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On the Tutte and Matching Polynomials for Complete Graphs

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Abstract. Let $T(G; X, Y)$ be the Tutte polynomial for graphs. We study the sequence $t_{a,b}(n) = T(K_n; a, b)$ where a, b are integers, and show that for every $\mu \in \mathbb{N}$ the sequence $t_{a,b}(n)$ is ultimately periodic modulo μ provided $a \neq 1 \mod \mu$ and $b \neq 1 \mod \mu$. This result is related to a conjecture by A. Mani and R. Stones from 2016. The theorem is a consequence of a more general theorem which holds for a wide class of graph polynomials definable in Monadic Second Order Logic. This gives also similar results for the various substitution instances of the bivariate matching polynomial and the trivariate edge elimination polynomial $\xi(G; X, Y, Z)$ introduced by I. Averbouch, B. Godlin and the second author in 2008. All our results depend on the Specker-Blatter Theorem from 1981, which studies modular recurrence relations of combinatorial sequences which count the number of labeled graphs.

1. Introduction

Boris (Boaz) Abramovich Trakhtenbrot (1921-2016) was one of the pioneers in recognizing the usefulness of Monadic Second Order Logic MSOL for treating situations in automata theory [1]. In [2], published in the Festschrift for Boaz' 85th birthday, Eldar Fischer and the second author gave an application of Monadic Second Order Logic to graph polynomials, by proving that for a graph polynomial

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$P(G;\bar{x})$ definable in MSOL and a sequence of recursively defined graphs $G_i : i \in \mathbb{N}$ the sequence of polynomials $P(G_i;\bar{x})$ satisfies a linear recurrence relation over the polynomial ring $\mathbb{Z}[\bar{x}]$.

Ernst Specker (born in 1920-2011), together with Christian Blatter (1935-2021), was the first to use Monadic Second Order Logic to prove a meta-theorem in counting combinatorics, [3]. In order to celebrate Boaz' centenary we shall give another application of Monadic Second Order Logic to MSOL-definable graph polynomials $P(G;\bar{x})$, where the proof again uses the Specker-Blatter Theorem. We fix $\bar{a} \in \mathbb{N}^r$ and look at the sequence $p(n,\bar{a}) = P(K_n;\bar{a})$ modulo $\mu \in \mathbb{N}$, where K_n is the complete graph on n vertices. We show that, under simple assumption on $\bar{a}$ and μ, this sequence is ultimately periodic modulo μ.

Our results are easy, but possibly unexpected, applications of the Specker-Blatter Theorem. They illustrate more the power of it, by using general methods of Monadic Second Order Logic MSOL, rather than applying combinatorial arguments specially tailored to a particular case.

1.1. Some graph polynomials

Let $G = (V(G), E(G))$ be a finite graph. We put $n(G) = |V(G)|$, $m(G) = |E(G)|$, $\kappa(G)$ is the number of connected components of G. From a logical point of view G can be represented in various ways. The vocabulary τ_{graph} consists of one binary relation symbol for $E(G)$ and the vocabulary τ_{hgraph} consists of two unary predicates, one for vertices $V(G)$ and one for edges $E(G)$ and one binary relation symbol for $R(G)$ for the incidence relation between edges and vertices. τ_{hgraph} is also suitable for representing hypergraphs. *Induced subgraphs* are substructures in the case of τ_{graph}. and *subgraphs* are substructures in the case of τ_{hgraph}. In other words, let $A \subseteq V(G)$ be a set of vertices. The *induced subgraph* of G generated by A is the graph $G[A] = (A, E(G) \cap A^2)$. On the other hand the graph (A, E) is a subgraph of G for any $F \subseteq E(G)$.

MSOL on graphs allows quantification over subsets of vertices only. MSOL on hypergraphs allows quantification over subsets of vertices and subsets of edges. MSOL on hypergraphs has the same expressive power as Second Order Logic SOL on graphs where second order quantification is *restricted to unary predicates and binary relations which are subsets of the edge relation*. We denote this version by GMSOL, for Guarded Monadic Second Order Logic. We assume the reader is familiar with Monadic Second Order Logic MSOL. For the Monadic Second Order theory of graphs the reader is referred to the encyclopedic [4]. CMSOL is the logic extending MSOL with modular counting quantifiers.

A *graph polynomial* $P(G,\bar{X})$ is a *function* P which associates with a graph G a polynomial $P(G,\bar{X}) \in \mathbb{Z}[\bar{X}]$, and which is *invariant* under graph isomorphisms (disregarding the labels). Here $\bar{X} = (X_1,\ldots,X_s)$.

Let us look at some examples:

(i) Let $i_k(G)$ denote the number of independents sets $A \subseteq V(G)$ of size k.
The *independence polynomial* $In(G,X)$ is defined as

$$In(G,X) = \sum_{k=0}^{n(G)} i_k(G) X^k$$

(ii) Let $c_k(G)$ denote the number of sets $A \subseteq V(G)$ which induce a clique of size k.
The *clique polynomial* $Cl(G, X)$ is defined as

$$Cl(G; X) = \sum_{k=0}^{n(G)} c_k(G) X^k$$

(iii) Let $\lambda \in \mathbb{N}$. $\chi(G, \lambda)$ denotes the number of proper λ-colorings of G.
By the well-known observation of G. Birkhoff (1912), $\chi(G, \lambda)$ is a polynomial in $\mathbb{N}[\lambda]$.
We denote by $\chi(G, X)$ the extensions of $\chi(G, \lambda)$ to a polynomial over the complex numbers, $\chi(G, X) \in \mathbb{C}[X]$. $\chi(G, X)$ is called the *chromatic polynomial of* G.

