P_m'[X]$ of P_m' in which elements of X are required to commute with elements of P_m'. Formally, let $Y = \Delta_m \mathbin{\dot\cup} \{0\} \mathbin{\dot\cup} X$ and $P_m'[X]$ the quotient of Y^* under the congruence generated by (i) the matching rules $\{\, p_i q_j = \delta_{i,j} \mid i,j < m\,\}$, (ii) the annihilation rules $y0 = 0$ and $0y = 0$ for $y \in Y$, and (iii) the commutation rules $\{\, xd = dx \mid x \in X, d \in \Delta_m\,\}$. The set Y^* can be decomposed into strings containing a 0, strings containing an opening bracket followed by a symbol of X or by a closing bracket, strings containing a symbol of X followed by a closing bracket, and strings consisting only of closing brackets followed by symbols of X followed by opening brackets, i.e.

$$Y^* = Y^*\{0\}Y^* \cup Y^*(P_m X \cup P_m Q_m \cup X Q_m)Y^* \cup Q_m^* X^* P_m^*.$$

A normal form $nf(w) \in Q_m^* X^* P_m^* \cup \{0\}$ for strings $w \in Y^*$ can hence be obtained: use the annihilation rules to replace $u0v$ by 0, use the commutation rules to move opening brackets $p_i \in P_m$ to the right and closing brackets $q_i \in Q_m$ to the left of elements of X^*, then use the matching rules to shorten up_iq_jv to uv or $u0v$, and repeat this process. I.e. for $i,j < m, i \neq j$ and $x \in X$, $u,v \in Y^*$ we put

$$\begin{array}{rclrclrcl} nf(up_ixv) & := & nf(uxp_iv), & nf(u0v) & := & 0, & nf(up_iq_iv) & := & nf(uv), \\ nf(uxq_iv) & := & nf(uq_ixv), & nf(1) & := & 1, & nf(up_iq_jv) & := & 0. \end{array}$$

We leave it to the readers to convince themselves that this amounts to a confluent rewriting system, so that $nf : Y^* \to Q_m^* X^* P_m^* \cup \{0\}$ is well-defined, and that

$$P_m'[X] \simeq (Q_m^* X^* P_m^* \cup \{0\}, \cdot, 1), \quad \text{where } u \cdot v := nf(uv). \tag{4}$$

The normal form nf on $P_m'[X]$ is the motivating idea behind the normal form theorem (Theorem 3.5) for elements of the tensor product $\mathcal{R}X^* \otimes_{\mathcal{R}} C_m'$ to be introduced in the next section. On the tensor product, regular sets $A \in \mathcal{R}X^*$ and (congruence classes of) regular sets $B \in \mathcal{R}\Delta_m$ commute with each other, and the tensor product is an $\mathcal{R}$-dioid structure, not just a monoid structure.

We notice that a suitable coding of $m \geq 2$ bracket pairs by two pairs extends to an embedding of C_m' in C_2'. In the context $p_0 \ldots q_0$, the code of any normal form $w \in Q_m^* P_m^*$ except 1 is annihilated.

Lemma 2.3. For $m \geq 2$ there is an embedding $\mathcal{R}$-morphism $g : C_m' \to C_2'$ such that for $i,j < m$,

$$g(p_i) \cdot g(q_j) = \delta_{i,j} \quad \text{and} \quad p_0 \cdot g(q_j) = 0 = g(p_i) \cdot q_0,$$

where we wrote p_i, q_j for the congruence class of $\{p_i\}, \{q_j\}$ in C_m' and C_2', respectively.

Proof:
Write $\Delta_m = P_m \dot\cup Q_m$ with $P_m = \{p_0, \dots, p_{m-1}\}$, $Q_m = \{q_0, \dots, q_{m-1}\}$, but for $\Delta_2 = P_2 \dot\cup Q_2$, use b, p for p_0, p_1 and d, q for q_0, q_1. Let $\overline{\cdot} : \Delta_m^* \to \Delta_2^*$ be the homomorphism generated by the coding of Δ_m in Δ_2^* by

$$\overline{p_i} = bp^{i+1}, \quad \overline{q_i} = q^{i+1}d, \quad \text{for } i < m.$$

The functor $\mathcal{R}$ lifts $\overline{\cdot}$ by $\overline{A} := \{\, \overline{w} \mid w \in A \,\}$ to a monotone homomorphism $\overline{\cdot} : \mathcal{R}\Delta_m^* \to \mathcal{R}\Delta_2^*$; since the supremum $\sum$ on $\mathcal{R}\mathcal{R}\Delta_m^*$ and $\mathcal{R}\mathcal{R}\Delta_2^*$ is the union of sets, $\overline{\cdot}$ is an $\mathcal{R}$-morphism. Let ρ_m be the $\mathcal{R}$-congruence on $\mathcal{R}\Delta_m^*$ generated by the (semiring) equations

$$p_i q_i = 1, \quad p_i q_j = 0, \quad \text{for } i \neq j < m.$$

Then clearly

$$\overline{p_i q_j}/\rho_2 = bp^{i+1}q^{j+1}d/\rho_2 = \delta_{i,j} = p_i q_j/\rho_m$$

and

$$(b \cdot \overline{q_j})/\rho_2 = bq^{j+1}d/\rho_2 = 0 = bp^{i+1}d/\rho_2 = (\overline{p_i} \cdot d)/\rho_2.$$

Extend the $\mathcal{R}$-morphism $\overline{\cdot} : \mathcal{R}\Delta_m^* \to \mathcal{R}\Delta_2^*$ to a map $g : C_m' \to C_2'$ by

$$g(A/\rho_m) := \overline{A}/\rho_2 \quad \text{for } A \in \mathcal{R}\Delta_m^*.$$

This map is well-defined and injective: by Proposition 2.2, A/ρ_m is represented by a set of strings in normal form, $\{\, nf(w) \mid w \in A \,\} \setminus \{0\} \subseteq Q_m^* P_m^*$, and $\overline{\cdot}$ maps $Q_m^* P_m^*$ injectively to a set of normal form strings of $Q_2^* P_2^*$.

Clearly, $g : C_m' \to C_2'$ is a monotone semiring morphism. Since $\cdot/\rho_m : \mathcal{R}\Delta_m^* \to C_m'$ is surjective, $g : C_m' \to C_2'$ is an $\mathcal{R}$-morphism: for each $U \in \mathcal{R}C_m'$ there is $V \in \mathcal{R}\Delta_m^*$ such that $U = \{\, A/\rho_m \mid A \in V \,\}$, hence

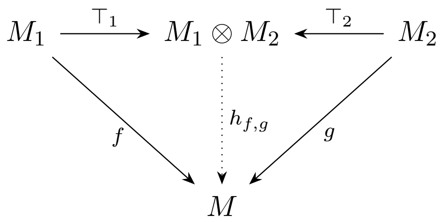
$$\begin{aligned} g(\sum U) &= g((\bigcup V)/\rho_m) = g(\bigcup V)/\rho_2 \\ &= (\bigcup \{\, g(A) \mid A \in V \,\})/\rho_2 \\ &= \sum \{\, g(A)/\rho_2 \mid A \in V \,\} \\ &= \sum \{\, g(A/\rho_m) \mid A \in V \,\} \\ &= \sum \{\, g(B) \mid B \in U \,\}. \end{aligned}$$

□

Based on Lemma 2.3, in the following we state most results only for $m = 2$.

2.2. The tensor product $K \otimes_{\mathcal{R}} C$ of $\mathcal{R}$-dioids K and C

Two maps $f : M_1 \to M \leftarrow M_2 : g$ to a monoid M are *relatively commuting* if $f(m_1)g(m_2) = g(m_2)f(m_1)$ for all $m_1 \in M_1$ and $m_2 \in M_2$. In a category whose objects have a monoid structure, a *tensor product* of two objects M_1 and M_2 is an object $M_1 \otimes M_2$ with two relatively commuting morphisms $\top_1 : M_1 \to M_1 \otimes M_2 \leftarrow M_2 : \top_2$ such that for any pair $f : M_1 \to M \leftarrow M_2 : g$ of relatively commuting morphisms there is a unique morphism $h_{f,g} : M_1 \otimes M_2 \to M$ with $f = h_{f,g} \circ \top_1$ and $g = h_{f,g} \circ \top_2$. That is, the diagram



can be uniquely completed as shown. Intuitively, the tensor product $M_1 \otimes M_2$ is the free extension of M_1 and M_2 in which elements of M_1 commute with those of M_2.

In the category of monoids, $M_1 \otimes M_2$ is the cartesian product $M_1 \times M_2$ with componentwise unit and product, and $h_{f,g}(m_1, m_2) = f(m_1) \cdot g(m_2)$. The category $\mathbb{D}\mathcal{R}$ of *-continuous Kleene algebras also has tensor products:

Theorem 2.4. (Theorem 4 of [5])
Let K_1, K_2 be $\mathcal{R}$-dioids and M_1, M_2 their multiplicative monoids. The tensor product of K_1, K_2 is

$$K_1 \otimes_{\mathcal{R}} K_2 := \mathcal{R}(M_1 \times M_2)/_{\equiv},$$

the quotient of the regular sets $\mathcal{R}(M_1 \times M_2)$ of the monoid product $M_1 \times M_2$ by the $\mathcal{R}$-congruence $\equiv$ generated by the "tensor product equations"

$$\{\, A \times B = \{(\sum A, \sum B)\} \mid A \in \mathcal{R}M_1, B \in \mathcal{R}M_2 \,\}.$$

Since the natural embeddings of M_1, M_2 in $M_1 \times M_2$ lift $A \in \mathcal{R}M_1$ and $B \in \mathcal{R}M_2$ to sets in $\mathcal{R}(M_1 \times M_2)$,

$$A \times B = (A \times \{1\})(\{1\} \times B) \in \mathcal{R}(M_1 \times M_2).$$

The $\mathcal{R}$-morphisms $\top_1 : K_1 \to K_1 \otimes_{\mathcal{R}} K_2 \leftarrow K_2 : \top_2$ are $\top_1(a) := \{(a, 1)\}/_{\equiv}$ for $a \in K_1$ and $\top_2(b) = \{(1, b)\}/_{\equiv}$ for $b \in K_2$. For a pair of commuting $\mathcal{R}$-morphisms $f : K_1 \to K \leftarrow K_2 : g$ to an $\mathcal{R}$-dioid K, the induced map is

$$h_{f,g}(R/_{\equiv}) := \sum\{\, f(a)g(b) \mid (a, b) \in R \,\}, \quad R \in \mathcal{R}(M_1 \times M_2).$$

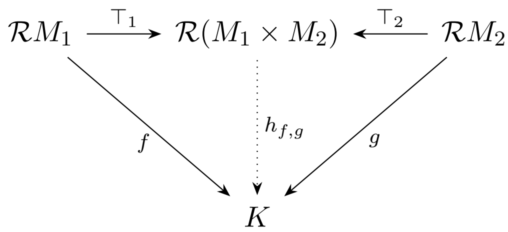
For $a \in K_1$ and $b \in K_2$, the tensor $\top_1(a)\top_2(b) = \{(a, b)\}/_{\equiv}$ is written $a \otimes b$, but when K_1 and K_2 are disjoint, we simply use ab. (If they are not disjoint, ab could also mean $(ab \otimes 1)$ or $(1 \otimes ab)$.) Notice that if $a = 0$ in K_1 or $b = 0$ in K_2, then $a \otimes b = 0$ in $K_1 \otimes_{\mathcal{R}} K_2$, for if, say, $a = 0$, then

$$\{(0, b)\} = \{(\sum \emptyset, \sum\{b\})\} \equiv \emptyset \times \{b\} = \emptyset.$$

It follows that $K_1 \otimes_{\mathcal{R}} K_2$ is trivial if K_1 or K_2 is trivial.

Proposition 2.5. (Proposition 7 of [5])
If M_1 and M_2 are monoids, then $\mathcal{R}M_1 \otimes_{\mathcal{R}} \mathcal{R}M_2 \simeq \mathcal{R}(M_1 \times M_2)$.

Proof: Let $\top_1(A) = A \times \{1\}$ for $A \in \mathcal{R}M_1$ and $\top_2(B) = \{1\} \times B$ for $B \in \mathcal{R}M_2$ in



and put $h_{f,g}(S) = \sum\{\, f(\{a\})g(\{b\}) \mid (a,b) \in S\,\}$ for $S \in \mathcal{R}(M_1 \times M_2)$ and commuting $\mathcal{R}$-morphisms f, g to an $\mathcal{R}$-dioid K. These satisfy the properties of a tensor product of $\mathcal{R}M_1$ and $\mathcal{R}M_2$, so the claim holds by the uniqueness of tensor products. ☐

In the following, for $\mathcal{R}$-dioids K_1, K_2, we also write $K_1 \times K_2$ for the product of their underlying multiplicative monoids, and for $R \in \mathcal{R}(K_1 \times K_2)$, we write $[R]$ instead of $R/_\equiv$. For $R, S \in \mathcal{R}(K_1 \times K_2)$, one has $[R] + [S] = [R \cup S]$, $[R][S] = [RS]$, and

$$[R]^* = \sum\{\, [R]^n \mid n \in \mathbb{N}\,\} = \sum\{\, [R^n] \mid n \in \mathbb{N}\,\} = [\bigcup\{\, R^n \mid n \in \mathbb{N}\,\}] = [R^*].$$

Notice also that $[R] = [\bigcup\{\, \{(a,b)\} \mid (a,b) \in R\}] = \sum\{\, a \otimes b \mid (a,b) \in R\,\}$.

The $\mathcal{R}$-morphisms in $\top_1 : K_1 \to K_1 \otimes_{\mathcal{R}} K_2 \leftarrow K_2 : \top_2$ are embeddings, unless one of K_1 and K_2 is trivial and the other is not:

Lemma 2.6. Let K_1 and K_2 be non-trivial $\mathcal{R}$-dioids. Then the tensor product

$$\top_1 : K_1 \to K_1 \otimes_{\mathcal{R}} K_2 \leftarrow K_2 : \top_2$$

is non-trivial, and $\top_1$ and $\top_2$ are embeddings.

Proof:
An element $(x,y) \in K_1 \times K_2$ is an *upper bound of* $R \subseteq K_1 \times K_2$, written $R \preceq (x,y)$, if $a \leq x$ and $b \leq y$ for all $(a,b) \in R$. Let $Z = \{\, (a,b) \in K_1 \times K_2 \mid a = 0 \text{ or } b = 0\,\}$. For $R, S \in \mathcal{R}(K_1 \times K_2)$, define $P(R,S)$ by

$$\forall (x,y),(a,b),(a',b')[(a,b)R(a',b') \setminus Z \preceq (x,y) \iff (a,b)S(a',b') \setminus Z \preceq (x,y)]. \qquad (5)$$

With $(a,b) = (a',b') = (1,1)$, (5) says that $R \setminus Z$ and $S \setminus Z$ have the same upper bounds in $K_1 \times K_2$. In particular, for $R \subseteq Z$ and $S \not\subseteq Z$, $P(R,S)$ is false, since $(0,0)$ is an upper bound of $R \setminus Z$, but not of $S \setminus Z$. We defer the proof of $\equiv \subseteq P$ to the appendix, Section 7. Then for any $(a,b) \notin Z$, $\{(0,0)\} \not\equiv \{(a,b)\}$, hence $0 \otimes 0 \neq a \otimes b$. As $(1,1) \notin Z$, $0 = 0 \otimes 0 \neq 1 \otimes 1 = 1$, so $K_1 \otimes_{\mathcal{R}} K_2$ is non-trivial. Furthermore, if (a,b) and (a',b') are different elements of $(K_1 \times K_2) \setminus Z$, then $a \otimes b \neq a' \otimes b'$, because $\{(a,b)\} \equiv \{(a',b')\}$ implies, via (5), that (a,b) is an upper bound of $\{(a',b')\}$ and (a',b') is an upper bound of $\{(a,b)\}$, so $(a,b) = (a',b')$. In particular, $\top_1$ and $\top_2$ are injective. ☐

Corollary 2.7. If $K_1 \otimes_{\mathcal{R}} K_2$ is the tensor product of non-trivial $\mathcal{R}$-dioids K_1 and K_2, then for all $a, a' \in K_1$ and $b, b' \in K_2$,

(i). if $a \otimes b = 0$, then $a = 0$ in K_1 or $b = 0$ in K_2,

(ii). if $0 \neq a \otimes b \leq a' \otimes b'$ in $K_1 \otimes_{\mathcal{R}} K_2$, then $0 \neq a \leq a'$ in K_1 and $0 \neq b \leq b'$ in K_2.

Proof:
For (i), if $a \neq 0$ and $b \neq 0$, then $\{(0,0)\} \not\equiv \{(a,b)\}$ by the previous proof, which just means that $a \otimes b \neq 0$ in $K_1 \otimes_{\mathcal{R}} K_2$. For (ii), suppose $0 \neq a \otimes b \leq a' \otimes b'$. Then $0 \neq a' \otimes b'$ too, and $a \neq 0 \neq a'$ in K_1 and $b \neq 0 \neq b'$ in K_2. Since $a \otimes b + a' \otimes b' = a' \otimes b'$, we have

$$\{(a,b),(a',b')\} \equiv \{(a',b')\},$$

and since $\equiv \subseteq P$ for the predicate P in the proof of Lemma 2.6, for any (x,y) we have, by (5),

$$\{(a,b),(a',b')\} \preceq (x,y) \Leftrightarrow \{(a',b')\} \preceq (x,y).$$

The right-hand side is true for $(x,y) = (a',b')$, so $(a,b) \leq (a',b')$ holds by the left-hand side. □

Corollary 2.8. Let f and g be injective $\mathcal{R}$-morphisms between non-trivial $\mathcal{R}$-dioids in

$$\begin{array}{ccccc}
K_1 & \xrightarrow{\top_1} & K_1 \otimes_{\mathcal{R}} K_2 & \xleftarrow{\top_2} & K_2 \\
\downarrow f & & \downarrow h_{(f\times g)} & & \downarrow g \\
K_1' & \xrightarrow{\top_1'} & K_1' \otimes_{\mathcal{R}} K_2' & \xleftarrow{\top_2'} & K_2'.
\end{array}$$

Then $h_{(f\times g)} : K_1 \otimes_{\mathcal{R}} K_2 \to K_1' \otimes_{\mathcal{R}} K_2'$, the induced $\mathcal{R}$-morphism for $\top_1' \circ f$ and $\top_2' \circ g$, is injective.