(iv) The *Tutte polynomial* $T(G; X, Y)$ is defined as

$$T(G; X, Y) = \sum_{A \subset E(G)} (X-1)^{\kappa(A)-\kappa(G)} \cdot (Y-1)^{|A|+\kappa(A)-|V(G)|} \tag{1}$$

where $\kappa(S)$ is the number of connected components of the spanning subgraph $G[S] = (V(G), S)$.

(v) The *matching polynomials* come in two versions: Let G be a graph and $m_k(G)$ be the number of matchings of size k of G. The *generating matching polynomial* $M(G : X)$ of G is defined as

$$M(G; X) = \sum_{k=0}^{\lfloor n/2 \rfloor} m_k(G) X^k$$

and the *matching defect* aka *acyclic polynomial* $\alpha(G; X)$ is defined as

$$\alpha(G; X) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k(G) X^{n-2k}$$

The two are related by the equations

$$\alpha(G; X) = X^n M(G; -X^{-2}) \text{ and } M(G; X) = (-i)^n X^{n/2} \alpha(G; iX^{-1/2})$$

There is also a bivariate version

$$\bar{M}(G; X, Y) = \sum_{k=0}^{\lfloor n/2 \rfloor} (X)^k m_k(G) Y^{n-2k}$$

where $\bar{M}(G; -1, Y) = \alpha(G; Y)$ and $\bar{M}(G; X, 1) = M(G; X)$.

(vi) In [5, 6] the authors introduce a most general edge elimination polynomial in three indeterminates $\xi(G; X, Y, Z)$.

$$\xi(G; X, Y, Z) = \sum_{(A \sqcup B) \subseteq E} X^{\kappa(A \sqcup B)-c(B)} \cdot Y^{|A|+|B|-c(B)} \cdot Z^{c(B)} \tag{2}$$

Here $c((V,E))$ is the number of connected components of (V,E) which have at least one edge. Both the matching polynomials and the Tutte polynomial are substitution instances of $\xi(G;X,Y,Z)$. In [7, 8] other trivariate graph polynomials are discussed which are equivalent to $\xi(G;X,Y,Z)$: Among them the subgraph counting polynomial $S(G,X,Y,Z)$ and the covered components polynomial $C(G;X,Y,Z)$. All these trivariate graph polynomials are substitution instances of each other. The subgraph counting polynomial $S(G;X,Y,Z)$ is defined as

$$S(G;X,Y,Z) = \sum_{H=(W,F)\subseteq G} X^{|W|}Y^{\kappa(H)}Z^{|F|}$$

where H ranges over all subgraphs of G.

The covered components polynomial $C(G;X,Y,Z)$ is defined as

$$C(G;X,Y,Z) = \sum_{A\subseteq E} X^{\kappa(([n],A))}Y^{|A|}Z^{c(([n],A))}$$

In [7] it is shown how $\xi(G;X,Y,Z)$ and $C(G;X,Y,Z)$ are related:

Proposition 1.1.

$$C(G;X,Y,Z) = \xi((G;X,Y,XYZ-XY) \text{ and } \xi((G;X,Y,Z) = C(G;X,Y,\frac{Z}{XY}+1)$$

A similar relation is given for $\xi(G;X,Y,Z)$ and $S(G;X,Y,Z)$ in [8] which is a bit more complicated, but no needed for this paper.

Remark 1.2. We note that both the Tutte polynomial and $\xi(G;,X,Y,Z)$ involve negative exponents, whereas $S(G;X,Y,Z)$ and $C(G;X,Y,Z)$ do not. This is the source of the difference between Theorem 1.5 and Theorem 1.6.

1.2. The case of $G = K_n$

Here we shall be concerned with computing a graph polynomial $P(G;\bar{X})$ in k indeterminates for the case where the graph G is K_n, the complete graph on n vertices. We define, for fixed non-negative integers $\bar{b}\in\mathbb{N}^k$ the sequence

$$P_n(\bar{b}) = P(K_n,\bar{b})$$

for fixed values $\bar{b}\in\mathbb{Z}^k$. Can we make some general statement about the sequence $P_n(\bar{b})$? In some cases computing $P_n(\bar{b})$ is very easy, using trivial observations or simple recurrence relations. However, the resulting graph polynomials may be unexpectedly complicated. The Tutte polynomial and the matching polynomial will illustrate this in the sequel.

We compute $P_n(\bar{b})$ first for some straightforward cases:

(i) *Independence polynomial*:

$$i(K_n,k) = \begin{cases} 0 & k\geq 2 \\ n & k=1 \\ 1 & k=0 \end{cases}$$

Hence,

$$In(K_n, b) = i_1(K_n) \cdot b + c_0(K_n) = nb + 1$$

(ii) *Clique polynomial*: $c_k(K_n) = \binom{n}{k}$, hence

$$Cl(K_n, b) = \sum_{k=0}^{n} \binom{n}{k} b^k = (b+1)^n$$

(iii) *Chromatic polynomial*: $\chi(K_n, b) = 0$ for $n \geq b + 1$.

1.3. The Tutte polynomial

The case of the Tutte polynomial is a bit more complicated, but of special interest. We note that

(i) $T(K_n; 2, 1)$ counts the number of forests on n vertices.

(ii) $T(K_n; 1, 1)$ counts the number of trees on n vertices.

(iii) $T(K_n; 1, 2)$ counts the number of connected graphs on n vertices.

The earliest computation of $T_n(X, Y) = T(K_n, X, Y)$ can be found in [9] which already gives a recursive computation. I. Gessel [10, 11, 12] and independently I. Pak [13], proved:

$$T(K_n; a, b) = \sum_{k=1}^{n} \binom{(n-1)}{(k-1)} \left(a + \sum_{i=1}^{k-1} b^i \right) T(K_{k-1}; 1, b) \cdot T(K_{n-k}; a, b)$$

I. Pak in [13] also lists many other evaluations of $T(K_n; X, Y)$ with their combinatorial interpretations.

However, in [14] it is noted that $T_n(X, Y)$ does not satisfy a linear recurrence which is independent of n.

Proposition 1.3. (N.L. Biggs, R.M. Damerell and D.A. Sand)
There is no linear recurrence relation which computes the sequence $T_n(a, b)$ for fixed $a, b \in \mathbb{Z}$.

1.4. The matching polynomial

From [15] we know that for the defect matching polynomial (aka the acyclic polynomial)

$$\alpha(K_n; X) = He_n(X)$$

where $He_n(X)$ are the probabilist's Hermite polynomials for $n \in \mathbb{N}$. The proof of this is due to C. Heilmann and E. Lieb [16]. From [17, Equations 3.4 and 3.8] one can derive[1] that the polynomials $He_n(X)$ satisfy the modular recurrence relation

$$He_{n+m}(X) = He_n(X) \cdot He_m(X) = He_n(X) \cdot X^m \pmod{\mu}.$$

and with $He_0(X) = 1$ and one gets $He_m(X) = X^m$.

[1] Thanks to V. Rakita for checking this.

Proposition 1.4. (Carlitz, 1953)
For every $a \in \mathbb{N}$ the sequence $\alpha(K_n; a) = He_n(a) = a^n$ is ultimately period modulo μ.

Our Theorem 1.8 below shows the ultimate periodicity modulo μ for $M(K_n, a)$ without using the connection to the Hermite polynomials.

1.5. Computing $P_n(\bar{b})$ modulo an integer μ

Let $\mu \in \mathbb{N}$. We compute $P_n(\bar{b})$ modulo μ and observe:

(i) For every $b \in \mathbb{Z}$ and $\mu \in \mathbb{N}$ the sequence $In(K_n, b) = nb + 1$ is *ultimately periodic*.

(ii) For every $b \in \mathbb{Z}$ and $\mu \in \mathbb{N}$ the sequence $Cl(K_n, b) = (b + 1)^n$ is *ultimately periodic*.

(iii) The sequence $\chi(K_n, b)$ is ultimately constant, hence *ultimately periodic*.

We shall see that this is the case for a very large class of graph polynomials subject to a definability condition in MSOL.

Our main results are for the Tutte polynomial $T(G; X, Y)$, the bivariate matching polynomial $\bar{M}(G; X)$ and the trivariate edge elimination polynomial $\xi(G; X, Y, Z)$.

Theorem 1.5. For every $a, b, \mu \in \mathbb{N}^+$ with $a, b > 1$, $\gcd(a - 1, \mu) = 1$ and $\gcd(b - 1, \mu) = 1$, the sequence $T(K_n, a, b)$ is ultimately periodic modulo μ.

Similarly, we also get for the edge elimination polynomial $\xi(G; X, Y, Z)$, the subgraph counting polynomial $S(G; X, Y, Z)$, and the covered components polynomial $C(G; X, Y, Z)$:

Theorem 1.6. (i) For every $a, b, c, \mu \in \mathbb{N}^+$ the sequences $S(K_n, a, b, c)$ and $C(K_n, a, b, c)$ are ultimately periodic modulo μ for every μ.

(ii) Assume ab divides c.
Then $\xi(K_n, a, b, c)$ is ultimately periodic modulo μ for every μ.

Remark 1.7. For $\xi(K_n, a, b, c)$ one needs an additional condition like in Theorem 1.5 due to the negative exponents in the definition 2. of $\xi(G; , X, Y, Z)$.

Theorem 1.8. For every $a, b, \mu \in \mathbb{N}^+$ the sequence $\bar{M}(K_n, a, b)$ is ultimately periodic modulo μ.

Theorem 1.8 can be generalized:

Theorem 1.9. If P is a graph polynomial definable in GMSOL_{hgraph} without order, then for every $\mu \in \mathbb{N}$ and every $a, b \in \mathbb{Z}$ the sequence $P(K_n, a, b)$ is ultimately periodic modulo μ.

Remark 1.10. We note that Theorem 1.5 is not a special case of Theorem 1.9, as the Tutte polynomial seems not to be GMSOL_{hgraph}-definable without an order on the vertices.

The three theorems only assert that the sequences are ultimately periodic modulo μ, without any indication of the length of the periodicity or the initial segment before the periodicity starts.

1.6. Methods

The proofs use tools from logic and combinatorics. In particular

- *Definability in Monadic Second Order Logic* MSOL;
- *Definability* in the extension CMSOL of MSOL, where modular counting quantifiers are added;
- The *Specker-Blatter Theorem*, which gives a sufficient condition on when certain CMSOL-definable density functions are ultimately periodic if considered modulo μ.

We assume that our readers are familiar with MSOL and first explain the *Specker-Blatter Theorem*.

1.7. Related results

The particular sequence $T_n(1,b) = T(K_n;1,b)$ has been analyzed by A. P. Mani and R.J. Stones, [18], for $\mu = p^k$ where p is an odd prime and $k \in \mathbb{N}$.

Let $\phi(n)$ be the *Euler totient function* which is defined as the number of integers $a \in \{1, 2, ..., m\} = [m]$ such that $gcd(a, m) = 1$, cf. [19].