Proof:
By Lemma 2.6, the $\mathcal{R}$-morphisms $\top_1, \top_2, \top_1', \top_2'$ are embeddings. The homomorphism $f \times g$: $K_1 \times K_2 \to K_1' \times K_2'$ lifts to a monotone homomorphism $(f \times g) : \mathcal{R}(K_1 \times K_2) \to \mathcal{R}(K_1' \times K_2')$. Since $\top_1' \circ f$ and $\top_2' \circ g$ are commuting, they induce an $\mathcal{R}$-morphism $h = h_{(f\times g)}$. For $R \in \mathcal{R}(K_1 \times K_2)$, it maps $[R] \in K_1 \otimes_{\mathcal{R}} K_2$ to

$$h([R]) = [(f \times g)(R)]' = \sum{}' \{ fa \otimes' gb \mid (a,b) \in R \} \in K_1' \otimes_{\mathcal{R}} K_2',$$

where $fa \otimes' gb = \top_1'(fa)\top_2'(gb)$ and $\sum'$ is the least upper bound of the $\mathcal{R}$-dioid $K_1' \otimes_{\mathcal{R}} K_2'$. In particular, for $(a,b) \in K_1 \times K_2$,

$$h(a \otimes b) = fa \otimes' gb.$$

By Lemma 2.6, h is monotone and injective on the image of $K_1 \times K_2$ under $\otimes$. To see that h is injective, suppose $R, S \in \mathcal{R}(K_1 \times K_2)$ and

$$[R] = \sum \{ a \otimes b \mid (a,b) \in R \} \neq \sum \{ a \otimes b \mid (a,b) \in S \} = [S].$$

Then $\{ a \otimes b \mid (a,b) \in R \}^{\downarrow} \neq \{ a \otimes b \mid (a,b) \in S \}^{\downarrow}$ by the definition of $\sum$ of $K_1 \otimes_{\mathcal{R}} K_2$. Since h is monotone and injective on the image of $K_1 \times K_2$ under $\otimes$,

$$\{ fa \otimes' gb \mid (a,b) \in R \}^{\downarrow} \neq \{ fa \otimes' gb \mid (a,b) \in S \}^{\downarrow}.$$

Then we must have

$$h([R]) = \sum{}' \{ fa \otimes' gb \mid (a,b) \in R \}^{\downarrow} \neq \sum{}' \{ fa \otimes' gb \mid (a,b) \in S \}^{\downarrow} = h([S]),$$

as otherwise $(f \times g)(R) \equiv' (f \times g)(S)$ for the $\mathcal{R}$-congruence $\equiv'$ on $\mathcal{R}(K_1' \times K_2')$ defining $K_1' \otimes_{\mathcal{R}} K_2'$, and $\equiv'$ were not the *least* $\mathcal{R}$-congruence on $\mathcal{R}(K_1' \times K_2')$ containing the tensor product equations. $\square$

We will mainly consider tensor products $K \otimes_{\mathcal{R}} C$ where $K = \mathcal{R}X^*$ and C is a polycyclic $\mathcal{R}$-dioid C_m' or bra-ket $\mathcal{R}$-dioid C_m. For $L \in \mathcal{R}X^*$, we have $\{ \{w\} \mid w \in L \} \in \mathcal{R}(\mathcal{R}X^*)$, and since $\top_1$ is an $\mathcal{R}$-morphism,

$$L \otimes 1 = \top_1(\bigcup \{ \{w\} \mid w \in L \}) = \sum \{ \{w\} \otimes 1 \mid w \in L \} \in \mathcal{R}X^* \otimes_{\mathcal{R}} C_2'.$$

The interest in Kleene algebras $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ comes from the fact that $\mathcal{C}X^*$, the set of context-free languages over X, embeds in $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$, via

$$L \in \mathcal{C}X^* \mapsto \sum L := \sum \{ \{w\} \otimes 1 \mid w \in L \} \in \mathcal{R}X^* \otimes_{\mathcal{R}} C_2',$$

cf. Theorem 17 of [11]. Notice that $L \otimes 1$ need not exist for non-regular L. Since all elements of $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ can be denoted by regular expressions over $X \mathbin{\dot\cup} \Delta_2$, every *context-free* set $L \subseteq X^*$ is represented by the value of a *regular* expression.

Example 2.9. Suppose $a, b \in X$. Then $L = \{ a^n b^n \mid n \in \mathbb{N} \} \in \mathcal{C}X^*$ is represented in $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ by the value of the regular expression $r_L := p_0(ap_1)^*(q_1 b)^* q_0$ over $X \mathbin{\dot\cup} \Delta_2$. Writing elements of X and Δ_2 for their values in $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$, we have

$$\begin{aligned} r_L &= \textstyle\sum \{ p_0(ap_1)^n (q_1 b)^m q_0 \mid n, m \in \mathbb{N} \} && \text{(*-continuity)} \\ &= \textstyle\sum \{ a^n p_0 p_1^n q_1^m q_0 b^m \mid n, m \in \mathbb{N} \} && \text{(relative commutativity)} \\ &= \textstyle\sum \{ a^n b^n \mid n \in \mathbb{N} \} && \text{(bracket match } p_i q_j = \delta_{i,j}\text{)}. \quad \triangleleft \end{aligned}$$

In the cases $K \otimes_{\mathcal{R}} C_m'$ of our main interest, where $K = \mathcal{R}X^*$ and the polycyclic $\mathcal{R}$-dioid $C_m' \simeq \mathcal{R}P_m'/\nu$ is a suitable quotient of $\mathcal{R}P_m'$, the tensor product construction can be replaced by a quotient construction. This is a consequence of the following extension of Proposition 2.5.

Theorem 2.10. Let M be a monoid and N a monoid with annihilating element 0. Then

$$\mathcal{R}M \otimes_{\mathcal{R}} (\mathcal{R}N/\nu) \simeq \mathcal{R}(M \times N)/\tilde{\nu},$$

where ν is the least $\mathcal{R}$-congruence on $\mathcal{R}N$ containing $(\{0\}, \emptyset)$ and $\tilde{\nu}$ is the least $\mathcal{R}$-congruence on $\mathcal{R}(M \times N)$ containing $(\{(1,0)\}, \emptyset)$.

Putting $R_\nu := \{ (A, B/\nu) \mid (A,B) \in R \}$ for $R \in \mathcal{R}(\mathcal{R}M \times \mathcal{R}N)$, the isomorphism is given by

$$\begin{aligned} [R_\nu] &\mapsto (S_R)/\tilde{\nu}, && \text{where } S_R := \bigcup \{ A \times B \mid (A,B) \in R \} \text{ for } R \in \mathcal{R}(\mathcal{R}M \times \mathcal{R}N), \\ S/\tilde{\nu} &\mapsto [(R_S)_\nu], && \text{where } R_S := \{ (\{m\}, \{n\}) \mid (m,n) \in S \} \text{ for } S \in \mathcal{R}(M \times N). \end{aligned}$$

Proof: This is an instance of Theorem 12 of [11]. □

For $A \in \mathcal{R}M$ and $B \in \mathcal{R}N$, the isomorphism maps $A \otimes B/\nu$ to $(A \times B)/\tilde{\nu}$, where B/ν is uniquely represented by $B \setminus \{0\}$ and $(A \times B)/\tilde{\nu}$ by $(A \times B) \setminus (A \times \{0\})$. As $C_m' \simeq P_m'/\nu$ by Proposition 2.2, an application of the theorem is

$$\mathcal{R}X^* \otimes_{\mathcal{R}} C_m' \simeq \mathcal{R}(X^* \times P_m')/\tilde{\nu}.$$

Moreover, since elements of X and P_m' commute in the monoid $P_m'[X]$ of (4),

$$\mathcal{R}(X^* \times P_m')/\tilde{\nu} \simeq \mathcal{R}(P_m'[X])/\nu.$$

It follows that an element of $\mathcal{R}X^* \otimes_{\mathcal{R}} C_m'$ has a unique representation by a subset of $Q_m^* X^* P_m^*$.

However, to state our results for arbitrary $\mathcal{R}$-dioids K, we do need the tensor product $K \otimes_{\mathcal{R}} C_m'$.

2.3. The centralizer $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ of C_2' in $K \otimes_{\mathcal{R}} C_2'$

In a monoid M, the *centralizer* $Z_C(M)$ of a set $C \subseteq M$ in M consists of those elements that commute with every element of C, i.e. the submonoid

$$Z_C(M) := \{\, m \in M \mid mc = cm \text{ for all } c \in C \,\}.$$

For example, the centralizer of Δ_m in $P_m'[X]$ is $X^* \cup \{0\}$.

In Section 3, we will, for non-trivial $\mathcal{R}$-dioids K, consider the representation of elements of $K \otimes_{\mathcal{R}} C_m'$ by automata. As $\top_1, \top_2$ in $\top_1 : K \to K \otimes_{\mathcal{R}} C_2' \leftarrow C_2' : \top_2$ are relatively commuting, for all $k \in K$ and $c \in C_2'$ we have

$$kc = \top_1(k) \cdot \top_2(c) = k \otimes c = \top_2(c) \cdot \top_1(k) = ck,$$

in $K \otimes_{\mathcal{R}} C_2'$, so $K \subseteq Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ (modulo $\top_1$). Moreover, $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ clearly is a semiring and, by *-continuity of $K \otimes_{\mathcal{R}} C_2'$, it is closed under *: if a commutes with $c \in C_2'$, then

$$c \cdot a^* = \sum \{\, c \cdot a^n \mid n \in \mathbb{N} \,\} = \sum \{\, a^n \cdot c \mid n \in \mathbb{N} \,\} = a^* \cdot c.$$

In fact, $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is an $\mathcal{R}$-dioid, by Proposition 24 of [11]. It has even stronger closure properties, see Theorem 2.11, (ii) below.

A *Chomsky algebra* (Grathwohl e.a. [9]) is an idempotent semiring D which is *algebraically closed*, i.e. every finite inequation system

$$x_1 \geq p_1(x_1, \ldots, x_k), \ldots, x_k \geq p_k(x_1, \ldots, x_k)$$

with polynomials $p_1, \ldots, p_k \in D[x_1, \ldots, x_k]$ has a least solution in D, where $\leq$ is the partial order on D defined by $a \leq b \iff a + b = b$. Semiring terms over an infinite set X of variables can be extended by a least-fixed-point operator μ, such that if t is a term and $x \in X$, $\mu x.t$ is a term. In a Chomsky algebra D with an assignment $h : X \to D$, the value of $\mu x.t$ is the least solution of $x \geq t$

with respect to h, i.e. the least $a \in D$ such that $x \geq t$ is true with respect to $h[x/a]$. A Chomsky algebra D is *μ-continuous*, if for all $a, b \in D$ and μ-terms t,

$$a \cdot \mu x.t \cdot b = \sum \{\, a \cdot t^n \cdot b \mid n \in \mathbb{N} \,\}$$

is true for all assignments $h : X \to D$, where $t^0 = 0$, $t^{n+1} = t[x/t^n]$. The *-continuity condition of $\mathcal{R}$-dioids is a special instance of the μ-continuity condition, where $c^* = \mu x.(cx + 1)$. The semiring $\mathcal{C}X^*$ of context-free languages over X is a μ-continuous Chomsky algebra. The μ-continuous Chomsky algebras, with fixed-point preserving semiring homomorphisms, form a category of dioids.

This category had been introduced as the category $\mathbb{D}\mathcal{C}$ of $\mathcal{C}$-dioids and $\mathcal{C}$-morphisms in [3] as follows; for the equivalence, see [12]. For monoids M, let $\mathcal{C}M$ be the semiring $(\mathcal{C}M, \cup, \cdot, \emptyset, \{1\})$ of context-free subsets of M. A *$\mathcal{C}$-dioid* $(M, \cdot, 1, \leq, \sum)$ is a partially ordered monoid $(M, \cdot, 1, \leq)$ with an operation $\sum : \mathcal{C}M \to M$ that is *$\mathcal{C}$-complete* and *$\mathcal{C}$-distributive*, i.e.

(i) for each $A \in \mathcal{C}M$, $\sum A$ is the least upper bound of A in M with respect to $\leq$,

(ii) for all $A, B \in \mathcal{C}M$, $\sum(AB) = (\sum A) \cdot (\sum B)$.

A *$\mathcal{C}$-morphism* is a monotone homomorphism between $\mathcal{C}$-dioids that preserves least upper bounds of context-free subsets. The above mentioned strong closure property of the centralizer of C_2' in $K \otimes_{\mathcal{R}} C_2'$ is that it is algebraically closed, which follows from (ii) of the following facts:

Theorem 2.11. (Theorem 27, Lemma 30, Lemma 31 of [11])
Let M be a monoid and K an $\mathcal{R}$-dioid.

(i). $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2') = \{\, [R] \mid R \in \mathcal{R}(K \times C_2'), R \subseteq K \times \{0, 1\} \,\}$.

(ii). $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is a $\mathcal{C}$-dioid.

(iii). The least-upper-bound operator $\sum : \mathcal{C}K \to Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is a surjective homomorphism.

(iv). The least-upper-bound operator $\sum : \mathcal{C}M \to Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2')$ is a $\mathcal{C}$-isomorphism.

While (i) gives a characterization of the elements in the centralizer of C_2' in $K \otimes_{\mathcal{R}} C_2'$, in Section 3 we provide descriptions of *all* elements of $K \otimes_{\mathcal{R}} C_2'$ via normal forms. As the proof of Theorem 2.11 is lengthy, we try to avoid using (i) - (iv) as far as possible. However, we need (i) in the following corollary, which in turn is used to give a simplified normal form for elements of the centralizer in Corollary 3.8, and we use (ii) in Example 3.12 and for the product case of Theorem 4.1.

A subset X of a partial order $(P, \leq)$ is *downward closed*, if for all $a, b \in P$, if $b \in X$ and $a \leq b$, then $a \in X$.

Corollary 2.12. If K is a non-trivial $\mathcal{R}$-dioid and has no zero divisors, then $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is a downward-closed subset of $K \otimes_{\mathcal{R}} C_2'$.

Proof:

Suppose $[R] \leq [S] \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ for $R, S \in \mathcal{R}(K \times C_2')$. By Theorem 2.11 (i), we can assume $S \subseteq K \times \{0,1\}$ and must show that there is $R' \in \mathcal{R}(K \times C_2')$ with $[R] = [R']$ and $R' \subseteq K \times \{0,1\}$. The projection from $K \times C_2'$ to K lifts to a homomorphism $\pi : \mathcal{R}(K \times C_2') \to \mathcal{R}K$, so $A := \pi(S) \in \mathcal{R}K$. Then

$$S \subseteq A \times \{0,1\} \in \mathcal{R}(K \times C_2'),$$

and for each $(k, c) \in R$,

$$k \otimes c \leq [R] \leq [S] \leq [A \times \{0,1\}] = [\{(\sum A, 1)\}] = (\sum A) \otimes 1.$$

If $0 \neq k \otimes c$, then $c \leq 1$ in C_2' by Corollary 2.7; by Proposition 2.2, $c \in \{0,1\}$, so $(k,c) \in K \times \{0,1\}$. If $0 = k \otimes c$, then by Corollary 2.7 again, either $c = 0$ and $(k,c) \in K \times \{0,1\}$, or else $k = 0$. Let $R' = R \setminus \{\, (0,c) \in R \mid c \in C_2' \,\}$. Then $R' \subseteq K \times \{0,1\}$ and $[R] = \sum\{\, k \otimes c \mid (k,c) \in R \,\}$ is the least upper bound of $\{\, k \otimes c \mid (k,c) \in R' \,\}$. We show by induction on the construction of $R \in \mathcal{R}(K \times C_2')$ that $R' \in \mathcal{R}(K \times C_2')$. This also gives $[R] = [R']$.