Proposition 1.11. (A.P. Mani and R.J. Stones)
Let p be a prime, and let k be a positive integer. For $b, n \in \mathbb{Z}$ such that $n \geq p^k$ and $b \neq 1 \mod p$, it holds that

$$T_n(1,b) = \mod p^k \begin{cases} b^{\frac{\phi(p^k)}{2}} C_{n-\phi(p^k}(b) & \text{if } p \geq 3 \text{ and } n > p \\ b^{\frac{\phi(p)}{2}} - 1 & \text{if } p \geq 3 \text{ and } n = p \\ 1 & \text{if } p = n = 2 \\ 2 & \text{if } p = k = 2 \text{ and } n = 4 \\ 0 & \text{otherwise} \end{cases}$$

This is much more informative than our Theorem 1.5 for the case $a = 1$ and $\mu = p^k$. In [18] they formulate also a conjecture for $T(K_n; a, b)$ for general a, but still for $\mu = p^k$.

Conjecture 1.12. (A.P. Mani and R.J. Stones)
Let p be an odd prime and let k be a positive integer.

(i) If $n \geq p^k$, $a, b \in \mathbb{Z}$, and $b \not\equiv 1 \mod p$, then modulo p we have

$$T(K_n; a, b) \equiv \mod p \begin{cases} b^{\frac{\phi(p)}{2}-1} & \text{if } p \geq 3, n = p \\ & \text{and } a = 1 \mod p \\ b^{\frac{\phi(p^k)}{2}} \cdot T(K_{n-\phi(p^k)}; a, b) & \text{otherwise} \end{cases}$$

(ii) If $n \geq p^k$, $a, b \in \mathbb{Z}$, and $b \equiv 1 \mod p$, then

$$T(K_n; a, b) \equiv \mod p^k \begin{cases} (n + a - 1)^{p^k} \cdot T(K_{n-p^k}; a, b) & n > p^k \\ (a-1)^{p^k - 1} & n = p^k \end{cases}$$

1.8. Outline of the paper

In Section 2 we present the Specker-Blatter Theorem. In Section 3 we prove Theorem 1.8, and in In Section 4 we prove Theorem 1.5 and 1.6. In Section 5 we prove Theorem 1.9. Finally, in Section 6 we present our conclusions and suggestions for further research.

2. The Specker-Blatter Theorem

The Specker-Blatter Theorem from 1980 is the first application of Logic to Combinatorial Counting. At the time the theorem was hardly noticed, mostly due to its unlucky placement for publication, [20, 3, 21]. An easily accessible place to find proofs and a survey of further developments is [22].

2.1. Counting graphs: The density function

A *labeled graph* G with n vertices will have $V(G) = \{0, 1, \dots, n-1\} = [n]$. There are $gr(n) = 2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}}$ many graphs with n vertices.

A graph property is a class of finite graphs closed under graph isomorphisms. For a graph property $\mathcal{P}$ denote by $\mathcal{P}^n$ the graphs with n vertices in $\mathcal{P}$, and by $d_{\mathcal{P}}(n) = \mid \mathcal{P}^n \mid$, the number of graphs G with $V(G) = [n]$ which are in $\mathcal{P}$. $d_{\mathcal{P}}(n)$ is called the *density function* of $\mathcal{P}$. The density function counts *labeled* graphs. Let G consist of the vertices $[n]$ and one single edge which is not a loop. There is, up to isomorphisms (disregarding the labels) one such graph, but there are $n(n-1)/2$ such labeled graphs.

A graph property $\mathcal{P}$ is hereditary if it is closed under induced subgraphs. $\mathcal{P}$ is monotone if it is closed under (not necessarily induced) subgraphs. If $\mathcal{P}$ is hereditary or monotone, the density function $d_{\mathcal{P}}(n)$ of $\mathcal{P}$ is also called the *speed* of $\mathcal{P}$, since it is an *ultimately monotone increasing function*, cf. [23, 24, 25]. Studying the possible growth rate of the speed of a graph property was initiated in 1994 by E. Scheinerman and J. Zito in [26]. E. Specker and C. Blatter already in 1980 studied under what conditions the density function of graph properties satisfy recurrence relations. The definition of the density function can be extended to relational structures of any vocabulary τ. However, we will consider only vocabularies τ without function symbols.

Examples 2.1. (i) If $\mathcal{P} = Graphs$ consists of all simple graphs,

$$d_{Graphs}(n) = 2^{\binom{n}{2}}$$

In the unlabeled case the function is rather complicated.

(ii) If $\mathcal{P} = LinOrd$ consists of all linear orders.

$$d_{LinOrd}(n) = n!$$

In the unlabeled case we have the constant function with value 1.

(iii) If $\mathcal{P} = SqGrids$ consists of all square grids,

$$d_{SqGrids}(n) = \begin{cases} \frac{n!}{4} & \text{if } n = m^2 \\ 0 & \text{else} \end{cases}$$

In the unlabeled case we have 1 instead of $n!$ in the above expression.

For some graph properties $\mathcal{P}$ the density functions satisfies a linear recurrence relation over $\mathbb{Z}$. However, this is not always the case.

Lemma 2.2. (Folklore)
Let $f : \mathbb{Z} \to \mathbb{Z}$ a function which satisfies a linear recurrence relation

$$f(n+1) = \sum_{i=0}^{k} a_i f(n-i)$$

over $\mathbb{Z}$. Then there is a constant $c \in \mathbb{Z}$ such that $f(n) \leq 2^{cn}$.

Corollary 2.3. For $\mathcal{C} \in \{Graphs, LinOrd, SqGrids\}$, $d_{\mathcal{C}}(n)$ does not satisfy a linear recurrence over $\mathbb{Z}$.

2.2. Modular counting

Let $\mu \in \mathbb{N}$.

Observation 2.4. For every $\mu \in \mathbb{N}$ and for large enough n we have $n! = 0 \pmod{\mu}$

Hence, for $n \geq N(\mu)$ we have

$$d_{LinOrd}(n+1) = d_{LinOrd}(n) \pmod{\mu}$$

and

$$d_{SqGrid}(n+1) = d_{SqGrid}(n) \pmod{\mu}$$

We say that a function $f(n)$ satisfies a *trivial modular recurrence* if for every μ there exists N_μ such that if $n > N_\mu$ then $f(n) \equiv 0 \pmod{\mu}$. Clearly, the two examples above satisfy trivial modular recurrences.

Observation 2.5. $f(n)$ satisfies a trivial modular recurrence iff there exist functions $g(n), h(n)$ with $g(n)$ tending to infinity such that $f(n) = g(n)! \cdot h(n)$.

In other words, trivial modular recurrences are always caused by some factor which is a factorial.

It is sometimes more intuitive to say of integer sequences with values in $[m]$ that they are ultimately periodic rather than to talk about modular linear recurrence.

Proposition 2.6. (Folklore)
Let a_n be an integer sequence.
a_n satisfies a linear recurrence relation modulo μ iff a_n is ultimately periodic modulo μ.

The following are two instructive examples. First, we look at the class of all graphs.

Example 2.7. The density function for all graphs is given by

$$d_{Graphs}(n+1) = 2^{\binom{n+1}{2}} = 2^{\binom{n}{2}} \cdot 2^n.$$

Therefore

$$d_{Graphs}(n+m) = d_{Graphs}(n) \cdot \prod_{i=0}^{m-1} 2^{n+i} = d_{Graphs}(n) \cdot 2^{nm} \cdot \prod_{i=0}^{m-1} 2^i$$

As $a^{p-1} = a \pmod{p}$ (Fermat's Little Theorem) we get with $a = 2^n$ and $m = p$ a prime

$$d_{Graphs}(n+p) = d_{Graphs}(n) \cdot \prod_{i=0}^{p-1} 2^i \pmod{p}$$

This is a non-trivial recurrence for $\mu = p$ a prime. It is also different for distinct primes p and p', In other words, the existence of the modular recurrence is non-uniform in p.

The second example is the class of graphs $EQ_2CLIQUE$ which consists of the graphs which are the disjoint unions of two equal-sized cliques.

Example 2.8. For density function $d_{EQ_2CLIQUE}(n)$ we have

$$d_{EQ_2CLIQUE}(n) = b_2(n) = \begin{cases} \frac{1}{2}\binom{2m}{m} & \text{for } n = 2m \\ 0 & \text{else} \end{cases}$$

The factor $\frac{1}{2}$ is there because we cannot distinguish the choice of the first clique from the choice of its complement.

The function $b_2(n)$ was studied by F. E. A. Lucas (1842-1891) in 1878, but not published at the time. It was found in his notes in the National Archive of France. A proof may be reconstructed from the hints in [19, Exercise 5.61].

Proposition 2.9. (Lucas, 1878)
For every n which is not a power of 2, we have $b_2(n) \equiv 0 \pmod{2}$, and for every n which is a power of 2 we have $b_2(n) \equiv 1 \pmod{2}$.
In particular, $b_2(n)$ is not ultimately periodic modulo 2.

We conclude that $d_{EQ_2CLIQUE}(n)$ is not ultimately periodic modulo 2.

Finally, here is an example, were the precise counting is known. It is originally due to Redfield, [27] and was rediscovered by R.C. Read and G. Polya, [28, 29, 30].

Example 2.10. Let $\mathcal{R}_d$ be the class of regular graphs of degree d and $d_{\mathcal{R}_d}(n)$ its density function.

It is not at all obvious that $d_{\mathcal{R}_d}(n)$ is ultimately periodic modulo μ, but it follows from the Specker-Blatter Theorem 2.12 below that, indeed, it is.

Proposition 2.11. (J.H. Redfield, 1927)
For $d = 3$ we have $d_{\mathcal{R}_3}(2n+1) = 0$ and

$$d_{\mathcal{R}_3}(2n) = \frac{(2n)!}{6^n}?\sum_{j,k}\frac{(-1)^j(6k-2j)!6^j}{(3k-j)!(2k-j)!(n-k)!}48^k\sum_i\frac{(-1)^i j!}{(j-2i)!i!}$$

An accessible proof can be found in [31, page 187].

2.3. MSOL-definable graph properties

We now look at the density functions of graph properties $\mathcal{P}$ which are definable in MSOL.

Theorem 2.12. (Specker-Blatter Theorem)
Let $\mathcal{P}$ be a graph property which is $MSOL$-definable and let $d_{\mathcal{P}}(n)$ be its density function.

- $d_{\mathcal{P}}(n)$ satisfies modular recurrence relations for each $\mu \in \mathbb{N}$, hence it is ultimately periodic modulo μ.
- This remains true for vocabularies τ with several binary edge relations and unary predicates on the vertices.

The Specker-Blatter Theorem does not hold if one allows quaternary relations in τ, [32, 33].

Theorem 2.13. (E. Fischer, 2003)
Let τ_0 consist of one quaternary relation. There is a class of $FOL(\tau_0)$-definable τ_0-structures $\mathcal{F}$ such that $d_{\mathcal{F}}$ is not ultimately periodic modulo 2.

The proof consists of a very clever encoding of $EQ_2CLIQUE$ using the quaternary relation.

The restriction to binary relations is not needed if the graph property $\mathcal{P}$ contains only graphs of bounded degree, see [34]. The same works for classes of τ-structures of bounded degree, where the degree is defined via the Gaifman graph of the structures.