If R is finite, so is R', therefore $R' \in \mathcal{R}(K \times C_2')$. Suppose for $R_i \in \mathcal{R}(K \times C_2')$, $i = 1, 2$, we have $R_i' = R_i \setminus \{\, (0,c) \mid c \in C_2' \,\} \in \mathcal{R}(K \times C_2')$. If $R = R_1 \cup R_2$, then $R' = R_1' \cup R_2' \in \mathcal{R}(K \times C_2')$. If $R = R_1 R_2$, then $R' \subseteq R_1' R_2'$, and since K has no zero divisors, $R_1' R_2' \subseteq R'$, so $R' = R_1' R_2' \in \mathcal{R}(K \times C_2')$. If $R = R_1^*$, then $R' = (\bigcup\{\, R_1^n \mid n \in \mathbb{N} \,\})' = \bigcup\{\, (R_1')^n \mid n \in \mathbb{N} \,\} = (R_1')^* \in \mathcal{R}(K \times C_2')$. □

2.4. Automata over a Kleene algebra

A *finite automaton* $\mathcal{A} = \langle S, A, F \rangle$ *with* n *states* over a Kleene algebra K consists of a transition matrix $A \in K^{n \times n}$ and two vectors $S \in \mathbb{B}^{1 \times n}$ and $F \in \mathbb{B}^{n \times 1}$, coding the initial and final states. The 1-step transitions from state $i < n$ to state $j < n$ are represented by $A_{i,j}$, and paths from i to j of finite length by $A^*_{i,j}$, where A^* is the iteration of A. The sum of paths leading from initial to final states defines an element of K,

$$L(\mathcal{A}) = S \cdot A^* \cdot F \ \in K.$$

The iteration M^* of $M \in K^{n \times n}$ is defined by induction on n: for $n = 1$ and $M = (k)$, $M^* = (k^*)$, and for $n > 1$,

$$M^* = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} F^* & F^*BD^* \\ D^*CF^* & D^*CF^*BD^* + D^* \end{pmatrix}, \tag{6}$$

where $F = A + BD^*C$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is any splitting of M in which A and D are square matrices of dimensions $n_1, n_2 < n$ with $n = n_1 + n_2$.

By Kleene's representation theorem, the set $\mathcal{R}X^*$ of regular subsets of X^* consists of the languages

$$L(\mathcal{A}) = S \cdot A^* \cdot F \ \subseteq X^*$$

of finite automata $\mathcal{A} = \langle S, A, F \rangle$, where for some $n \in \mathbb{N}$, $A \in (\mathcal{F}X^*)^{n \times n}$, $S \in \mathbb{B}^{1 \times n}$, $F \in \mathbb{B}^{n \times 1}$ and $\mathcal{F}X^*$ is the set of finite subsets of X^*.

For various notions of Kleene algebra, Conway showed that the set $K^{n\times n}$ of $n \times n$-matrices over K with matrix addition, multiplication and iteration as defined above and zero and unit matrices $0_n, 1_n \in K^{n\times n}$ form a Kleene algebra

$$Mat_{n,n}(K) = (K^{n\times n}, +, \cdot, ^*, 0_n, 1_n)$$

and used this to prove Kleene's representation theorem, see [2]. For the notion of Kleene algebra used here, the same has been done by Kozen in [8]. We are mostly working with $\mathcal{R}$-dioids, i.e. *-continuous Kleene algebras, and will often make use of *-continuity on the matrix level in Section 3. In fact, the $n \times n$-matrices over a *-continuous Kleene algebra form a *-continuous Kleene algebra:

Theorem 2.13. (Kozen [7], Chapter 7.1.)
If K is a *-continuous Kleene algebra, so is $Mat_{n,n}(K)$, for $n \geq 1$.

We remark that $Mat_{n,n}(K)$ can be reduced to the tensor product of K with $Mat_{n,n}(\mathbb{B})$, but we will use this only in connection with bra-ket $\mathcal{R}$-dioids in Section 5.2.

Proposition 2.14. For any $\mathcal{R}$-dioid K and $n \geq 1$, $Mat_{n,n}(K) \simeq K \otimes_{\mathcal{R}} Mat_{n,n}(\mathbb{B})$.

Proof (sketch)**:**
One shows that $I_K : K \to Mat_{n,n}(K) \leftarrow Mat_{n,n}(\mathbb{B}) : \mathit{Id}$ has the properties of a tensor product, where $I_K(a) := a1_n$ for $a \in K$ and $\mathit{Id}(B) = B$ for $B \in \mathbb{B}^{n\times n}$. For relatively commuting $\mathcal{R}$-morphisms $f : K \to D \leftarrow Mat_{n,n}(\mathbb{B}) : g$ to an $\mathcal{R}$-dioid D, the unique $\mathcal{R}$-morphism with $f = h_{f,g} \circ I_K$ and $g = h_{f,g} \circ \mathit{Id}$ is defined by

$$h_{f,g}(A) := \sum \{\, f(A_{i,j})g(E_{(i,j)}) \mid i, j < n \,\}, \quad \text{for } A \in K^{n\times n},$$

where $E_{(i,j)} \in \mathbb{B}^{n\times n}$ is the matrix with 1 only in line i, row j. The claim then follows by the uniqueness of tensor products. □

For any $\mathcal{R}$-dioid K, we next prove Kleene's representation theorem for $K \otimes_{\mathcal{R}} C_2'$: any element of $K \otimes_{\mathcal{R}} C_2'$ is the "language" $L(\mathcal{A}) = SA^*F$ of a finite automaton $\mathcal{A} = \langle S, A, F\rangle$ over $K \otimes_{\mathcal{R}} C_2'$. This follows the proofs by Conway and Kozen; the point here is how transitions by elements of C_2' in the transition matrix A can be reduced to transitions by generators $c \in \Delta_2$ of C_2'.

For $a \in K$ and $c \in C_2'$, we write a and c also for their images in $K \otimes_{\mathcal{R}} C_2'$, likewise ac for their product in $K \otimes_{\mathcal{R}} C_2'$. From now on, for $\Delta_2 = P_2 \mathbin{\dot\cup} Q_2$ we use $P_2 = \{b, p\}$ instead of $\{p_0, p_1\}$ and $Q_2 = \{d, q\}$ instead of $\{q_0, q_1\}$, unless stated otherwise.

Theorem 2.15. Let K be an $\mathcal{R}$-dioid, i.e. a *-continuous Kleene-algebra, and C_2' the polycyclic Kleene algebra over Δ_2. For each $\varphi \in K \otimes_{\mathcal{R}} C_2'$ there are $n \in \mathbb{N}$, $S \in \mathbb{B}^{1\times n}, F \in \mathbb{B}^{n\times 1}$, $U \in \{0, b, p\}^{n\times n}$, $V \in \{0, d, q\}^{n\times n}$ and $X \in K^{n\times n}$ such that

$$\varphi = S(U + X + V)^*F.$$

Proof:
Since $\varphi = [R]$ for some $R \in \mathcal{R}(K \times C_2')$, by induction on the construction of R we build an automaton $\mathcal{A}_R = \langle S, A, F\rangle$ over $K \otimes_{\mathcal{R}} C_2'$ such that $L(\mathcal{A}_R) = [R]$ and A splits as $U + X + V$ as in the claim.

- $R = \emptyset$: Let $\mathcal{A}_R = \langle S, A, F\rangle$ be the automaton of dimension 1 with $S = (0)$, $A = (0)$, $F = (0)$. Then $L(\mathcal{A}_R) = 0 = [\emptyset]$. We have $A = U + X + V$ with 1×1 zero matrices U, X, V.

- $R = \{(k, c)\}$ with $k \in K, c \in C_2'$: Since $\{(k, c)\} = \{(k, 1)\} \cdot \{(1, c)\}$, by the product case below we may assume $k = 1$ or $c = 1$. In the case $R = \{(k, 1)\}$, let $\mathcal{A}_R = \langle S, A, F\rangle$ consist of

$$S = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $A^* = A$, since $A^0 \leq A = A^2$, hence $L(\mathcal{A}_R) = A_{1,2} = k1 = [\{(k, 1)\}]$. The splitting is

$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = U + X + V.$$

For the case $R = \{(1, c)\}$, the element $c \in C_2'$ is the congruence class of a set $C \in \mathcal{R}\Delta_2^*$ under the $\mathcal{R}$-congruence ρ_2' generated by the match relations, so we can view c as a regular expression in the letters of Δ_2. By the tensor product equations of $K \otimes_{\mathcal{R}} C_2'$,

$$\{(1, c_1 + c_2)\} \equiv \{1\} \times \{c_1, c_2\} = \{(1, c_1)\} \cup \{(1, c_2)\},$$

and since $\{(1, c_1c_2)\} = \{(1, c_1)\}\{(1, c_2)\}$ and $\{(1, c_1^*)\} = \{(1, c_1)\}^*$, we can construct $\mathcal{A}_R$ by induction on the cases $R = R_1 \cup R_2$, $R = R_1R_2$, and $R = R_1^*$ below. In the remaining cases, c is 0, 1 or a letter from Δ_2. Let $\mathcal{A}_R = \langle S, A, F\rangle$ consist of

$$S = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $A^* = A$ and $L(\mathcal{A}_R) = A_{1,1} = c = [\{(1, c)\}]$. If $c \in Q_2 = \{d, q\}$, the splitting of A is

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} = U + X + V.$$

If $c \in P_2 = \{b, p\}$, we switch the roles of U and V. If c is 0 or 1, let

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = U + X + V.$$

- $R = R_1 \cup R_2$: For $i = 1, 2$, let $\mathcal{A}_{R_i} = \langle S_i, A_i, F_i\rangle$ be an automaton of dimension n_i such that

$$L(\mathcal{A}_{R_i}) = S_iA_i^*F_i = [R_i].$$

Construct $\mathcal{A}_R = \langle S, A, F\rangle$ of dimension $n_1 + n_2$ by

$$S = \begin{pmatrix} S_1 & S_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.$$

By the recursion formula for iteration matrices,

$$\begin{array}{rcl} L(\mathcal{A}_R) & = & SA^*F = \begin{pmatrix} S_1 & S_2 \end{pmatrix} \cdot \begin{pmatrix} A_1^* & 0 \\ 0 & A_2^* \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \\ & = & S_1 A_1^* F_1 + S_2 A_2^* F_2 \\ & = & [R_1] + [R_2] = [R_1 \cup R_2] = [R]. \end{array}$$

The given splittings $A_1 = U_1 + X_1 + V_1$ and $A_2 = U_2 + X_2 + V_2$ combine to a suitable splitting of A by

$$A = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} + \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} + \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} = U + X + V.$$

- $R = R_1 R_2$: For $i = 1, 2$, let $\mathcal{A}_{R_i} = \langle S_i, A_i, F_i\rangle$ be an automaton of dimension n_i such that

$$L(\mathcal{A}_{R_i}) = S_i A_i^* F_i = [R_i].$$

Construct $\mathcal{A}_R = \langle S, A, F\rangle$ of dimension $n_1 + n_2$ by

$$S = \begin{pmatrix} S_1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & F_1 S_2 \\ 0 & A_2 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ F_2 \end{pmatrix}.$$

By the recursion formula for iteration matrices,

$$\begin{array}{rcl} L(\mathcal{A}_R) & = & SA^*F \\ & = & \begin{pmatrix} S_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} A_1^* & A_1^* F_1 S_2 A_2^* \\ 0 & A_2^* \end{pmatrix} \cdot \begin{pmatrix} 0 \\ F_2 \end{pmatrix} \\ & = & S_1 A_1^* F_1 S_2 A_2^* F_2 \\ & = & [R_1][R_2] = [R_1 R_2] = [R]. \end{array}$$

The given splittings $A_1 = U_1 + X_1 + V_1$ and $A_2 = U_2 + X_2 + V_2$ combine to the splitting

$$A = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} + \begin{pmatrix} X_1 & F_1 S_2 \\ 0 & X_2 \end{pmatrix} + \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} = U + X + V.$$

- $R = R_1^*$: Suppose $\mathcal{A}_{R_1} = (S_1, A_1, F_1)$, is an automaton such that

$$L(\mathcal{A}_{R_1}) = S_1 A_1^* F_1 = [R_1].$$

Let $\mathcal{A}_{R^+} = \langle S, A, F \rangle$ be $\langle S_1, A_1 + F_1 S_1, F_1 \rangle$. By equalities in Kleene algebras,

$$\begin{aligned} L(\mathcal{A}_{R^+}) &= S_1(A_1 + F_1 S_1)^* F_1 \\ &= S_1 A_1^*(F_1 S_1 A_1^*)^* F_1 \\ &= S_1 A_1^* F_1 (S_1 A_1^* F_1)^* \\ &= [R_1][R_1]^* \\ &= [R_1][R_1^*] = [R_1^+], \end{aligned}$$

The splitting $A = U + X + V$ is obtained from the splitting $A_1 = U_1 + X_1 + V_1$ by $U = U_1$, $X = X_1 + F_1 S_2$ and $V = V_1$. Finally, put $\mathcal{A}_{R^*} = \mathcal{A}_{\{(1,1)\} \cup R^+}$ and split its transition matrix as shown for the case $\mathcal{A}_{R_1 \cup R_2}$. □

3. Normal form theorems for $K \otimes_{\mathcal{R}} C'_2$ with $\mathcal{R}$-dioid K

In the representation of elements φ of $K \otimes_{\mathcal{R}} C'_2$ as $\varphi = L(\mathcal{A}) = SA^*F$ by automata $\mathcal{A} = \langle S, A, F \rangle$ with $A = U + X + V$ in Theorem 2.15, $A^* = (U + X + V)^*$ admits arbitrary sequences of opening brackets U with closing brackets V. We aim at a normal form for $(U + X + V)^*$ where brackets are mainly occurring in a balanced way. To this end, we now look at ways to express a Dyck-language with a single bracket pair u, v in a Kleene algebra.

3.1. Least solutions of some polynomial inequations in Kleene algebras

We first show that in any Kleene algebra K, if they exist, least solutions of two fixed-point inequations that might be used to define Dyck's language $D_1(X)$ with $X = \{x_1, \dots, x_n\} \subseteq K$, namely

$$y \geq (x_1 + \ldots + x_n + uyv)^* \quad \text{and} \quad y \geq 1 + x_1 + \ldots + x_n + uyv + yy,$$

are related, where $u, v \in K \setminus X$ represent a pair of brackets. It is then shown that $(u + X + v)^* = (Nv)^* N (uN)^*$, where $N \in K$ is the least solution of $y \geq (X + uyv)^*$ corresponding to $D_1(X)$. Except for the balanced bracket occurrences in N, in $(Nv)^* N (uN)^*$ all occurrences of the closing bracket v are to the left of all occurrences of the opening bracket u. This is similar to the normal form $\mathit{nf}(w) \in Q_1^* P_1^* \cup \{0\}$ in the polycyclic monoid P'_1 of Section 2.1 with $P_1 = \{u\}$ and $Q_1 = \{v\}$, i.e. the normal forms on $\{u, v\}^*$ modulo the congruence generated by $uv = 1$, and its extension to $\mathit{nf}(w) \in Q_1^* X^* V_1^* \cup \{0\}$ for $w \in P'_1[X]$ where elements of X commute with those of $P_1 \cup Q_1$.

Proposition 3.1. Let K be a Kleene algebra and $u, x, v \in K$. If $y \geq (x + uyv)^*$ has a least solution N, then $N = (x + uNv)^*$ and N is the least solution of $y \geq 1 + x + uyv + yy$. If $y \geq 1 + x + uyv + yy$ has a least solution D, then $D = 1 + x + uDv + DD$ and D is the least solution of $y \geq (x + uyv)^*$.

Proof:
Let f and h be defined by $f(y) = x + uyv$ and $h(y) = 1 + x + uyv + yy$. (i) If $y \geq h(y)$, then

$y \geq f(y)$ and $y \geq 1 + yy$, hence $y \geq y^*$ by axioms of Kleene algebra, and so $y \geq y^* \geq f(y)^*$ by monotonicity of *. (ii) Conversely, if $y \geq f(y)^*$, then $f(y)^* \geq h(f(y)^*)$, because

$$h(f(y)^*) \leq 1 + x + uyv + f(y)^* f(y)^* \leq f(y) + f(y)^* \leq f(y)^*.$$

It follows that if N is the least solution of $y \geq f(y)^*$, then by (i), any solution of $y \geq h(y)$ satisfies $y \geq N$, and by (ii), $f(N)^*$ is a solution of $y \geq h(y)$, so $f(N)^* = N$ is the least solution of $y \geq h(y)$.

If D is the least solution of $y \geq h(y)$, then by (ii), any solution of $y \geq f(y)^*$ satisfies $y \geq f(y)^* \geq D$, and by (i), $D \geq f(D)^*$. Hence D is the least solution of $y \geq f(y)^*$. Then $D = f(D)^*$ and hence $D = DD = 1 + f(D) + DD = h(D)$. □

Theorem 3.2. Let K be a Kleene algebra and $x, u, v \in K$. If $y \geq (x + uyv)^*$ has a least solution N in K, then $(u + x + v)^* = (Nv)^* N (uN)^*$.

Proof:
Let $N = \mu y.(x + uyv)^*$ and $n = (u + x + v)^*$. We first show $N \leq n$, by showing that n solves $(x + uyv)^* \leq y$. By monotonicity of $+$, $\cdot$, and *,

$$x + unv \leq x + un^*v \leq n + nn^*n = (1 + nn^*)n = n^*n \leq n^* = n,$$

hence $(x + unv)^* \leq n^* = n$. So $N \leq n$, from which

$$(Nv)^* N (uN)^* \quad \leq \quad (u + x + v)^*$$

follows using $u, v, N \leq n$ and $(nn)^* = n^* = n = nnn$.

Now consider the reverse inequality, $(u + x + v)^* \leq (Nv)^* N (uN)^*$: As $(x + uNv)^* = N$ by Proposition 3.1, we have $(x + uNv)N + 1 \leq N$. Using this and Kleene algebra identities like $(ab)^*a = a(ba)^*$, we show that $(Nv)^* N (uN)^*$ solves $(u + x + v)z + 1 \leq z$ in z:

$$\begin{aligned}
&(u + x + v)(Nv)^* N (uN)^* + 1 \\
&= (u + x + v)N(vN)^*(uN)^* + 1 \\
&= uN(vN)^*(uN)^* + xN(vN)^*(uN)^* + vN(vN)^*(uN)^* + 1 \\
&= uN(1 + vN(vN)^*)(uN)^* + xN(vN)^*(uN)^* + vN(vN)^*(uN)^* + 1 \\
&= uN(uN)^* + uNvN(vN)^*(uN)^* + xN(vN)^*(uN)^* + vN(vN)^*(uN)^* + 1 \\
&= (x + uNv)N(vN)^*(uN)^* + uN(uN)^* + vN(vN)^*(uN)^* + 1 \\
&= (x + uNv)N(vN)^*(uN)^* + (1 + vN(vN)^*)(uN)^* \\
&= (x + uNv)N(vN)^*(uN)^* + (vN)^*(uN)^* \\
&= ((x + uNv)N + 1)(vN)^*(uN)^* \\
&\leq N(vN)^*(uN)^* \\
&= (Nv)^* N (uN)^*.
\end{aligned}$$

Since $(u+x+v)^*$ is the least solution of $(u+x+v)z+1 \leq z$, the claim $(u+x+v)^* \leq (Nv)^* N (uN)^*$ is shown. □

It is worth noticing that these results are generic to Kleene algebras and do not require the *-continuity property. They are all conditioned on the *existence* of the relevant least-fixed-points, and it is for existence that *-continuity will come into play.

3.2. Normal form theorems

Let $\mathcal{A} = \langle S, A, F \rangle$ be an automaton with $A = U + X + V$ as in Theorem 2.15, representing an element $\varphi = L(\mathcal{A}) = SA^*F$ of $K \otimes_{\mathcal{R}} C_2'$. We first show that there is a least solution of $y \geq (UyV + X)^*$ in $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$, which is related to Dyck's context-free language $D \subseteq \{U, X, V\}^*$ of balanced strings of matrices, with U as "opening bracket" and V as "closing bracket". Namely, if concatenation is interpreted by matrix multiplication and the empty sequence as unit matrix, D becomes a context-free subset of $(K \otimes_{\mathcal{R}} C_2')^{n \times n}$ and the least solution of $y \geq (UyV + X)^*$ its least upper bound.

Lemma 3.3. Let K be an $\mathcal{R}$-dioid, $n \in \mathbb{N}$, $X \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n \times n}$, $U \in \{0, b, p\}^{n \times n}$ and $V \in \{0, d, q\}^{n \times n}$. In $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$,

$$y \geq (UyV + X)^* \tag{7}$$

has a least solution, namely $N := b(Up + X + qV)^*d$, and $N \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n \times n}$.

When multiplying b, d, p, q with $n \times n$-matrices, we identify them with corresponding diagonal matrices.[2]

Proof:
Let D and D' be the Dyck languages over $\{U, X, V\}$ and $\{Up, X, qV\}$ with brackets U, V and Up, qV, respectively. By interpreting concatenation as matrix multiplication and the empty sequence as unit matrix, elements of D and D' belong to $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$. To simplify the notation, we write T for $Up + X + qV$ and Z for $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$.

Claim 3.3. Every $A \in D$ evaluates in $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$ to an element of $Z^{n \times n}$.

Proof:
This is clear for $A = 1$ and $A = X$, and if $A, B \in D$ evaluate to $A, B \in Z^{n \times n}$, then $AB \in Z^{n \times n}$, because Z is a semiring. Finally, consider $A = UBV$ with $B \in Z^{n \times n}$. Since elements of Z and C_2' commute with each other in $K \otimes_{\mathcal{R}} C_2'$, we have

$$(UBV)_{ij} = \sum_{k,l=1}^{n} U_{ik}(B_{kl}V_{lj}) = \sum_{k,l=1}^{n} B_{kl}(U_{ik}V_{lj}),$$

and since $U_{ik} \in \{0, b, p\}$ and $V_{lj} \in \{0, d, q\}$, we obtain $U_{ik}V_{lj} \in \{0, 1\}$, hence $(UBV)_{ij} \in Z$, and so $A \in Z^{n \times n}$. $\triangleleft$

It follows that $bAd = A = pAq$ for each $A \in D$ and $\sum(\{U, X, V\}^m \cap D) \in Z^{n \times n}$ for each $m \in \mathbb{N}$.

[2] The proof will show that N is the least upper bound of a context-free set D of $n \times n$-matrices over $Z = Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ (with $DD \subseteq D$) and the least solution of the matrix inequation $y \geq 1 + X + UyV + yy$. Alternatively, by Theorem 2.11, (ii), Z is a $\mathcal{C}$-dioid, and by [10], its $n \times n$ matrix semiring also is. Hence D has a least upper bound $\sum D$ and $(\sum D)(\sum D) = \sum(DD) \leq \sum D$. Since U, V are not matrices over Z, one needs additional arguments to show that $\sum D$ is the least solution of the matrix inequation in $Mat_{n,n}(Z)$ and least in $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$. Our proof here is more elementary and uses properties of $\mathcal{R}$-dioids only.

Claim 3.3. $bT^m d = \sum(\{U, X, V\}^m \cap D)$ and $bT^m d \leq T^m$, for each $m \in \mathbb{N}$.

Proof:
Let $A' \in D'$ be obtained from $A \in D$ by replacing factors U by Up and factors V by qV. Then as matrices, $A' = A$: clearly $1' = 1$ and $X' = X$, and by induction, for $A, B \in D$, $(AB)' = A'B' = AB$ and $(UAV)' = UpA'qV = UpAqV = UAV$, as A belongs to $Z^{n\times n}$ by claim 3.3. Moreover, if $A \in D \cap \{U, X, V\}^m$, then the matrix value of $A' \in \{Up, X, qV\}^m \cap D'$ is a summand of T^m and thus $A = A' \leq T^m$. By monotonicity, $A = bAd \leq bT^m d$. It follows that

$$\sum(\{U, X, V\}^m \cap D) \leq T^m \quad \text{and} \quad \sum(\{U, X, V\}^m \cap D) \leq bT^m d.$$

To show the reverse of the second inequation, let $A' \in \{Up, X, qV\}^m$ be a summand of $T^m = (Up + X + qV)^m$ that is not obtained from any $A \in \{U, X, V\}^m \cap D$ by this substitution. Then $bA'd = 0$, because $A' \in (D'qV)^* D'(UpD')^* \setminus D'$ and b, d commute with factors from D' (with values in $Z^{n\times n}$), so in $bA'd$, b can be moved over factors to the right, until it meets q and gives $bq = 0$, or d can be moved over factors to the left until it meets p and gives $pd = 0$. It follows that $bT^m d \leq \sum(\{U, X, V\}^m \cap D) \leq T^m$. ◁

By *-continuity, claim 3.3 implies that the set D of matrices obtained from the context-free language $D \subseteq \{U, X, V\}^*$ has a least upper bound in $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$:

$$\begin{aligned} N = bT^* d &= \sum\{\, bT^m d \mid m \in \mathbb{N}\,\} \\ &= \sum\{\, D \cap \{U, X, V\}^m \mid m \in \mathbb{N}\,\} = \sum D. \end{aligned}$$

Claim 3.3. $N \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n\times n}$.

Proof:
We have seen $bT^m d \in Z^{n\times n}$ for each $m \in \mathbb{N}$. So for each $c \in C_2'$, $c(bT^m d) = (bT^m d)c$ and

$$cN = cbT^* d = \sum\{\, cbT^m d \mid m \in \mathbb{N}\,\}) = \sum\{\, bT^m dc \mid m \in \mathbb{N}\,\}) = bT^* dc = Nc,$$

since $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$ is *-continuous. It follows that each entry of N commutes with c. ◁

Claim 3.3. N is the least solution of $y \geq (UyV + X)^*$ in $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$.

Proof:
We show that N is the least solution of $y \geq 1 + X + UyV + yy$ and apply Proposition 3.1. By claim 3.3, we get $1 + X \leq N$ and since $bT^m d$ is a finite sum of balanced sequences of length m over $\{U, X, V\}$, by distributivity $UbT^m dV$ is a sum of balanced sequences of length $m + 2$, hence

$$UbT^m dV \leq \sum(\{U, X, V\}^{m+2} \cap D) = bT^{m+2} d \leq N.$$

Thus by *-continuity, $UNV = UbT^*dV = \sum\{UbT^m dV \mid m \in \mathbb{N}\} \leq N$. It remains to show $NN \leq N$. By *-continuity,

$$NN = \sum_{k\in\mathbb{N}} bT^k dN = \sum_{k,l\in\mathbb{N}} bT^k dbT^l d.$$

By claim 3.3 and claim 3.3, $bT^k d \in Z^{n\times n}$, so $(bT^k d)bT^l d = b(bT^k d)T^l d$, and $bT^k d \leq T^k$, whence

$$NN = \sum_{k,l\in\mathbb{N}} b(bT^k d)T^l d \leq \sum_{k,l\in\mathbb{N}} bT^k T^l d = N.$$

Therefore, N is a solution of $y \geq 1+X+UyV+yy$. To show that it is the least solution, suppose $y \in Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$ satisfies $y \geq 1+X+UyV+yy$. As $N = \sum D$, it is sufficient to show $A \leq y$ for each $A \in D$. This is clear for 1 and X, and if $A, B \in D$ satisfy $A, B \leq y$, then $UAV \leq UyV \leq y$ and $AB \leq yy \leq y$ by monotonicity. So y is an upper bound of D. ◁

By the last two claims, the Lemma is proven. □

Example 3.4. In the most simple case $n = 1$, with $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2') \simeq K \otimes_{\mathcal{R}} C_2'$, suppose $U = b, V = d$ and $X = x \in K$. Then $N = b(bp+x+qd)^*d = \sum D$ for Dyck's language $D \subseteq \{b,x,d\}^*$. The proof shows $N = \sum D \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$. ◁

Theorem 3.5. (First Normal Form)
Let K be an $\mathcal{R}$-dioid. For each $\varphi \in K \otimes_{\mathcal{R}} C_2'$ there are $n \in \mathbb{N}$, $S \in \mathbb{B}^{1\times n}$, $F \in \mathbb{B}^{n\times 1}$, $U \in \{0,b,p\}^{n\times n}$, $V \in \{0,d,q\}^{n\times n}$ and $X \in K^{n\times n}$ such that

$$\varphi = S(NV)^*N(UN)^*F,$$

where $N \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n\times n}$ is the least solution of $y \geq (UyV + X)^*$ in $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$.

For $n = 1$, N commutes with U and V, so $(NV)^k N(UN)^l = V^k NU^l$, and by *-continuity, $(NV)^*N(UN)^* = V^*NU^*$. This is related to the normal form for the extension $P_m'[X]$ of the polycyclic monoid P_m' in Section 2.1.

Proof:
By definition of $K \otimes_{\mathcal{R}} C_2'$, there is $R \in \mathcal{R}(K \times C_2')$ such that $\varphi = [R]$. As in Theorem 2.15, by induction on R one constructs an automaton $\langle S, A, F\rangle$ with

$$\varphi = [R] = L(\langle S, A, F\rangle) = SA^*F$$

and a transition matrix $A \in (K \otimes_{\mathcal{R}} C_2')^{n\times n}$ of the form $A = U + X + V$ where $U \in \{0,b,d\}^{n\times n}$, $X \in K^{n\times n}$ and $V \in \{0,d,q\}^{n\times n}$, for some n. By Lemma 3.3, $y \geq (UyV + X)^*$ has a least solution N in $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$, and

$$N \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n\times n}.$$

By Theorem 3.2, this N allows us to write A^* as

$$A^* = (U + X + V)^* = (NV)^*N(UN)^*$$

and obtain the normal form $\varphi = [R] = SA^*F = S(NV)^*N(UN)^*F$. □

While Theorem 3.5 gives a generic normal form for an element φ of $K \otimes_{\mathcal{R}} C_2'$, it is not straightforward to compute the matrix N occurring in the normal form of φ. The following example demonstrates how N is obtained from an automaton for φ through the construction of Lemma 3.3. In Section 4 we will show how to compute a normal form inductively from a regular expression φ.

Example 3.6. Let $P_2 = \{p_0, p_1\}$, $Q_2 = \{q_0, q_1\}$, and $K = \mathcal{R}\{a, b\}^* \otimes_{\mathcal{R}} C_2'$. The element $\varphi = (ap_1)^*(q_1 b)^* \in K$ is represented as $\varphi = L(\mathcal{A}) = SA^*F$ by the automaton $\mathcal{A} = \langle S, A, F\rangle$ of Figure 1 with initial state 1 and accepting state 3.

$$\left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & 1 & 0 \\ p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1 \\ 0 & 0 & b & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

Figure 1. $\mathcal{A} = \langle S, A, F\rangle$

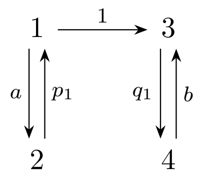


Figure 2. Graph of A

The iteration A^* of A calculated using the formula (6) can be read off from the graph: the entry $(A^*)_{i,j}$ describes the labellings on paths from node i to node j. Hence, with $\bar{a} = ap_1$ and $\bar{b} = q_1 b$, we have

$$A^* = \begin{pmatrix} \bar{a}^* & \bar{a}^* a & \bar{a}^* \bar{b}^* & \bar{a}^* \bar{b}^* q_1 \\ p_1 \bar{a}^* & 1 + p_1 \bar{a}^* a & p_1 \bar{a}^* \bar{b}^* & p_1 \bar{a}^* \bar{b}^* q_1 \\ 0 & 0 & \bar{b}^* & \bar{b}^* q_1 \\ 0 & 0 & b\bar{b}^* & 1 + b\bar{b}^* q_1 \end{pmatrix}.$$

To obtain the normal form $(NV)^* N (UN)^*$ of A^*, split A as $U + X + V$ with

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & a & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To determine $N = p_0(Up_1 + X + q_1 V)^* q_0$, let $\widetilde{A} = (Up_1 + X + q_1 V)$ and read off $\widetilde{A}^*$ from the graph of $\widetilde{A}$, obtaining a copy of A^* with $\tilde{a} = ap_1^2, \tilde{b} = q_1^2 b, p_1^2, q_1^2$ instead of $\bar{a}, \bar{b}, p_1, q_1$, respectively. The entries of N are then

$$N_{i,j} = p_0 (\widetilde{A}^*)_{i,j} q_0.$$

The resulting matrix is as follows, writing $\widehat{L}$ for $\sum L$ with $L = \{\, a^n b^n \mid n \in \mathbb{N}\,\}$,

$$N = p_0 \begin{pmatrix} \tilde{a}^* & \tilde{a}^* a & \tilde{a}^* \tilde{b}^* & \tilde{a}^* \tilde{b}^* q_1^2 \\ p_1^2 \tilde{a}^* & 1 + p_1^2 \tilde{a}^* a & p_1^2 \tilde{a}^* \tilde{b}^* & p_1^2 \tilde{a}^* \tilde{b}^* q_1^2 \\ 0 & 0 & \tilde{b}^* & \tilde{b}^* q_1^2 \\ 0 & 0 & b\tilde{b}^* & 1 + b\tilde{b}^* q_1^2 \end{pmatrix} q_0 = \begin{pmatrix} 1 & a & \widehat{L} & a\widehat{L} \\ 0 & 1 & \widehat{L}b & \widehat{L} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b & 1 \end{pmatrix}.$$

For example, $N_{1,3} = p_0\tilde{a}^*\tilde{b}^*q_0 = p_0(ap_1^2)^*(q_1^2b)^*q_0 = \widehat{L}$ is calculated as in Example 2.9. It follows that

$$NV = \begin{pmatrix} 0 & 0 & 0 & \widehat{L}q_1 \\ 0 & 0 & 0 & \widehat{L}bq_1 \\ 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & bq_1 \end{pmatrix}, \qquad UN = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p_1 & p_1a & p_1\widehat{L} & p_1a\widehat{L} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which imply $(NV)^* = 1 + NV(bq_1)^*$ and $(UN)^* = 1 + (p_1a)^*UN$. By matrix multiplication, one obtains the normal form $(NV)^*N(UN)^* = A^*$.

To determine N, one can also use that N is the least solution of $y \geq (UyV + X)^*$ in $Mat_{4,4}(K)$, hence $N = (UNV + X)^*$. Let e_i be the unit column vector with 1 in the i-th row, 0 else, e_i' its transpose row vector. Then e_ie_j' is the 4×4-matrix with 1 at (i, j), 0 else, and $e_i'e_j$ the 1×1-matrix with entry $\delta_{i,j}$. Since

$$UNV = (e_2p_1e_1')(\sum_{1 \leq i,j \leq 4} e_iN_{i,j}e_j')(e_3q_1e_4') = e_2p_1N_{1,3}q_1e_4' = e_2N_{1,3}e_4',$$

the graph of $X + UNV$ is that of X with additional edge $2 \stackrel{N_{1,3}}{\longrightarrow} 4$, from which one can read off $(X + UNV)^*$ as

$$(X + UNV)^* = \begin{pmatrix} 1 & a & 1 + aN_{1,3}b & aN_{1,3} \\ 0 & 1 & N_{1,3}b & N_{1,3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b & 1 \end{pmatrix} = N.$$

Since N is the least solution of $y \geq (UyV + X)^*$, $N_{1,3}$ is the least solution of $y_{1,3} \geq 1 + ay_{1,3}b$, i.e. $\mu x(1 + axb) = \sum L$ for $L = \{ a^nb^n \mid n \in \mathbb{N} \} \in \mathcal{C}K$, leading to the matrix N shown above. $\triangleleft$

3.3. Reduced normal form

We conjectured in [6] that the normal form $S(NV)^*N(UN)^*F$ for $\varphi \in K \otimes_{\mathcal{R}} C_2'$ given in Theorem 3.5 can be simplified to SNF for elements $\varphi \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$. We can now prove this under the additional assumption that K is non-trivial and has no zero divisors.