The Gaifman graph of a τ-structure $\mathcal{A}$ is the (undirected, loop-free) graph $G_{\mathcal{A}}$ with vertex set A, the universe of $\mathcal{A}$ and an edge between two distinct vertices $a, binA$ iff there exists an $R \in \tau$ and a tuple $(a_1, .., a_r) \in R^A$ such that $a, b \in \{a_1, .., a_r\}$, cf. [35].

3. The matching polynomial of the complete graph

Here we prove

Theorem 1.8: For all $a, b, \mu \in \mathbb{N}$. $\bar{M}(K_n; a, b)$ is ultimately periodic modulo μ.
In particular $M(K_n; a)$ is ultimately periodic modulo μ.

Proof:
First we prove it for $M(K_n; a)$ and note first that $M(K_n; a)$ is of the form

$$M(K_n; a) = P_\psi(K_n, a) = \sum_{F \subseteq E(G): \psi(E,F)} a^{|F|}$$

where $\psi(E, F)$ says that E is the edge relation of K_n and $F \subseteq E$ is a set of independent edges of K_n. We interpret a^F as the set of functions $f : F \to [a]$. Each f induces a partition of F with

$$F_f(i) = \{e \in F : f(e) = i\}.$$

We have

$$a^{|F|} = |\{f : F \to [a]\}|.$$

Next we look at the density function of

$$g_a(n) = |\{U_1, \ldots U_a \subseteq [n]^2, F \subseteq [n]^2 : \phi_1(\bar{U}, F), \psi(E, F)\}|$$

where $\phi_1(\bar{U})$ says: $U_1, \ldots, U_a$ partition F and $\psi(E, F)$ says that E is the edge relation of K_n and $F \subseteq E$ is a set of independent edges of K_n. $g_a(n)$ encodes the computation of $M(K_n; a)$.

Claim 3.1.

$$g_a(n) = P_\psi(K_n, a) = \sum_{F \subseteq E(G):\psi(E,F)} a^{|F|} = M(K_n; a)$$

All the relation symbols of ϕ_1 and ψ are binary, therefore we can apply the Specker-Blatter Theorem and conclude that $g_a(n)$ is ultimately periodic modulo every $\mu \in \mathbb{N}$, and so is $M(K_n; a)$.

The proof for $\bar{M}(K_n; a, b)$ is similar.

$$\bar{M}(K_n; a, b) = \sum_{k=0}^{\lfloor n/2 \rfloor} (X)^k m_k(G) Y^{n-2k} = \sum_{F \subseteq E(G):\psi(E,F)} (a)^{|F|} m_k(G) b^{|dF|}$$

where dF is the set of vertices in $[n]$ not covered by F. If F has k edges, then dF has $n - 2k$ vertices. □

4. The Tutte polynomial of a complete graph

Now we prove

Theorem 1.5
For every $a, b, \mu \in \mathbb{N}^+$ with $a, b > 1$, $\gcd(a - 1, \mu) = 1$ and $\gcd(b - 1, \mu) = 1$, $T_n(a, b) = T(K_n, a, b)$ is ultimately periodic modulo μ.

Proof:
We first rewrite the Tutte polynomial as

$$T(G; X, Y) = \frac{1}{(Y-1)^{|V(G)|} \cdot (X-1)^{\kappa(G))}} \sum_{A \subseteq E(G)} (X-1)^{\kappa(A)} \cdot (Y-1)^{\kappa(A)+|A|}$$

and put $G = K_n$. Now $|V(K_n)| = n$ and $|E(K_n)| = \binom{n}{2}$, and $\kappa(K_n) = 1$.

$$T(K_n; X, Y) = \frac{1}{(Y-1)^n \cdot (X-1)} \sum_{A \subseteq E(G)} (X-1)^{\kappa(A)} \cdot (Y-1)^{\kappa(A)+|A|}.$$

Like in the proof Theorem 1.8 we interpret a^A as the set of functions $f : A \to [a]$. Each f induces a partition of A with

$$A_f(i) = \{e \in A : f(e) = i\}.$$

We have

$$a^{|A|} = |\{f : A \to [a]\}|.$$

In the case of $a^{\kappa(A)}$ the sets $A_f(i)$ have to be A-closed, in order to partition the connected components of the spanning induced by the set of edges of A. Next look at the function

$$f_{a,b}(n) = (b-1)^n \cdot (a-1) \cdot T(K_n, a, b) = \sum_{A \subseteq E(G)} (a-1)^{\kappa(A)} \cdot (b-1)^{\kappa(A)} \cdot (b-1)^{|A|}$$

as a MSOL-definable density function:

$$f_{a,b}(n) = |\{\bar{U}, \bar{R}, \bar{S} \subseteq [n] : \phi_1(\bar{U}, A), \phi_2(\bar{R}, A), \phi_3(\bar{S}), A \subseteq E\}|$$

where for $\bar{U} = (U_1, \ldots, U_{a-1})$ and $\bar{R} = (R_1, \ldots, R_{b-1})$ are unary relations and $\bar{S} = (S_1, \ldots, S_{b-1})$ are binary, and

(i) each $U_1, \ldots, U_{a-1}, R_1, \ldots, R_{b-1}, S_1, \ldots, S_{b-1} \subseteq [n]$;

(ii) $\phi_1(\bar{U}, A)$ says: $U_1, \ldots, U_{a-1}$ partitions the connected components of $G[A]$, and each U_i is a disjoint union of connected components of the graph $G = ([n], A)$;

(iii) $\phi_2(\bar{R}, A)$ says: $R_1, \ldots, R_{b-1}$ also partitions the connected components of $G = ([n], A)$,

(iv) and $\phi_3(\bar{S})$ says: $S_1, \ldots, S_{b-1}$ partitions $A \subseteq [n]^2$.