Lemma 3.7. Let $m \geq 2$, $g : C_m' \to C_2'$ the $\mathcal{R}$-embedding of Lemma 2.3, and K an $\mathcal{R}$-dioid. There is an $\mathcal{R}$-embedding $\overline{\cdot} : K \otimes_{\mathcal{R}} C_m' \to K \otimes_{\mathcal{R}} C_2'$, given by

$$\overline{[R]} = \sum \{ a \cdot g(b) \mid (a, b) \in R \} \quad \text{for } R \in \mathcal{R}(K \times C_m),$$

which maps $Z_{C_m'}(K \otimes_{\mathcal{R}} C_m')$ to $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$; for $m = 2$, it is the identity on $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$.

Proof:
Let $\overline{\cdot}$ be the induced injective $\mathcal{R}$-morphism $h_{(f\times g)}$ for the embeddings $f = Id_K$ and $g : C'_m \to C'_2$ in

$$\begin{array}{ccccc} K_1 & \xrightarrow{\top'_1} & K_1 \otimes_{\mathcal{R}} C'_m & \xleftarrow{\top'_2} & C'_m \\ \downarrow{\scriptstyle f} & & \vdots\, {\scriptstyle h_{(f\times g)}} & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{\top_1} & K \otimes_{\mathcal{R}} C'_2 & \xleftarrow{\top_2} & C'_2. \end{array}$$

according to Corollary 2.8. For $R \in \mathcal{R}(K \times C'_m)$, the element $[R]' \in K \otimes_{\mathcal{R}} C'_m$ is mapped to

$$\overline{[R]'} = [(f\times g)(R)] = \sum\{\, f(a) \otimes g(b) \mid (a, b) \in R\,\}.$$

By Theorem 2.11 (i), each element of $Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m)$ is the congruence class $[R]'$ of some relation $R \in \mathcal{R}(K \times C'_m)$ with $R \subseteq K \times \{0, 1\}$. Since $g(0) = 0$ and $g(1) = 1$, we have $[(f\times g)(R)] = [R]$. Hence $\overline{\cdot}$ restricts to an $\mathcal{R}$-morphism $\overline{\cdot} : Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m) \to Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$. For $m = 2$, this is the identity on $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$, since $\top_1 = \top'_1$, $\top_2 = \top'_2$ and $Id_K \times g$ leaves R fixed. □

Corollary 3.8. (Reduced Normal Form)
Let K be a non-trivial $\mathcal{R}$-dioid without zero divisors. Let $\varphi = SA^*F \in K \otimes_{\mathcal{R}} C'_2$ with $A = U + X + V$ and n, S, F, U, X, V and N as in Theorem 3.5. If $\varphi \in Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$, then $\varphi = SNF$.

Proof:
Suppose $\varphi = SA^*F \in Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$. Since φ is a finite sum of entries of A^*, by Corollary 2.12, all summands belong to $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$. Therefore, $\varphi = SA^*F = SNF$ is shown if for all $i, j < n$

$$A^*{}_{i,j} \in Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2) \implies A^*{}_{i,j} = N_{i,j}. \qquad (8)$$

Let $\overline{\cdot} : K \otimes_{\mathcal{R}} C'_2 \to K \otimes_{\mathcal{R}} C'_2$ be the $\mathcal{R}$-morphism of Lemma 3.7 On $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$, it is the identity. Applying $\overline{\cdot}$ entrywise to matrices we get

$$\begin{aligned} \overline{A^*} &= \overline{(NV)^*N(UN)^*} \\ &= (\overline{NV})^*\overline{N}(\overline{UN})^* \\ &= (N\overline{V})^*N(\overline{U}N)^*. \end{aligned}$$

Notice $\overline{U} \in \{\overline{0}, \overline{b}, \overline{p}\}^{n\times n} = \{0, bp, bp^2\}^{n\times n}$ and $\overline{V} \in \{\overline{0}, \overline{d}, \overline{q}\}^{n\times n} = \{0, qd, q^2d\}^{n\times n}$. By Lemma 2.3 and b, d as diagonal matrices, $b\overline{V} = 0 = \overline{U}d$, so $b(N\overline{V}) = Nb\overline{V} = 0 = \overline{U}dN = (\overline{U}N)d$, hence $b(N\overline{V})^* = b$ and $(\overline{U}N)^*d = d$. For $(A^*)_{i,j} \in Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ this gives

$$\begin{aligned} (A^*)_{i,j} &= (\overline{A^*})_{i,j} = b(\overline{A^*})_{i,j}d \\ &= (b\overline{A^*}d)_{i,j} = (b(N\overline{V})^*N(\overline{U}N)^*d)_{i,j} \\ &= (bNd)_{i,j} \\ &= N_{i,j}. \end{aligned}$$

We thus have shown (8). □

Notice that in the useful cases where $K = \mathcal{R}M$ for a monoid M, indeed K is non-trivial and has no zero divisors.

In the special case of $Z_{C_m'}(\mathcal{R}X^* \otimes_{\mathcal{R}} C_m')$, the elements of the centralizer of C_m' have previously been chararcterized as follows:

Theorem 3.9. (Corollary 28 of [11])
For $m > 2$ and $\varphi \in \mathcal{R}X^* \otimes_{\mathcal{R}} C_m'$, we have $\varphi \in Z_{C_m'}(\mathcal{R}X^* \otimes_{\mathcal{R}} C_m')$ iff there is a regular expression r over $X \dot\cup (\Delta_m \setminus \{p_0, q_0\})$ such that $\varphi = p_0 r q_0$.

To prove this, one codes the $m > 2$ bracket pairs by the two pairs p_1, q_1 and p_2, q_2 to get a regular expression r in $\overline{p_i} = p_1 p_2^{i+1}$ and $\overline{q_j} = q_2^{j+1} q_1$, and then has p_0, q_0 as a fresh bracket pair to eliminate the unbalanced strings using $p_0 r q_0$. One can do the same for $m = 2$:

> We have $\varphi \in Z_{C_2'}(\mathcal{R}X^* \otimes_{\mathcal{R}} C_2')$ iff there is a regular expression r over $X \dot\cup \Delta_2$ with p_0 only as part of $p_0 p_1$ and q_0 only as part of $q_1 q_0$, such that $\varphi = p_0 r q_0$.

For any $m \geq 2$, the first normal form theorem 3.5 holds as well with C_m' instead of C_2'. If the automaton $\langle S, A, F\rangle$ for φ has no transitions under p_0 and q_0, then $\varphi \mapsto p_0 \varphi q_0$ is a projection on the centralizer:

Corollary 3.10. Suppose $\varphi = SA^*F \in K \otimes_{\mathcal{R}} C_m'$ is represented by an automaton $\langle S, A, F\rangle$ not using p_0, q_0, i.e. $U \in \{0, p_1, \ldots, p_{m-1}\}^{n\times n}$ and $V \in \{0, q_1, \ldots, q_{m-1}\}^{n\times n}$ in $A = U + X + V$. If $S(NV)^*N(UN)^*F$ is the normal form of φ, then

$$p_0 \varphi q_0 = SNF \in Z_{C_m'}(K \otimes_{\mathcal{R}} C_m').$$

Proof:
By the assumption on U and V, for the diagonal matrix versions of p_0, q_0 we have $p_0 V = 0 = U q_0$, and since N commutes with p_0 and q_0, we get $p_0(NV)^* = p_0$ and $(UN)^* q_0 = q_0$. Hence

$$p_0 A^* q_0 = p_0 (NV)^* N (UN)^* q_0 = p_0 N q_0 = N,$$

and thus $p_0 \varphi q_0 = p_0 SA^*F q_0 = S p_0 A^* q_0 F = SNF \in Z_{C_m'}(K \otimes_{\mathcal{R}} C_m')$. □

3.4. Second normal form

Corollary 3.10 can be extended by admitting that $\varphi = SA^*F \in K \otimes_{\mathcal{R}} C_m'$ is given by an automaton $\langle S, A, F\rangle$ whose transition matrix A contains transitions by $q_0 p_0$ in addition to those by elements of K and $\Delta_m \setminus \{p_0, q_0\}$. This is useful to combine representations $p_0 r_i q_0 = \sum L_i$ of $L_i \in \mathcal{C}X^*$, $i = 1, 2$, in $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ to a representation $p_0 r_1 q_0 p_0 r_2 q_0 = (\sum L_1)(\sum L_2) = \sum (L_1 L_2)$ of $L_1 L_2$, as will be exemplified below.

Theorem 3.11. (Second Normal Form)
Let K be an $\mathcal{R}$-dioid, $m \geq 2$ and $\varphi \in K \otimes_{\mathcal{R}} C'_m$ be given in matrix form $\varphi = S(U+X+V+W\pi)^*F$, where $\pi = q_0p_0$ and for some $n \geq 0$,

$$\begin{array}{llllllll} S & \in & \{0,1\}^{1\times n}, & X & \in & K^{n\times n}, & U & \in \{0, p_1, \ldots, p_{m-1}\}^{n\times n}, \\ F & \in & \{0,1\}^{n\times 1}, & W & \in & \{0,1\}^{n\times n}, & V & \in \{0, q_1, \ldots, q_{m-1}\}^{n\times n}. \end{array}$$

Then there is a least solution N of $y \geq (UyV + X)^*$ in $Mat_{n,n}(K \otimes_{\mathcal{R}} C'_m)$, and

$$p_0\varphi q_0 = SN(WN)^*F \ \in Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m).$$

Proof:
Let $A = U + X + V$. By Theorem 3.5, there is $N = \mu y.(UyV + X)^* \in (Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m))^{n\times n}$ with

$$A^* = (U + X + V)^* = (NV)^*N(UN)^*.$$

As in the proof of Corollary 3.10, we obtain

$$p_0A^*q_0 = p_0(NV)^*N(UN)^*q_0 = p_0Nq_0 = N,$$

and therefore in the Kleene algebra $Mat_{n,n}(K \otimes_{\mathcal{R}} C'_m)$, using identities $(a + b)^* = a^*(ba^*)^*$ and $(ab)^*a = a(ba)^*$ of Kleene algebra,

$$\begin{array}{lll} p_0(A + W\pi)^*q_0 & = & p_0A^*(W\pi A^*)^*q_0 \\ & = & p_0A^*(q_0Wp_0A^*)^*q_0 \\ & = & p_0A^*q_0(Wp_0A^*q_0)^* \\ & = & N(WN)^* \in (Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m))^{n\times n}. \end{array}$$

Because S, N, W and F commute with p_0 and q_0, it follows that

$$p_0\varphi q_0 = Sp_0(A + W\pi)^*q_0F = SN(WN)^*F \in Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m).$$

Notice also that $\pi\varphi\pi = q_0p_0\varphi q_0p_0 = \pi p_0\varphi q_0$. □

Example 3.12. Consider $\varphi = (ap_1)^*(q_1b)^* \in K := \mathcal{R}\{a, b\}^* \otimes_{\mathcal{R}} C'_2$ of Example 3.6 and its automaton $\langle S, A, F\rangle$ with $A = U + X + V$ and graph as shown in Figure 2. We have seen that $p_0\varphi q_0 = \sum L$ represents $L = \{ a^nb^n \mid n \in \mathbb{N}\} \in \mathcal{C}\{a, b\}^*$ in K. By Corollary 3.10, $p_0\varphi q_0$ is the projection of φ to the centralizer $Z_{C'_2}K$. Using $\pi = q_0p_0$, we claim that the projection of $\psi = \varphi\pi\varphi$ to the centralizer represents LL in K, i.e. $p_0\psi q_0 = \sum(LL) \in Z_{C'_2}K$. To obtain an automaton $\langle\tilde{S}, \tilde{A}, \tilde{F}\rangle$ for ψ, connect the graph of A with a copy of itself by an edge labelled by π, to get the graph of $\tilde{A}$ shown in Figure 3.

The automaton of ψ is $\langle\tilde{S}, \tilde{A}, \tilde{F}\rangle$ and has 8 states, with initial state 1 coded by $\tilde{S}_{1,1} = 1$, accepting state 7 coded by $\tilde{F}_{7,1} = 1$, and transition matrix $\tilde{A} = \tilde{U} + (\tilde{X} + \pi W) + \tilde{V}$ shown in Figure 4.

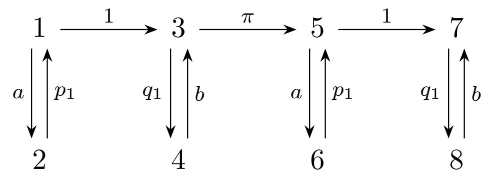


Figure 3. Graph of $\tilde{A}$

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0\\
p_1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&p_1&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{pmatrix}
+
\begin{pmatrix}
0&a&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&\pi&0&0&0\\
0&0&b&0&0&0&0&0\\
0&0&0&0&0&a&1&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&b&0
\end{pmatrix}
+
\begin{pmatrix}
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&q_1&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&q_1\\
0&0&0&0&0&0&0&0
\end{pmatrix}.
$$

Figure 4. Transition matrix $\tilde{A} = \tilde{U} + (\tilde{X} + \pi W) + \tilde{V}$

For $N = \mu y.(\tilde{U}y\tilde{V} + \tilde{X})^* = p_0(\tilde{U}p_1 + \tilde{X} + q_1\tilde{V})^* q_0 \in (Z_{C_2'}(\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'))^{8\times 8}$, the proof shows that

$$
\begin{aligned}
(\tilde{U} + \tilde{X} + \tilde{V})^* &= (N\tilde{V})^* N(\tilde{U}N)^*,\\
p_0(\tilde{U} + \tilde{X} + \tilde{V})^* q_0 &= N,\\
p_0(\tilde{A})^* q_0 &= N(WN)^*.
\end{aligned}
$$

Since the graph of $(\tilde{U}p_1 + \tilde{X} + q_1\tilde{V})$ consists of two disconnected components isomorphic to that of A of Figure 2 above, the matrix N obtained from its transitive reflexive hull is, using $\widehat{L} = p_0(ap_1^2)^*(q_1^2 b)^* q_0$ as in Example 3.6,

$$
N = p_0(\tilde{U}p_1 + \tilde{X} + q_1\tilde{V})^* q_0 =
\begin{pmatrix}
1&a&\widehat{L}&a\widehat{L}&0&0&0&0\\
0&1&\widehat{L}b&\widehat{L}&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&b&1&0&0&0&0\\
0&0&0&0&1&a&\widehat{L}&a\widehat{L}\\
0&0&0&0&0&1&\widehat{L}b&\widehat{L}\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&b&1
\end{pmatrix}.
$$

The matrix W is the boolean 8×8-matrix with 1 only at $W_{3,5}$, so

$$
\begin{aligned}
p_0(\tilde{A}^*)q_0 &= N(WN)^* \\
&= N \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a & \widehat{L} & a\widehat{L} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}^* = N \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & a & \widehat{L} & a\widehat{L} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
&= \begin{pmatrix}
1 & a & \widehat{L} & a\widehat{L} & \widehat{L} & a\widehat{L} & \widehat{L}\widehat{L} & \widehat{L}a\widehat{L} \\
0 & 1 & \widehat{L}b & \widehat{L} & \widehat{L}b & \widehat{L}ba & \widehat{L}b\widehat{L} & \widehat{L}ba\widehat{L} \\
0 & 0 & 1 & 0 & 1 & a & \widehat{L} & a\widehat{L} \\
0 & 0 & b & 1 & b & ba & b\widehat{L} & ba\widehat{L} \\
0 & 0 & 0 & 0 & 1 & a & \widehat{L} & a\widehat{L} \\
0 & 0 & 0 & 0 & 0 & 1 & \widehat{L}b & \widehat{L} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b & 1
\end{pmatrix}.
\end{aligned}
$$

Hence $p_0\psi q_0 = p_0\tilde{S}\tilde{A}^*\tilde{F}q_0 = p_0(\tilde{A}^*)_{1,7}q_0 = (N(WN)^*)_{1,7} = \widehat{L}\widehat{L}$. As shown in Example 2.9,

$$\widehat{L} = p_0(ap_1^2)^*(q_1^2b)^*q_0 = p_0(ap_1)^*(q_1b)^*q_0 = p_0\varphi q_0 = \sum L \in Z_{C_2'}K,$$

and since $Z_{C_2'}K$ is a $\mathcal{C}$-dioid by Theorem 2.11 (ii), $\widehat{L}\widehat{L} = (\sum L)(\sum L) = \sum(LL)$. So, $p_0\psi q_0 = p_0\varphi q_0 p_0\varphi q_0$ represents LL in K. This can also be seen using *-continuity as in Example 2.9. $\triangleleft$

4. Combining normal forms by Kleene algebra operations

We here show that normal forms for elements of $K \otimes_{\mathcal{R}} C_2'$ can be defined directly by induction on the regular operations, using the representation by automata only implicitly.

Theorem 4.1. Let K be an $\mathcal{R}$-dioid. For every $\varphi \in K \otimes_{\mathcal{R}} C_2'$ there are $n \geq 1$, $S \in \mathbb{B}^{1\times n}$, $F \in \mathbb{B}^{n\times 1}$, $U \in \{0, p_0, p_1\}^{n\times n}$, $V \in \{0, q_0, q_1\}^{n\times n}$, and $N \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n\times n}$ such that

$$\varphi = S(NV)^*N(UN)^*F.$$

Moreover, N is the least solution of $y \geq (UyV + N)^*$ in $Mat_{n,n}(Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))$.

Proof:
For the second claim, we only show $N \geq (UNV + N)^*$, since any $y \geq (UyV + N)^*$ is above N. The first claim is shown by induction on φ, choosing S, U, N, V, F from an implicit automaton $\langle S, A, F\rangle$ for $\varphi = SA^*F$ (cf. Theorem 2.15) with $A = U + X + V$ and $N = \mu y.(UyV + X)^*$.

$\varphi \in \{0, 1\}$**:** Put $n = 1, U = V = (0)$ and $F = N = (1)$. Then $(UNV + N)^* \leq N$ and

$$(NV)^*N(UN)^* = (0)^*N(0)^* = N = (1).$$

We have $S(NV)^*N(UN)^*F = \varphi$ if we take $S = (0)$ for $\varphi = 0$ and $S = (1)$ for $\varphi = 1$.

In the remaining cases, this instance of the recursion formula (6) for matrix iteration is used often:

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^* = \begin{pmatrix} A^* & A^*BD^* \\ 0 & D^* \end{pmatrix}. \tag{9}$$

For the remaining generators, i.e. the images in $K \otimes_{\mathcal{R}} C_2'$ of $k \in K$ or $p_0, p_1, q_0, q_1 \in C_2'$, let $n = 2$ and S, U, N, V, F as shown below.

$\varphi = k \in K$**:** Here,

$$\begin{aligned} & S(NV)^*N(UN)^*F \\ = \;& \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)^* \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \right)^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ = \;& \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = k = \varphi. \end{aligned}$$

By (9), $(UNV + N)^* = N^* = N$.

$\varphi \in \{p_i, q_i\}$**:** If φ is an opening bracket p_i, or, respectively, a closing bracket q_i, let U, N, V be

$$\begin{pmatrix} 0 & p_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{respectively} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & q_i \\ 0 & 0 \end{pmatrix}.$$

Then, using S and F as for $\varphi = k$ above, $S(NV)^*N(UN)^*F = SU^*F = SUF = p_i$ and, respectively, $S(NV)^*N(UN)^*F = SV^*F = SVF = q_i$.

For φ among $\varphi_1 + \varphi_2$, $\varphi_1 \cdot \varphi_2$, and φ_1^+, suppose that for $i = 1, 2$, by induction we have $n_i \geq 1$ and S_i, U_i, V_i, F_i and N_i such that $\varphi_i = S_i(N_iV_i)^*N_i(U_iN_i)^*F_i$ and $N_i = \mu y.(U_iyV_i + N_i)^*$.

$\varphi = \varphi_1 + \varphi_2$**:** Let $n = n_1 + n_2$ and S, U, N, V, F as shown in

$$\begin{aligned}
&S(NV)^*N(UN)^*F \\
&= \begin{pmatrix} S_1 & S_2 \end{pmatrix} \left(\begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \right)^* \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \left(\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \right)^* \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \\
&= \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} (N_1V_1)^* & 0 \\ 0 & (N_2V_2)^* \end{pmatrix} \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} (U_1N_1)^* & 0 \\ 0 & (U_2N_2)^* \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \\
&= \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} (N_1V_1)^*N_1(U_1N_1)^* & 0 \\ 0 & (N_2V_2)^*N_2(U_2N_2)^* \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \\
&= S_1(N_1V_1)^*N_1(U_1N_1)^*F_1 + S_2(N_2V_2)^*N_2(U_2N_2)^*F_2 \\
&= \varphi_1 + \varphi_2.
\end{aligned}$$

For the second claim, from $(U_iN_iV_i + N_i)^* \leq N_i$ we obtain

$$(UNV + N)^* = \begin{pmatrix} U_1N_1V_1 + N_1 & 0 \\ 0 & U_2N_2V_2 + N_2 \end{pmatrix}^* \leq \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} = N.$$

$\varphi = \varphi_1 \cdot \varphi_2$**:** Notice that entries of $N_1F_1S_2N_2$ belong to the centralizer, and if z is an $n_1 \times n_2$ matrix of elements x of the centralizer, so is U_1zV_2, because its entries are 0 or sums of elements $p_ixq_j = x \cdot \delta_{i,j}$, which belong to the centralizer. Hence, $f(z) = N_1U_1zV_2N_2 + N_1F_1S_2N_2$ defines a monotone map

$$f : (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n_1 \times n_2} \to (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n_1 \times n_2},$$

By Theorem 2.11 (ii), $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is a $\mathcal{C}$-dioid, hence a Chomsky algebra, so f has a least pre-fixpoint α, i.e. the system of n_1n_2 polynomial inequations

$$z \geq N_1U_1zV_2N_2 + N_1F_1S_2N_2 \tag{10}$$

has α as least solution[3]. Let $n = n_1 + n_2$ and S, U, N, V, F as in

$$\begin{aligned}
&S(NV)^*N(UN)^*F \\
&= \begin{pmatrix} S_1 & 0 \end{pmatrix} \left(\begin{pmatrix} N_1 & \alpha \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \right)^* \begin{pmatrix} N_1 & \alpha \\ 0 & N_2 \end{pmatrix} \left(\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} N_1 & \alpha \\ 0 & N_2 \end{pmatrix} \right)^* \begin{pmatrix} 0 \\ F_2 \end{pmatrix}
\end{aligned}$$

[3] The entries of α can be given as regular μ-terms from the polynomials of (10), as shown in [10], or as regular expressions in the parameters of (10) and the brackets of C_2', by a method presented in Theorem 15 and Example 6 of [11]. Alternatively, by Lemma 3.3, using $A = U_1p + X_1 + qV_1$ and $D = U_2p + X_2 + qV_2$ with automaton $\langle S_i, U_i + X_i + V_i, F_i \rangle$ for φ_i,

$$N = b(Up + X + qV)^*d = b \begin{pmatrix} A & F_1S_2 \\ 0 & D \end{pmatrix}^* d = b \begin{pmatrix} A^* & A^*F_1S_2D^* \\ 0 & D^* \end{pmatrix} d = \begin{pmatrix} N_1 & bA^*F_1S_2D^*d \\ 0 & N_2 \end{pmatrix},$$

so we also have $\alpha = bA^*F_1S_2D^*d$, but it is not obvious that this is a matrix of elements from the centralizer.

$$\begin{aligned}
&= \begin{pmatrix} S_1 & 0 \end{pmatrix} \begin{pmatrix} N_1V_1 & \alpha V_2 \\ 0 & N_2V_2 \end{pmatrix}^* \begin{pmatrix} N_1 & \alpha \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} U_1N_1 & U_1\alpha \\ 0 & U_2N_2 \end{pmatrix}^* \begin{pmatrix} 0 \\ F_2 \end{pmatrix} \\
&= \begin{pmatrix} S_1 & 0 \end{pmatrix} \begin{pmatrix} (N_1V_1)^* & (N_1V_1)^*\alpha V_2(N_2V_2)^* \\ 0 & (N_2V_2)^* \end{pmatrix} \begin{pmatrix} N_1 & \alpha \\ 0 & N_2 \end{pmatrix} \\
&\qquad \begin{pmatrix} (U_1N_1)^* & (U_1N_1)^*U_1\alpha(U_2N_2)^* \\ 0 & (U_2N_2)^* \end{pmatrix} \begin{pmatrix} 0 \\ F_2 \end{pmatrix} \\
&= \begin{pmatrix} S_1(N_1V_1)^* & S_1(N_1V_1)^*\alpha V_2(N_2V_2)^* \end{pmatrix} \begin{pmatrix} N_1 & \alpha \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} (U_1N_1)^*U_1\alpha(U_2N_2)^*F_2 \\ (U_2N_2)^*F_2 \end{pmatrix} \\
&= S_1(N_1V_1)^*[N_1(U_1N_1)^*U_1\alpha + \alpha + \alpha V_2(N_2V_2)^*N_2](U_2N_2)^*F_2.
\end{aligned}$$

By [10], the μ-continuity of $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ lifts to the matrix level, so

$$\begin{aligned}
\alpha &= \sum\{\, (N_1U_1)^k(N_1F_1S_2N_2)(V_2N_2)^k \mid k \in \mathbb{N}\,\} \\
&= \sum\{\, N_1(U_1N_1)^kF_1S_2(N_2V_2)^kN_2 \mid k \in \mathbb{N}\,\}
\end{aligned}$$

and

$$\begin{aligned}
&S(NV)^*N(UN)^*F \\
&\quad= S_1(N_1V_1)^*[N_1(U_1N_1)^*U_1\alpha + \alpha + \alpha V_2(N_2V_2)^*N_2](U_2N_2)^*F_2 \\
&\quad= \sum\{\, S_1(N_1V_1)^*N_1(U_1N_1)^*U_1N_1(U_1N_1)^kF_1S_2(N_2V_2)^kN_2(U_2N_2)^*F_2 \mid k \in \mathbb{N}\,\} \\
&\qquad + \sum\{\, S_1(N_1V_1)^*N_1(U_1N_1)^kF_1S_2(N_2V_2)^kN_2(U_2N_2)^*F_2 \mid k \in \mathbb{N}\,\} \\
&\qquad + \sum\{\, S_1(N_1V_1)^*N_1(U_1N_1)^kF_1S_2(N_2V_2)^kN_2V_2(N_2V_2)^*N_2(U_2N_2)^*F_2 \mid k \in \mathbb{N}\,\} \\
&\quad= \sum\{\, S_1(N_1V_1)^*N_1(U_1N_1)^kF_1S_2(N_2V_2)^lN_2(U_2N_2)^*F_2 \mid k,l \in \mathbb{N}\,\} \\
&\quad= S_1(N_1V_1)^*N_1(U_1N_1)^*F_1 \cdot S_2(N_2V_2)^*N_2(U_2N_2)^*F_2 \\
&\quad= \varphi_1 \cdot \varphi_2.
\end{aligned}$$

For the second claim,

$$(UNV + N)^* = \begin{pmatrix} U_1N_1V_1 + N_1 & U_1\alpha V_2 + \alpha \\ 0 & U_2N_2V_2 + N_2 \end{pmatrix}^* = \begin{pmatrix} N_1 & N_1(U_1\alpha V_2 + \alpha)N_2 \\ 0 & N_2 \end{pmatrix}.$$

Since $N_iN_i \leq N_i^* \leq N_i$, we have $N_1\alpha N_2 \leq \alpha = N_1U_1\alpha V_2N_2$, hence $(UNV + N)^* \leq N$.

$\varphi = \varphi_1^+$: Let n, S, U, V, F be n_1, S_1, U_1, V_1, F_1. Since N_1 is the least solution of $y \geq (UyV + N_1)^*$, by Theorem 3.2 $(N_1V)^*N_1(UN_1)^* = A_1^*$ for $A_1 = U + N_1 + V$, so $\varphi_1 = SA_1^*F$. Then

$$S(A_1 + FS)^*F = SA_1^*(FSA_1^*)^*F = SA_1^*F(SA_1^*F)^* = \varphi_1\varphi_1^* = \varphi_1^+.$$

By Lemma 3.3, $y \geq (UyV + N_1 + FS)^*$ has a least solution N, and $N \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n\times n}$. Using Theorem 3.2 again,

$$S(NV)^*N(UN)^*F = S(U + N_1 + FS + V)^*F = S(A_1 + FS)^*F = \varphi^+.$$

For the second claim, by definition of N we have $N = (UNV + N_1 + FS)^*$, so

$$\begin{aligned}(UNV + N)^* &\leq (UNV + (UNV + N_1 + FS)^*)^* \\ &= (UNV + UNV + N_1 + FS)^* \leq N.\end{aligned}$$

The case φ_1^* is treated via $\varphi_1^* = 1 + \varphi_1^+$. □

5. Bra-ket $\mathcal{R}$-dioids C_m and the completeness property

The *bra-ket $\mathcal{R}$-dioid C_m* is the quotient $\mathcal{R}\Delta_m^*/\rho_m$ of $\mathcal{R}\Delta_m^*$ by the $\mathcal{R}$-congruence ρ_m generated by the relations

$$\{\, p_i q_j = \delta_{i,j} \mid i, j < m \,\} \cup \{q_0 p_0 + \ldots + q_{m-1} p_{m-1} = 1\}.$$

While the match equations can be interpreted in monoids with an annihilating element 0, such as the polycyclic monoid P_m', the completeness equation $1 = \sum_{i<m} q_i p_i$ is a semiring equation.

The name *bra-ket* $\mathcal{R}$-dioid comes by analogy to a notation in quantum mechanics, where a quantum state is represented by a vector ψ of a Hilbert space $\mathcal{H}$, written $|\psi\rangle$ and called a *ket*. Elements f of the dual space $\mathcal{H}^*$ of (continuous) linear functions on $\mathcal{H}$ can uniquely be represented by elements $\varphi \in \mathcal{H}$, via $f(\psi) = \langle\varphi, \psi\rangle$ for all $\psi \in \mathcal{H}$, where $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to F$ is the inner product on $\mathcal{H}$ to the underlying field F. The element of $\mathcal{H}^*$ represented by φ is written $\langle\varphi|$ and called a *bra*.

Suppose $\mathcal{H}$ has finite dimenson m and let $|0\rangle, \ldots, |m-1\rangle$ be a basis of $\mathcal{H}$ of unit column vectors and $\langle 0|, \ldots, \langle m-1|$ a basis of $\mathcal{H}^*$ of unit row vectors. If the application of $\langle i| = (i_0, \ldots, i_{m-1}) \in \mathcal{H}^*$ to the vector $|j\rangle$ with row values $j_0 = \delta_{0,j} \ldots, j_{m-1} = \delta_{m-1,j}$ is written as juxtaposition, we get the bracket match- and mismatch equation for the inner product,

$$\langle i||j\rangle = \langle i, j\rangle = \sum\{\, i_k j_k \mid k < m \,\} = \delta_{i,j}.$$

The outer product $|j\rangle\langle i|$ of q_j and p_i is a linear operator on $\mathcal{H}$ and represented by the $m \times m$ matrix

$$|j\rangle\langle i| = \big(j_k i_l\big).$$

In particular, $|i\rangle\langle i|$ is a projection to the subspace spanned by $|i\rangle$ and represented by the $m \times m$-matrix with 1 on the i-th position on the diagonal and 0 otherwise. The combination of the projections gives the identity operator

$$|0\rangle\langle 0| + \ldots + |m-1\rangle\langle m-1| = 1,$$

represented by the unit matrix of dimension m, corresponding to the completeness equation. This interpretation of $\langle i||j\rangle$ and $|j\rangle\langle i|$ has to be combined with an interpretation of $|i\rangle|j\rangle$ as a tensor in the 2-particle space $\mathcal{H} \otimes \mathcal{H}$. Here, opening and closing brackets are interpreted by different kinds of objects and strings of brackets are interpreted in several ways. A uniform interpretation of brackets and bracket concatenation is given below.

5.1. The bra-ket $\mathcal{R}$-dioid C_m and matrix algebras

For applications to context-free languages $L \subseteq X^*$, the $\mathcal{R}$-dioids C_m and C_m' as factors C of $\mathcal{R}X^* \otimes_{\mathcal{R}} C$ arise by an interpretation of brackets p_i as pushing and q_i as popping symbol i from a stack. Then $p_i q_i$ leaves the stack unchanged, $p_i q_j$ for $j \neq i$ aborts the computation, and $q_i p_i$ succeeds iff i is on top of the stack. The completeness equation $\sum_{i<m} q_i p_i = 1$ of C_m says that one of the symbols $i < m$ is always on top of the stack (including 0 as end marker). This originally seemed necessary to have $Z_C(\mathcal{R}X^* \otimes_{\mathcal{R}} C)$ be isomorphic to the $\mathcal{C}$-dioid of context-free languages over X^*, but as shown in [11], $C = C_m'$ is sufficient.

More precisely, a uniform interpretation of brackets as binary relations on a countably infinite set and bracket concatenation as relation product, i.e. an interpretation of C_m in $Mat_{\omega,\omega}(\mathbb{B})$, is as follows:

Example 5.1. Let $e_0, e_1, \dots$ be the unit vectors of size $\omega \times 1$ and $e_0^t, e_1^t, \dots$ their transposed vectors of size $1 \times \omega$. Each $e_k e_l^t$ is a boolean square matrix of dimension ω, representing the relation $\{(k,l)\} \subseteq \omega \times \omega$, and so, for $i, j < m$, we can interpret brackets p_i and q_j in $\mathbb{B}^{\omega\times\omega}$ by

$$p_i = \sum_{k<\omega} e_k e_{mk+i}^t, \qquad q_j = \sum_{k<\omega} e_{mk+j} e_k^t,$$

representing the relations $\{\, (k, mk+i) \mid k \in \mathbb{N} \,\}$ and $\{\, (mk+j, k) \mid k \in \mathbb{N} \,\}$, respectively. Concatenation of brackets is boolean matrix multiplication, corresponding to relation composition, so

$$p_i q_j = \delta_{i,j}$$

holds, with 0 and 1 for the zero and unit square matrices of dimension ω. The matrix $q_i p_i$ represents the subrelation $\{\, (mk+i, mk+i) \mid k \in \mathbb{N} \,\}$ of the identity, so the completeness equation

$$\sum_{i<m} q_i p_i = 1$$

also holds. In a similar spirit, one can think of $\Gamma = \{0, \dots, m-1\}$ as a stack alphabet, Γ^* as the set of possible stack contents w (with top of the stack on the left), and let p_i be the graph of the operation “push symbol i”, q_i the graph of “pop symbol i”, $\cdot$ the relation product, $+$ the union of relations, 0 the empty relation and 1 the identity relation on Γ^*. Then clearly $p_i q_j = \delta_{i,j}$ holds, but, since one cannot pop from the empty stack ϵ, $e := \sum_{i<m} q_i p_i$ is the identity relation on the non-empty stack Γ^+ only, so $e < 1 = e + \{(\epsilon, \epsilon)\}$. To obtain $e = 1$, one can treat 0 as a special symbol, pad all stack contents $w \in \Gamma^*$ by an ω-sequence of 0's to $w0^\omega$, and interpret the operations as binary relations on the new stack $\Gamma^* 0^\omega$. ◁

A remarkable consequence of the completeness equation is the following:

Theorem 5.2. C_m is isomorphic to its own matrix Kleene algebra $Mat_{m,m}(C_m)$.