All the formulas $\phi_1(\bar{U}, A), \phi_2(\bar{R}, A), \phi_3(\bar{S})$ are in MSOL and contain only unary and binary relation symbols. It follows by Theorem 2.12 that $f_{a,b}(n)$ is ultimately periodic modulo every $\mu \in \mathbb{N}$.

Claim 4.1.

$$f_{a,b}(n) = \sum_{A \subseteq E(G)} (a-1)^{\kappa(A))} \cdot (b-1)^{\kappa(A)} \cdot (b-1)^{|A|}$$

Now we need a lemma.

Lemma 4.2. Let $d_1(n), d_2(n)$ be integer functions.

(i) Let $c, \mu \in \mathbb{N}^+$.
Assume $c \cdot d_1(n)$ is ultimately periodic modulo μ and $\gcd(c, \mu) = 1$.
Then $d_1(n)$ is ultimately periodic modulo μ.

(ii) Let $t, \mu \in \mathbb{N}^+$ with $\gcd(t, \mu) = 1$.
Assume that $t^n \cdot d_2(n)$ is ultimately periodic modulo μ.
Then $d_2(n)$ is ultimately periodic modulo μ.

Proof:
(i) is left to the reader.
(ii): Since t and μ are relatively prime, t has a multiplicative inverse g modulo μ: $g \cdot t \equiv 1(\mod \mu)$. The product of two ultimately periodic sequence is ultimately periodic, hence this is true for the product of g and $t^n \cdot d_2(n)$. But we have

$$g^n \cdot t^n \cdot d_2(n) \equiv d_2(n)(\mod \mu),$$

hence $d_2(n)$ is ultimately periodic. □

To complete the proof of Theorem 1.5 we note that $\kappa(E) = 1$ and $|V| = n$, and we put:

$$d_1(n) = T(K_n, a, b) \text{ and } d_2(n) = (b-1) \cdot T(K_n, a, b)$$

We have $f_{a,b}(n) = (a-1)^n \cdot d_2(n)$ is ultimately periodic modulo μ for $\gcd((a-1), \mu) = 1$. By the Lemma 4.2(ii) we have that $d_2(n)$ ultimately periodic modulo μ for $\gcd((a-1), \mu) = 1$. By the Lemma 4.2(i) we have that $d_1(n)$ ultimately periodic modulo μ for $\gcd((b-1), \mu) = 1$. □

The proofs of Theorem 1.6 for $S(G; a, b, c)$ and $C(G; a, b, c)$ are similar as for $f_{a,b}(n)$.
For $\xi(G; a, b, c)$ we note that $\xi((G; X, Y, Z) = C(G; X, Y, \frac{Z}{XY} + 1)$ from Proposition 1.1.
To be able to use now $C(G; a, b, \frac{c}{ab} + 1)$ one requires that ab divides c so that $\frac{c}{ab}$ is a non-negative integer.

5. Theorem 1.9 and its limitations

The logic CMSOL is the extension of MSOL by adding *modular counting quantifiers*. Let $\phi(x)$ be a formula with free variable x. A modular counting quantifier $C_{\mu,k}x\phi(x)$ says that there are, modulo μ, k-many elements satisfying $\phi(x)$. Using Ehrenfeucht-Fraïssé games for MSOL one can show that $C_{\mu,k}x\phi(x)$ is not expressible in MSOL. However, for $\phi(x)$ a formula in SOL the formula $C_{\mu,k}x\phi(x)$ it is expressible in SOL. We say that there is an equivalence relation E on the set defined by $\phi(x)$ which has exactly one equivalence class of size k and all the other non-empty equivalence classes are of size μ. There are other ways of defining $C_{\mu,k}x\phi(x)$ in SOL. Instead of the existence of one binary relation we can also assert that there are μ disjoint subsets of equal size of the set defined by $\phi(x)$ and the complement of their union has size k. But then we need the existence of a binary relation which expresses that the unary predicates are of equal size.

CMSOL is like MSOL but modular counting quantifiers are allowed. The syntax of CMSOL is obtained from the syntax of MSOL by allowing also quantification with $C_{\mu,k}x$. The meaning function of MSOL is then naturally extended to CMSOL. CGMSOL is defined analogously by adding modular counting quantifiers to GMSOL.

It was shown in [34, 22] that the Specker-Blatter Theorem also holds for CMSOL for relational structures with relations of arity at most two, and for CGMSOL graphs.

5.1. CMSOL-definable graph polynomials

We now discuss how Theorems 1.5 and 1.8 can be extended.

A univariate graph polynomial of the form

$$P_\phi(G;X) = \sum_{A \subseteq V(G):\phi(E,A)} X^{|A|}$$

is CMSOL definable if $\phi(A, E)$ is a CMSOL-formula in the language of graphs with an additional predicate for A.

A univariate graph polynomial of the form

$$P_\psi(G;X) = \sum_{F \subseteq E(G):\psi(E,F)} X^{|F|}$$

is CGMSOL definable if $\psi(F, E)$ is a CGMSOL-formula in the language of graphs with an additional predicate for A or for $F \subseteq E$. The independence polynomial and the clique polynomial are of the first form. The matching polynomials α and M are of the second form.

Analyzing the proof of Theorem 1.8 immediate gives Theorem 1.9:

> If P is a graph polynomial definable in CGMSOL_{graph} without order, then for every $\mu \in \mathbb{N}$ and every $a, b \in \mathbb{Z}$
> the sequence $P(K_n, a, b)$ is ultimately periodic modulo μ.