Proof:
Define $\hat{\cdot} : C_m \to Mat_{m,m}(C_m)$ and $\check{\cdot} : Mat_{m,m}(C_m) \to C_m$ by

$$\hat{a}_{ij} := p_i a q_j \quad \text{for } a \in C_m, \qquad \text{and} \qquad \check{A} = \sum_{i,j<m} q_i A_{ij} p_j \quad \text{for } A \in Mat_{m,m}(C_m).$$

These maps are inverse to each other, because for $a \in C_m$ and $A \in Mat_{m,m}(C_m)$,

$$\check{\hat{a}} = \sum_{i,j} q_i \hat{a}_{ij} p_j = \sum_{i,j} q_i p_i a q_j p_j = (\sum_i q_i p_i) a (\sum_j q_j p_j) = a,$$

$$(\check{A})\hat{}_{kl} = (\sum_{i,j} q_i A_{ij} p_j)\hat{}_{kl} = p_k (\sum_{i,j} q_i A_{ij} p_j) q_l = p_k q_k A_{kl} p_l q_l = A_{kl}.$$

Let 0_m be the zero and 1_m the unit matrix of dimension $m \times m$. Clearly, $\hat{\cdot}$ is a semiring morphism, by

$$\hat{0} = \big(p_i 0 q_j\big) = 0_m,$$

$$\hat{1} = \big(p_i 1 q_j\big) = \big(\delta_{ij}\big) = 1_m,$$

$$\hat{a} + \hat{b} = \big(p_i a q_j\big) + \big(p_i b q_j\big) = \big(p_i (a+b) q_j\big) = \widehat{a+b},$$

$$\hat{a} \cdot \hat{b} = \big(\sum_k p_i a q_k p_k b q_j\big) = \big(p_i a (\sum_k q_k p_k) b q_j\big) = \big(p_i a b q_j\big) = \widehat{ab}\,.$$

We leave it to the reader to check that the inverse $\check{\cdot}$ also is a semiring morphism. Since they preserve $+$, these maps are monotone and order isomorphisms. To see that they are Kleene algebra morphisms, let $a \in C_m$ and $A \in Mat_{m,m}(C_m)$. Then $a^* = \mu x.g_a(x)$ and $A^* = \mu x.h_A(x)$ are the least pre-fixpoints of the monotone maps $g_a : C_m \to C_m$ and $h_A : Mat_{m,m}(C_m) \to Mat_{m,m}(C_m)$ defined by $g_a(x) = ax + 1$ and $h_A(x) = Ax + 1_m$. For $f = \hat{\cdot} : C_m \to Mat_{m,m}(C_m)$ we have

$$(f \circ g_a)(x) = \widehat{ax+1} = \hat{a}\hat{x} + \hat{1} = (h_{\hat{a}} \circ f)(x).$$

It follows that

$$\widehat{a^*} = f(\mu x.g_a(x)) = \mu x.h_{\hat{a}}(f(x)) = \hat{a}^*.$$

Likewise, for the inverse $f^{-1} = \check{\cdot} : Mat_{m,m}(C_m) \to C_m$ we have

$$(f^{-1} \circ h_A)(x) = (Ax + 1_m)\check{} = \check{A}\check{x} + 1 = (g_{\check{A}} \circ f^{-1})(x),$$

which implies $(A^*)\check{} = \check{A}^*$. □

Corollary 5.3. The Kleene subalgebra of C_m generated by $\{ q_i p_j \mid i,j < m \}$ is isomorphic to $Mat_{m,m}(\mathbb{B})$. Moreover, $C_m \simeq C_m \otimes_{\mathcal{R}} Mat_{m,m}(\mathbb{B})$.

Proof:
Let $E_{(i,j)}$ be the $m \times m$ boolean matrix with 1 only at position (i,j). The first claim holds since the isomorphism $\check{\cdot} : Mat_{m,m}(C_m) \to C_m$ maps a generator $E_{(i,j)}$ of $Mat_{m,m}(\mathbb{B})$ to $q_i p_j$. The second claim follows from $C_m \simeq Mat_{m,m}(C_m)$ and Proposition 2.14. □

Similar to Lemma 2.3, we can code C_m in C_2 for $m > 2$:

Proposition 5.4. For $m > 2$ there is an embedding $\mathcal{R}$-morphism $g : C_m \to C_2$ such that for $i, j < m$,

$$g(p_i) \cdot g(q_j) = \delta_{i,j} \quad \text{and} \quad g(\sum_{i<m} q_i p_i) = 1,$$

writing p_i and q_j for the congruence classes $\{p_i\}/\rho_m$ and $\{q_j\}/\rho_m$ in C_m.

Proof:
Let ρ_m be the $\mathcal{R}$-congruence on $\mathcal{R}\Delta_m^*$ generated by the match- and completeness equations for $\mathcal{R}\Delta_m$ and ρ_2 the corresponding $\mathcal{R}$-congruence on $\mathcal{R}\Delta_2^*$. Writing again $\Delta_2 = \{b, p, d, q\}$, we modify the coding $\overline{\cdot}$ of Δ_m in Δ_2^* of Lemma 2.3 by putting

$$\overline{p_i} = \begin{cases} bp^i, & i < m-1 \\ p^i, & i = m-1 \end{cases} \quad \text{and} \quad \overline{q_i} = \begin{cases} q^i d, & i < m-1 \\ q^i, & i = m-1. \end{cases}$$

This extends to a homomorphism from Δ_m^* to Δ_2^* and lifts to an $\mathcal{R}$-morphism $\overline{\cdot} : \mathcal{R}\Delta_m^* \to \mathcal{R}\Delta_2^*$. Clearly, the match equations $\overline{p_i}\,\overline{q_j} = \delta_{i,j}$ for $i, j < m$ hold in $C_2 = \mathcal{R}\Delta_2^*/\rho_2$. In C_2, we have $1 = db + qp = \overline{q_0 p_0} + q^1 p^1$, and since for $1 \leq i < m-1$

$$q^i p^i = q^i(db + qp)p^i = q^i dbp^i + q^i qpp^i = \overline{q_i p_i} + q^{i+1} p^{i+1},$$

it follows that

$$1 = \overline{q_0 p_0} + q^1 p^1 = \sum_{i<m-1} \overline{q_i p_i} + q^{m-1} p^{m-1} = \sum_{i<m} \overline{q_i p_i}.$$

So the completeness equation of C_m also holds under the coding in C_2. Hence a map $g : \mathcal{R}\Delta_m^*/\rho_m \to \mathcal{R}\Delta_2^*/\rho_2$ is well-defined by $g(A/\rho_m) = \overline{A}/\rho_2$ for $A \in \mathcal{R}\Delta_m^*$. As in Lemma 2.3, it is an $\mathcal{R}$-morphism and satisfies $g(p_i) \cdot g(q_j) = \delta_{i,j}$ and $1 = g(\sum_{i<m} q_i p_i)$. (But the additional property $p_0 \cdot g(q_i) = 0 = g(p_i) \cdot q_0$ of Lemma 2.3 only holds for $i > 0$, since $p_0 \overline{q_0} = bq^0 d = 1 = bp^0 d = \overline{p_0} q_0$ in C_2.)

To see that g is injective, first notice that $\overline{\cdot} : \mathcal{R}\Delta_m^* \to \mathcal{R}\Delta_2^*$ is injective: any $w \in \Delta_2^*$ in the image of $\overline{\cdot}$ can uniquely be parsed into a word of $\{\overline{p_0}, \ldots, \overline{p_{m-1}}, \overline{q_0}, \ldots, \overline{q_{m-1}}\}^*$, so there is a unique $v \in \Delta_m^*$ with $w = \overline{v}$. It is therefore sufficient to show

$$\text{for all } A, B \in \mathcal{R}\Delta_m^*(\overline{A}\,\rho_{2,n}\,\overline{B} \Rightarrow A/\rho_m = B/\rho_m), \tag{11}$$

where $\rho_{2,n}$ is the n-th stage of the inductive definition of ρ_2. This is done by induction on n. If $\overline{A}\,\rho_{2,0}\,\overline{B}$, either $\overline{A} = \overline{B}$, in which case $A = B$, or $\overline{A}\,\rho_{2,0}\,\overline{B}$ is a match equation or the completeness equation of ρ_2, in which case (A, B) is the corresponding match or completeness equation of ρ_m, so $A/\rho_m = B/\rho_m$. If $\overline{A}\,\rho_{2,n+1}\,\overline{B}$ is obtained by symmetry from $\overline{B}\,\rho_{2,n}\,\overline{A}$ or by transitivity from $\overline{A}\,\rho_{2,n}\,\overline{C}$ and $\overline{C}\,\rho_{2,n}\,\overline{B}$, the claim follows from symmetry resp. transitivity of ρ_m.

If $\overline{A}\,\rho_{2,n+1}\,\overline{B}$ is obtained from $\overline{A_1}\,\rho_{2,n}\,\overline{B_1}$ and $\overline{A_2}\,\rho_{2,n}\,\overline{B_2}$ by $\overline{A} = \overline{A_1}\,\overline{A_2}$ and $\overline{B} = \overline{B_1}\,\overline{B_2}$, then

$$A/\rho_m = (A_1 A_2)/\rho_m = A_1/\rho_m A_2/\rho_m = B_1/\rho_m B_2/\rho_m = (B_1 B_2)/\rho_m = B/\rho_m$$

by induction. The argument is similar if $\overline{A} = \overline{A_1} \cup \overline{A_2}$ and $\overline{B} = \overline{B_1} \cup \overline{B_2}$.

Suppose $\bigcup U' \, \rho_{2,n+1} \bigcup V'$, where $U', V' \in \mathcal{R}(\mathcal{R}\Delta_2^*)$ contain only regular sets of words in the image of $\overline{\cdot} : \Delta_m^* \to \Delta_2^*$ and $(U'/\rho_2)^{\downarrow} = (V'/\rho_2)^{\downarrow}$ in stage n. As these regular sets of words are also regular sets of words over $\overline{\Delta_m}$, there are $U, V \in \mathcal{R}(\mathcal{R}\Delta_m^*)$ such that $U' = \{ \overline{A} \mid A \in U \}$, $V' = \{ \overline{B} \mid B \in V \}$, and for each $A \in U$ there is $B \in V$ with $\overline{A} \cup \overline{B} \, \rho_{2,n} \, \overline{B}$ and for each $B \in V$ there is $A \in U$ with $\overline{B} \cup \overline{A} \, \rho_{2,n} \, \overline{A}$. By induction, $\overline{A \cup B} = \overline{A} \cup \overline{B} \, \rho_{2,n} \, \overline{B}$ implies $A/\rho_m \leq B/\rho_m$ and $\overline{B} \cup \overline{A} \, \rho_{2,n} \, \overline{A}$ implies $B/\rho_m \leq A/\rho_m$, so that $(U/\rho_m)^{\downarrow} = (V/\rho_m)^{\downarrow}$ and therefore $\bigcup U/\rho_m = \bigcup U/\rho_m$. Since $\bigcup U' = \overline{\bigcup U}$ and $\bigcup V' = \overline{\bigcup V}$, the claim is proven. □

5.2. Relativizing the completeness property

Let $m \geq 2$ and $e := \sum_{i<m} q_i p_i$. For the tensor product $K \otimes_{\mathcal{R}} C_m$ of an $\mathcal{R}$-dioid K and C_m, the completeness equation $e = 1$ can be used to show that every element of the centralizer of C_m is the least upper bound of some context-free subset of K, i.e. that

$$\sum : \mathcal{C}K \to Z_{C_m}(K \otimes_{\mathcal{R}} C_m)$$

is surjective. Also, Lemma 3.3 is a bit easier to prove for C_2 than for C_2', as we can use $db \leq 1$ to prove $NN \leq N$ in Claim 3.3. We do not go into this here, but observe that in suitable contexts, $e = 1$ in a sense holds in the polycyclic algebras C_m' as well. For example, as $p_i e = p_i$ for $p_i \in P_m$ and $eq_j = q_j$ for $q_j \in Q_m$, in C_m' we have

$$p_0 e q_0 = p_0 q_0 = 1 = p_0 1 q_0.$$

This can be generalized to a relativized form of the completeness property. Basically, for any regular expression $\varphi(x)$ in an unknown x, elements of K and brackets of C_m' other than p_0, q_0, the two elements $\varphi(e), \varphi(1) \in K \otimes_{\mathcal{R}} C_m'$ are suprema of regular sets that differ only by elements from the centralizer $Z_{C_m'}(K \otimes_{\mathcal{R}} C_m')$ weighted by factors from $\{q_1, \ldots, q_{m-1}\}^* \{p_1, \ldots, p_{m-1}\}^* \setminus \{1\}$, and these reduce to 0 in the context $p_0 \ldots q_0$ of a fresh pair of brackets.

Theorem 5.5. (Relative Completeness)
Let K be an $\mathcal{R}$-dioid. For any $\varphi(x) = \varphi(\pi, p_1, \ldots, p_{m-1}, q_1, \ldots, q_{m-1}, x) \in (K \otimes_{\mathcal{R}} C_m')[x]$ in which p_0 and q_0 occur only in $\pi = q_0 p_0$,

$$p_0 \varphi(e) q_0 = p_0 \varphi(1) q_0 \quad \text{and} \quad p_0 \varphi(1) q_0 \in Z_{C_m'}(K \otimes_{\mathcal{R}} C_m').$$

Proof:
Let $\varphi(x) = S(A + W\pi)^* F$ be given by an automaton $\langle S, A + W\pi, F \rangle$, where $A = U + X + V + Yx$, S, U, X, V, W, F and n are as in Theorem 3.11, and $Y \in \{0,1\}^{n \times n}$. Then

$$p_0 \varphi(x) q_0 = p_0 S(A + W\pi)^* F q_0 = S p_0 (A + W\pi)^* q_0 F.$$

Recall that on the right and in the following, p_i and q_j are identified with corresponding diagonal matrices of dimension n. As in the proof of Theorem 3.11,

$$p_0 (A + W\pi)^* q_0 \quad = \quad p_0 A^* q_0 (W p_0 A^* q_0)^*.$$

It is sufficient to show that $p_0 A^* q_0$ does not depend on the choice of $x \in \{1, e\}$. Using $\alpha = U + X + V$,

$$p_0 A^* q_0 \;=\; p_0(\alpha + Yx)^* q_0 = p_0 \alpha^* (Yx\alpha^*)^* q_0.$$

Let $N \in (Z_{C_m}(K \otimes_{\mathcal{R}} C_m))^{n\times n}$ be as in Theorem 3.11, so that with *-continuity on the matrix level,

$$\begin{aligned}\alpha^* &= (U + X + V)^* = (NV)^* N(UN)^* \\ &= \sum\{\, (NV)^k N(NU)^l \mid k, l \in \mathbb{N}\,\}.\end{aligned}$$

There are $U_i, V_j \in \mathbb{B}^{n\times n}$ such that $U = \sum_{0<i<m} U_i p_i$ and $V = \sum_{0<j<m} q_j V_j$. As the q_j and p_i commute with N and boolean matrices,

$$\begin{aligned}(NV)^k N(UN)^l &= (\sum_{0<j<m} q_j N V_j)^k N(\sum_{0<i<m} U_i N p_i)^l \\ &= \sum_{0<j_1,\ldots,j_k,i_1,\ldots,i_l<m} q_{j_1}\cdots q_{j_k} p_{i_l}\cdots p_{i_1} N V_{j_1}\cdots N V_{j_k} N U_{i_l} N \cdots U_{i_1} N.\end{aligned}$$

Let $P = P_m \setminus \{p_0\}$ and $Q = Q_m \setminus \{q_0\}$. For $v = q_{j_1}\cdots q_{j_k} \in Q^*$ and $u = p_{i_l}\cdots p_{i_1} \in P^*$, put

$$N_{vu} = N V_{j_1}\cdots N V_{j_m} N U_{i_l} N \cdots U_{i_1} N,$$

so that

$$\alpha^* = \sum\{\, (NV)^k N(UN)^l \mid k, l \in \mathbb{N}\,\} = \sum\{\, vuN_{vu} \mid u \in P^*, v \in Q^*\,\}.$$

By *-continuity, it follows that

$$\begin{aligned}&p_0\alpha^*(Ye\alpha^*)^* q_0 \\ &= \sum\{\, p_0\alpha^*(Ye\alpha^*)^k q_0 \mid k \in \mathbb{N}\,\} \\ &= \sum\{\, p_0 v_0 u_0 N_{v_0u_0} \ldots Yev_k u_k N_{v_ku_k} q_0 \mid k \in \mathbb{N}, u_0,\ldots,u_k \in P^*, v_0,\ldots,v_k \in Q^*\,\} \\ &= \sum\{\, p_0 v_0 u_0 \ldots ev_k u_k q_0 N_{v_0u_0} \ldots YN_{v_ku_k} \mid k \in \mathbb{N}, u_0,\ldots,u_k \in P^*, v_0,\ldots,v_k \in Q^*\,\},\end{aligned}$$

where the final step holds since the $ev_{i+1}u_{i+1}$ commute with $N_{v_0u_0}Y\cdots N_{v_iu_i}Y$.

To show $p_0\alpha^*(Ye\alpha^*)^* q_0 = p_0\alpha^*(Y\alpha^*)^* q_0$, it therefore is sufficient that e can be replaced by 1 in the summands $p_0 v_0 u_0 \ldots ev_k u_k q_0 N_{v_0u_0}\ldots YN_{v_ku_k}$, i.e. that

$$p_0 v_0 u_0 e v_1 u_1 \ldots e v_k u_k = p_0 v_0 u_0 v_1 u_1 \ldots v_k u_k. \tag{12}$$

For $k = 0$, equation (12) is obvious. For $0 < k$, put $w_j = v_0 u_0 \ldots v_j u_j$ and, by induction, assume

$$p_0 v_0 u_0 e v_1 u_1 \ldots e v_j u_j = p_0 w_j$$

for some $j < k$. Since $w_j \in Q^*P^* \cup \{0\}$, we distinguish three cases. If $w_j = 1$, then $p_0 w_j e = p_0 e = p_0 = p_0 w_j$, so $p_0 w_j e v_{j+1} u_{j+1} = p_0 w_j v_{j+1} u_{j+1} = p_0 w_{j+1}$. If $w_j \in Q^+$, then $p_0 w_j = 0$, so $p_0 w_j e v_{j+1} u_{j+1} = p_0 w_{j+1}$. If $w_j \in Q^*P^+ \cup \{0\}$, then $w_j e = w_j$, so $p_0 w_j e v_{j+1} u_{j+1} = p_0 w_{j+1}$.

It follows that $p_0 v_0 u_0 e v_1 u_1 \ldots e v_{j+1} u_{j+1} = p_0 w_{j+1}$, and by induction, (12). Thus we have shown $p_0 \alpha^*(Y e \alpha^*)^* q_0 = p_0 \alpha^*(Y \alpha^*)^* q_0$ and thereby $p_0 \varphi(e) q_0 = p_0 \varphi(1) q_0$.

Moreover, since $p_0 w_k q_0 \in \{0, 1\}$ for all k, $p_0 \alpha^*(Y \alpha^*)^* q_0$ is the least upper bound of a regular set of $n \times n$-matrices over $Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m)$. It follows that, for $x = 1$,

$$p_0 A^* q_0 = p_0 \alpha^*(Y \alpha^*)^* q_0 \in (Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m))^{n \times n},$$

and therefore $p_0 \varphi(1) q_0 = S p_0 A^* q_0 (W p_0 A^* q_0)^* F \in Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m)$. □

It therefore seems that at least for applications to formal languages, where we can use a special pair p_0, q_0 of brackets to annihilate words of $\{q_1, \ldots, q_{m-1}\}^* \{p_1, \ldots, p_{m-1}\}^*$, the completeness equation is of little help.

6. Conclusion

The tensor product $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_m$ of the algebra $\mathcal{R}X^*$ of regular sets of X^* with the polycyclic Kleene algebra C'_m based on $m \geq 2$ bracket pairs is a *-continuous Kleene algebra subsuming an isomorphic copy of the algebra $\mathcal{C}X^*$ of context-free sets of X^*, the centralizer $Z_{C'_m}(\mathcal{R}X^* \otimes_{\mathcal{R}} C'_m)$ of C'_m.

We have investigated $K \otimes_{\mathcal{R}} C'_m$ for arbitrary *-continuous Kleene algebras K. Every element $\varphi \in K \otimes_{\mathcal{R}} C'_m$ is the value SA^*F of an automaton $\langle S, A, F \rangle$ whose transition matrix $A = U + X + V$ splits into transitions by opening brackets (and 0's) in U, transitions by elements of K in X, and transitions by closing brackets (and 0's) in V. Our main result is a normal form theorem saying that $A^* = (NV)^* N (UN)^*$, where N is the least solution of $y \geq (UyV + X)^*$ in $Mat_{n,n}(K \otimes_{\mathcal{R}} C'_m)$, corresponding to Dyck's language $D \subseteq \{U, X, V\}^*$ with bracket pair U, V, and N has entries in $Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m)$. If $\varphi = SA^*F$ belongs to the centralizer of C'_2 in $K \otimes_{\mathcal{R}} C'_2$, and K has no zero divisors, then $SA^*F = SNF$. It remains open whether the non-existence of zero divisors is a necessary assumption. These normal forms generalize a simpler normal form for elements of the polycyclic monoid $P'_m[X]$.

Our main result had been obtained earlier (unpublished) by the first author with the bra-ket Kleene algebra C_m instead of C'_m. For the brackets $p_0, \ldots, q_{m-1}$, we no longer need the completeness equation $1 = q_0 p_0 + \ldots + q_{m-1} p_{m-1}$ of C_m, but only the match- and mismatch equations $p_i q_j = \delta_{i,j}$ of C'_m. It is also shown that in the context $p_0 \ldots q_0$, in $K \otimes_{\mathcal{R}} C'_m$ this equation can be assumed to hold.

The two sets of cases of greatest interest are specializations of $\mathcal{R}M \otimes_{\mathcal{R}} C$ to the monoids $M = X^*$ and $M = X^* \times Y^*$ and to $C = C'_2$ and $C = C_2$. Applications, for $M = X^*$, include recognition of languages over an alphabet of inputs X, while for $M = X^* \times Y^*$, they include parsing or translation of languages over X, where Y may denote an alphabet of actions (such as parse tree building operations), or an alphabet of outputs. With the results established here, we have laid a foundation for an algebraic study of recognition, parsing and translation algorithms for context-free languages over X, that we hope to analyze in greater depth in later publications.

In addition, given the close relation between C'_2 and C_2 and stack machines, it is natural to enquire as to whether $\mathcal{R}M \otimes_{\mathcal{R}} C$ may provide a representation for 2-stack machine languages and relations, where $C = C'_2 \otimes_{\mathcal{R}} C'_2$ or $C = C_2 \otimes_{\mathcal{R}} C_2$, and, thus, a basis for a calculus for recursively enumerable languages and relations. We also hope to elaborate this in a future publication.

7. Appendix

We here complete the proof of Lemma 2.6 by showing that $\equiv \subseteq P$. We repeat that $P(R,S)$ is

$$\forall (x,y),(a,b),(a',b')[(a,b)R(a',b') \setminus Z \preceq (x,y) \iff (a,b)S(a',b') \setminus Z \preceq (x,y)], \tag{13}$$

where $Z = \{\, (a,b) \in K_1 \times K_2 \mid a = 0 \text{ or } b = 0 \,\}$ and $R \preceq (x,y)$ says that (x,y) is an upper bound of $R \subseteq K_1 \times K_2$.

Proof:
Let $\equiv_n$ be the n-th stage in the inductive definition of $\equiv$, where $\equiv_0$ consists of those (R,S) where $R = S$ or where they are a tensor product equation, i.e. $R = A \times B$ and $S = \{(\sum A, \sum B)\}$ for some $A \in \mathcal{R}K_1, B \in \mathcal{R}K_2$, and $\equiv_{n+1}$ adds pairs to $\equiv_n$ by the closure conditions for symmetry, transitivity, sum, product and supremum. To prove $\equiv \subseteq P$, it is sufficient to show $\equiv_n \subseteq P$ by induction on n.

Suppose $R \equiv_0 S$. If $R = S$, then $P(R,S)$ is clear, since P is reflexive. Otherwise, $R \equiv_0 S$ is a tensor product equation, i.e. there are $A \in \mathcal{R}K_1$ and $B \in \mathcal{R}K_2$ such that $R = A \times B$ and $S = \{(\sum A, \sum B)\}$. Let $(x,y),(a,b),(a',b') \in K_1 \times K_2$. To show

$$(a,b)(A \times B)(a',b') \setminus Z \preceq (x,y) \iff (a,b)\{(\sum A, \sum B)\}(a',b') \setminus Z \preceq (x,y), \tag{14}$$

we first observe that, since for a rectangle $A' \times B' \subseteq K_1 \times K_2$,

$$A' \times B' \subseteq Z \quad \iff \quad A' \subseteq \{0\} \vee B' \subseteq \{0\},$$

either both sets $(a,b)(A \times B)(a',b')$ and $(a,b)\{(\sum A, \sum B)\}(a',b')$ are subsets of Z or both are not:

$$\begin{aligned}
(a,b)(A \times B)(a',b') \subseteq Z \quad &\iff \quad aAa' \subseteq \{0\} \vee bBb' \subseteq \{0\} \\
&\iff \quad \sum aAa' = 0 \vee \sum bBb' = 0 \\
&\iff \quad (a(\sum A)a', b(\sum B)b') \in Z \\
&\iff \quad (a,b)\{(\sum A, \sum B)\}(a',b') \subseteq Z.
\end{aligned}$$

If both of these sets are subsets of Z, then clearly (14) holds. Otherwise, both $(a,b)(A \times B)(a',b') \setminus Z$ and $(a,b)\{(\sum A, \sum B)\}(a',b') \setminus Z$ are non-empty. Since for rectangles $A' \times B' \not\subseteq Z$,

$$A' \times B' \setminus Z \preceq (x,y) \quad \iff \quad A' \times B' \preceq (x,y),$$

the claim (14) is implied by the following:

$$\begin{aligned}
(a,b)(A \times B)(a',b') \preceq (x,y) \quad &\iff \quad (aAa' \times bBb') \preceq (x,y) \\
&\iff \quad aAa' \preceq x \wedge bBb' \preceq y \\
&\iff \quad \sum aAa' \leq x \wedge \sum bBb' \leq y \\
&\iff \quad (a,b)\{(\sum A, \sum B)\}(a',b') \preceq (x,y).
\end{aligned}$$

Suppose $R \equiv_{n+1} S$ is obtained from $S \equiv_n R$ by the condition to close $\equiv$ under symmetry. By induction, $P(S, R)$ holds, and since P is an equivalence relation, $P(R, S)$ also holds.

Suppose $R \equiv_{n+1} S$ is obtained from $R \equiv_n T$ and $T \equiv_n S$ by the condition to close $\equiv$ under transitivity. By induction, $P(R, T)$ and $P(T, S)$, and since P is an equivalence relation, $P(R, S)$.

Suppose $R_1 \cup R_2 \equiv_{n+1} S_1 \cup S_2$ is obtained from $R_1 \equiv_n S_1$ and $R_2 \equiv_n S_2$ by the condition to close $\equiv$ under union. By induction, $P(R_1, S_1)$ and $P(R_2, S_2)$, and hence, for all $(a, b), (a', b')$ and (x, y),

$$\begin{aligned}
&(a,b)(R_1 \cup R_2)(a',b') \setminus Z \preceq (x,y) \\
&\quad\iff\quad (a,b)R_1(a',b') \setminus Z \preceq (x,y) \wedge (a,b)R_2(a',b') \setminus Z \preceq (x,y) \\
&\quad\iff\quad (a,b)S_1(a',b') \setminus Z \preceq (x,y) \wedge (a,b)S_2(a',b') \setminus Z \preceq (x,y) \\
&\quad\iff\quad (a,b)(S_1 \cup S_2)(a',b') \setminus Z \preceq (x,y),
\end{aligned}$$

which shows $P(R_1 \cup R_2, S_1 \cup S_2)$.

Suppose $R_1R_2 \equiv_{n+1} S_1S_2$ is obtained from $R_1 \equiv_n S_1$ and $R_2 \equiv_n S_2$ by the condition to close $\equiv$ under products. Let $(a, b), (a', b'), (x, y) \in K_1 \times K_2$ and assume $(a, b)R_1R_2(a', b') \setminus Z \preceq (x, y)$. By induction, $P(R_1, S_1)$, and hence, exploiting the universal quantification in (13),

$$(a,b)S_1R_2(a',b') \setminus Z \preceq (x,y).$$

Since, by induction, we also have $P(R_2, S_2)$, this similarly gives $(a, b)S_1S_2(a', b') \setminus Z \preceq (x, y)$. In the same way, from $(a, b)S_1S_2(a', b') \setminus Z \preceq (x, y)$ one gets $(a, b)R_1R_2(a', b') \setminus Z \preceq (x, y)$. Taken together, this shows $P(R_1R_2, S_1S_2)$.

Suppose $\bigcup\mathcal{U} \equiv_{n+1} \bigcup\mathcal{V}$ comes from $\mathcal{U}, \mathcal{V} \in \mathcal{R}(\mathcal{R}(K_1 \times K_2))$ with $(\mathcal{U}/_{\equiv})^{\downarrow} = (\mathcal{V}/_{\equiv})^{\downarrow}$ in stage n, i.e.

$$\forall R \in \mathcal{U}\, \exists S \in \mathcal{V} (R \cup S \equiv_n S) \wedge \forall S \in \mathcal{V}\, \exists R \in \mathcal{U} (S \cup R \equiv_n R),$$

by the condition to close $\equiv$ under suprema. By induction,

$$\forall R \in \mathcal{U}\, \exists S \in \mathcal{V}\, P(R \cup S, S) \wedge \forall S \in \mathcal{V}\, \exists R \in \mathcal{U}\, P(S \cup R, R). \tag{15}$$

Let $(a, b), (a', b'), (x, y) \in K_1 \times K_2$, and assume $(a, b)(\bigcup\mathcal{U})(a', b') \setminus Z \preceq (x, y)$, i.e.

$$\forall R \in \mathcal{U}((a,b)R(a',b') \setminus Z \preceq (x,y)).$$

To show $(a, b)(\bigcup\mathcal{V})(a', b') \setminus Z \preceq (x, y)$, let $S \in \mathcal{V}$. By (15), there is $R \in \mathcal{U}$ with $P(S \cup R, R)$, hence

$$(a,b)(S \cup R)(a',b') \setminus Z \preceq (x,y) \iff (a,b)R(a',b') \setminus Z \preceq (x,y).$$

Since the right-hand side is true, we get $(a, b)S(a', b') \setminus Z \preceq (x, y)$ from the left-hand side. This shows $\forall S \in \mathcal{V}((a, b)S(a', b') \setminus Z \preceq (x, y))$, i.e. $(a, b)(\bigcup\mathcal{V})(a', b') \setminus Z \preceq (x, y)$. The reverse implication

$$(a,b)(\bigcup\mathcal{U})(a',b') \setminus Z \preceq (x,y) \Leftarrow (a,b)(\bigcup\mathcal{V})(a',b') \setminus Z \preceq (x,y)$$

is shown by a symmetric argument. Therefore, we have $P(\bigcup\mathcal{U}, \bigcup\mathcal{V})$. □

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References

[1] Chomsky N, Schützenberger M. The algebraic theory of context free languages. In: Braffort P, Hirschberg D (eds.), Computer Programming and Formal Systems. 1963 pp. 118–161.

[2] Conway JH. Regular Algebra and Finite Machines. Chapman and Hall, London, 1971.

[3] Hopkins M. The Algebraic Approach I: The Algebraization of the Chomsky Hierarchy. In: Berghammer R, Möller B, Struth G (eds.), Relational Methods in Computer Science/Applications of Kleene Algebra, LNCS 4988. Springer Verlag, Berlin Heidelberg, 2008 pp. 155–172.

[4] Hopkins M. The Algebraic Approach II: Dioids, Quantales and Monads. In: Berghammer R, Möller B, Struth G (eds.), Relational Methods in Computer Science/Applications of Kleene Algebra, LNCS 4988. Springer Verlag, Berlin Heidelberg, 2008 pp. 173–190.

[5] Hopkins M, Leiß H. Coequalizers and Tensor Products for Continuous Idempotent Semirings. In: Desharnais J, Guttmann W, Joosten S (eds.), 17th Int. Conf. on Relational and Algebraic Methods in Computer Science (RAMiCS 2018), LNCS 11194. Springer Nature Switzerland AG, Cham, 2018 pp. 37–52. doi: 10.1007/978-3-030-02149-8_3

[6] Hopkins M, Leiß H. Normal Forms for Elements of the *-continuous Kleene Algebras $K \otimes_{\mathcal{R}} C_2'$. In: Glück R, Santocanale L, Winter M (eds.), 20th Int.Conf. on Relational and Algebraic Methods in Computer Science, RAMiCS 2023, Augsburg, Germany, April 3-6, 2023, volume 13896 of *LNCS*. Springer, 2023 pp. 122–139. doi:10.1007/978-3-031-28083-2_8

[7] Kozen D. The Design and Analysis of Algorithms. Springer-Verlag, New York, 1991.

[8] Kozen D. A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events. *Information and Computation*, 1994. **110**(2):366–390.

[9] Grathwohl NBB, Henglein F, Kozen D. Infinitary Axiomatization of the Equational Theory of Context-Free Languages. In: Baelde D, Carayol A (eds.), Fixed Points in Computer Science (FICS 2013), volume 126 of *EPTCS*. 2013 pp. 44–55. doi:10.4204/EPTCS.126.4.

[10] Leiß H. The Matrix Ring of a μ-Continuous Chomsky Algebra is μ-Continuous. In: Regnier L, Talbot JM (eds.), 25th EACSL Annual Conference on Computer Science Logic (CSL 2016), Leibniz International Proceedings in Informatics. Leibniz-Zentrum für Informatik, Dagstuhl Publishing, 2016 pp. 1–16. doi: 10.4230/LIPIcs.CSL.2016.1.

[11] Leiß H. An Algebraic Representation of the Fixed-Point Closure of *-continuous Kleene Algebras – A Categorical Chomsky-Schützenberger-Theorem. *Mathematical Structures in Computer Science*, 2022. **32**(6):686–725. doi:10.1017/S0960129522000329.

[12] Leiß H, Hopkins M. C-Dioids and μ-Continuous Chomsky Algebras. In: Desharnais J, Guttmann W, Joosten S (eds.), 17th Int. Conf. on Relational and Algebraic Methods in Computer Science, RAMiCS 2018. Springer Nature Switzerland AG, Cham, 2018 pp. 21–36. doi:10.1007/978-3-030-02149-8_2