The proof of Theorem 1.5 also works for the graph polynomials listed in Theorem 1.6. Unfortunately, we have not found interesting applications of Theorem 1.9. In many cases $P(K_n, a, b)$ can be shown directly to satisfy linear recurrence relations over $\mathbb{Z}$.

Examples 5.1. (i) If $\mathcal{P}$ is a graph property which does not contain any complete graph and is definable in CMSOL for graphs and $\phi_{\mathcal{P}}(A, E)$ says that A induces a graph $G[A] \in \mathcal{P}$ then

$$P_{\phi_{\mathcal{P}}}(G;X) = \sum_{A \subseteq V(G):\phi_{\mathcal{P}}(E,A)} X^{|A|}$$

trivializes for $P_{\phi_{\mathcal{P}}}(K_n; X)$.

(ii) If instead, $\phi'_{\mathcal{P}}(A, E')$ says that (A, E') is a subgraph $\mathcal{P}$ $P_{\phi'_{\mathcal{P}}}(K_n; X)$ may be non-trivial.

(iii) The domination polynomial $D(G; X)$ is obtained by taking $\phi_{dom}(A, E)$ which says that A is a dominating set for E. In [36, 37] it is shown that $D(K_{n+1}; X) = D(K_n; X)(X + 1) + X$. For $X = a$ this is a linear recurrence relation, hence Theorem 1.9 gives nothing new.

(iv) The univariate interlace polynomial $q(G; X)$ from [38] is CMSOL-definable, as shown in [39]. But $q(K_n; X) = 2^{n-1}X$, hence $q(K_{n+1}; X) = 2Xq(K_n; X)$. Again Theorem 1.9 gives nothing new.

Problem 5.2. Find more graph polynomials $P(G; \bar{a})$ where for $\bar{a} \in \mathbb{N}^k, \mu \in \mathbb{N}$ the sequence $P(K_n; \bar{a})$ is not obviously ultimately periodic modulo μ.

Problem 5.3. Find interesting cases of graph polynomials where Theorem 1.9 gives a non-trivial result.

We have seen that Theorem 1.9 also holds for multivariate polynomials. The Tutte polynomial is not of this form, because of the term $(X-1)^{\kappa(S)}$. It is of this form in the language of ordered graphs. However, the Specker-Blatter Theorem formulated for ordered structures trivializes, because the ordering adds a factor of the form $n!$, hence the sequence satisfies a trivial modular recurrence.

5.2. Why complete graphs?

Complete labeled graphs on n vertices are definable in FOL in the empty vocabulary and are unique, not only up to isomorphisms. The same is true about the empty graph.

Proposition 5.4. Assume $\phi(\bar{x}, \bar{y})$ is a formula of CMSOL over the empty vocabulary which defines a unique edge relation E_ϕ on k-tuples of a set $[n]$. Then the graph $G = ([n]^k, E_\phi)$ is either a complete graph or the empty graph.

The way we used the Specker-Blatter Theorem in the proof of Theorem 1.9 did require that the function

$$g_a(n) = |\{U_1, \dots U_a \subseteq [n], F \subseteq [n]^2 : \phi_1(\bar{U}, F), \psi(F)\}|$$

evaluates the matching polynomial at a for $G = K_n$. If instead of $G = K_n$ we use some $G = ([n]^k, E_\phi)$ with $k = 1$ we can modify $g_a(n)$ and still apply the Specker-Blatter Theorem to it, but the modified version $g_{a,\phi}(n)$ will not evaluate the matching polynomial anymore. Let $sp(\phi, n)$ be the number of labeled graphs of the form $G = ([n], E_\phi)$. Even if all the graphs $G_n = ([n], E_\phi)$ are isomorphic, $g_{a,\phi}(n)$ computes

$$g_{a,\phi}(n) = M(G_n; a) \cdot sp(\phi, n)$$

which will be ultimately periodic modulo μ. However, this does not suffice to conclude that $M(G_n; a)$ is ultimately periodic modulo μ.

Facit: Our proofs of Theorems 1.5, 1.9 and 1.8 only work for $G = K_n$ or $G = \bar{K_n}$.

6. Conclusions

Inspired by the paper of A.P. Mani and R.J. Stones [18] we have examined modular recurrences for the matching and the Tutte polynomial of a complete graph. We have noted that the existence of modular recurrence relation follows from the Specker-Blatter Theorem, without explicitly describing the exact modular recurrence. The conjectures of A.P. Mani and R.J. Stones are more ambitious, as they give a precise statement about how these modular recurrences look in the case of the Tutte polynomial. We also noted that our approach via the Specker-Blatter Theorem 2.12 works for other graph polynomials definable in variants of MSOL. Most strikingly, it works for the trivariate edge elimination polynomial $\xi(G; X, Y, Z)$, which is the most general edge elimination polynomial, and generalizes both the matching polynomials and the Tutte polynomial.

Our result suggests that the use of the Specker-Batter Theorem to CMSOL-definable (CGMSOL-definable) graph polynomials should be further investigated. In [40, 22] a logic-free version of the Specker-Batter Theorem is discussed, where instead of the graph property $\mathcal{P}$ being definable in CMSOL one only requires that $\mathcal{P}$ has finite Specker rank. Here we note that there uncountably many graph properties of finite Specker rank, but there only countably many CMSOL-definable graph properties. CMSOL-definable graph properties have always finite Specker rank, but the upper bound on the Specker rank which follows from CMSOL-definability is very often exponentially bigger than the true Specker rank. On the other hand the parameters which characterize the ultimate periodicity depend only polynomially on the Specker rank, hence the structure of the modular recurrences can be more precisely described using the Specker rank.

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