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Right Buchberger Algorithm over Bijective Skew PBW Extensions

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Abstract. In this paper we present a right version of the Buchberger algorithm over skew Poincaré-Birkhoff-Witt extensions (skew PBW extensions for short) defined by Gallego and Lezama [5]. This algorithm is an adaptation of the left case given in [3]. In particular, we developed a right version of the division algorithm and from this we built the right Gröbner bases theory over bijective skew PBW extensions. The algorithms were implemented in the SPBWE library developed in Maple, this paper includes an application of these to the membership problem. The theory developed here is fundamental to complete the `SPBWE` library and thus be able to implement various homological applications that arise as result of obtaining the right Gröbner bases over skew PBW extensions.

Keywords: Non-commutative computational algebra, skew PBW extensions, Buchberger algorithm, Gröbner bases, `SPBWE` library, Maple.
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1. Skew PBW extensions

In this section we introduce the *bijective skew PBW extensions* whose are the fundamental topic in this paper. Skew PBW extensions include well known classes of Ore algebras, operator algebras and also a lot of quantum rings and algebras. The skew PBW extensions have been extensively studied,

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see [3], these are being implemented in the SPBWE library developed in Maple, see [1] and [4]. The main purpose of this paper is to present the theory needed to develop the right Gröbner theory and generate their respective algorithms implemented in Maple through SPBWE library., i.e., implementing the division algorithm and Buchberger algorithm in the right case, similar works have been implemented, see Fajardo [1] and [4], Fajardo-Lezama [2], Gasiorek et al., [7], Simson-Wojewódzki [11], Simson [12], [13] and [14].

Definition 1.1. Let R and A be rings, we say that A is a skew PBW extension of R (also called σ-PBW extension), if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left R-free module with basis

$$\mathrm{Mon}(A) := \mathrm{Mon}\{x_1, \dots, x_n\} = \{x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

(iii) For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1}$$

(iv) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{2}$$

In this case the extension is denoted by $A = \sigma(R)\langle x_1, \dots, x_n \rangle$, and R is called the ring of coefficients of the extension A.

Remark 1.2. Each element $f \in A - \{0\}$ has a unique representation in the form $f = c_1 X_1 + \cdots + c_t X_t$, with $c_i \in R - \{0\}$ and $X_i \in \mathrm{Mon}(A)$, $1 \leq i \leq t$.

The following proposition (see [3], Proposition 1.1.3) justifies the notation given in Definition 1.1 of the skew PBW extensions.

Proposition 1.3. Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a skew PBW extension of R. Then, for $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for every $r \in R$.

Definition 1.4. Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a skew PBW extension of R. A is called bijective if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i, j \leq n$.

Definition 1.5. Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a skew PBW extension of R, with endomorphisms $\sigma_i, 1 \leq i \leq n$. We will use the following notation.

(i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and if A is bijective $\sigma^{-\alpha} := \sigma_n^{-\alpha_n} \cdots \sigma_1^{-\alpha_1}$.], Moreover, if $\beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$, then $\alpha+\beta := (\alpha_1+\beta_1, ..., \alpha_n+\beta_n)$.

(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) Let $0 \neq f \in A$; if $f = c_1X_1 + \cdots + c_tX_t$, with $X_i \in \mathrm{Mon}(A)$ and $c_i \in R - \{0\}$, then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.

The following characterization of skew PBW extensions was given in [6].

Theorem 1.6. Let A be a ring of a left polynomial type over R with respect to $\{x_1, \ldots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions hold:

(a) For every $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$ there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R - \{0\}$ and $p_{\alpha,r} \in A$ such that

$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{3}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if r is left invertible, then r_α is left invertible.

(b) For every $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{4}$$

where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

Remark 1.7.

(i) Let $\theta, \gamma, \beta \in \mathbb{N}^n$ and $c \in R$. Then we have the following identities:

$$\sigma^\theta(c_{\gamma,\beta}) c_{\theta,\gamma+\beta} = c_{\theta,\gamma} c_{\theta+\gamma,\beta}, \tag{5}$$

$$\sigma^\theta(\sigma^\gamma(c)) c_{\theta,\gamma} = c_{\theta,\gamma} \sigma^{\theta+\gamma}(c). \tag{6}$$

(ii) One concludes from Theorem 1.6 that if A is bijective, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}^n$.

2. Orders on $\mathrm{Mon}(A^m)$

In this section we will compile some results taken from [3] that will be used in the theory of reduction and the theory of Gröbner for the right case.

Definition 2.1. (a) We define in $\mathrm{Mon}(A)$ the *deglex order* by the formulas:

$$x^\alpha \succeq x^\beta \iff \begin{cases} x^\alpha = x^\beta \\ \text{or} \\ x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta| \\ \text{or} \\ x^\alpha \neq x^\beta, |\alpha| = |\beta| \text{ but } \exists\, i \text{ with } \quad \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases}$$

It is clear that the deglex order is a total order.

(b) If $x^\alpha \succeq x^\beta$ and $x^\alpha \neq x^\beta$ we write $x^\alpha \succ x^\beta$.

(c) Assume that the element $f \in A \setminus \{0\}$ has the unique form $f = c_1x^{\alpha_1} + \cdots + c_tx^{\alpha_t}$, with $c_i \in R - \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$. We define the monomial x^{α_1} to be the *leader monomial* of f and we write $\mathrm{lm}(f) := x^{\alpha_1}$; c_1 is the *leader coefficient* of f, $\mathrm{lc}(f) := c_1$ and $c_1x^{\alpha_1}$ is the *leader term* of f denoted by $\mathrm{lt}(f) := c_1x^{\alpha_1}$. If $f = 0$, we define $\mathrm{lm}(0) := 0, \mathrm{lc}(0) := 0, \mathrm{lt}(0) := 0$ and we set $X \succ 0$ for any $X \in \mathrm{Mon}(A)$.

2.1. Monomial orders in skew PBW extensions

Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of R and let $\succeq$ be a total order defined on $\mathrm{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$ we will write $x^\alpha \succ x^\beta$. $x^\beta \preceq x^\alpha$ means that $x^\alpha \succeq x^\beta$. Let $f \neq 0$ be a polynomial of A. If

$$f = c_1X_1 + \cdots + c_tX_t,$$

with $c_i \in R - \{0\}$ and $X_1 \succ \cdots \succ X_t$ are the monomials of f, then $\mathrm{lm}(f) := X_1$ is the *leading monomial* of f, $\mathrm{lc}(f) := c_1$ is the *leading coefficient* of f and $\mathrm{lt}(f) := c_1X_1$ is the *leading term* of f. If $f = 0$, we define $\mathrm{lm}(0) := 0, \mathrm{lc}(0) := 0, \mathrm{lt}(0) := 0$, and we set $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. Thus, we extend $\succeq$ to $\mathrm{Mon}(A) \cup \{0\}$.

Definition 2.2. Let $\succeq$ be a total order on $\mathrm{Mon}(A)$, it is said that $\succeq$ is a monomial order on $\mathrm{Mon}(A)$ if the following conditions hold:

(i) For every $x^\beta, x^\alpha, x^\gamma, x^\lambda \in \mathrm{Mon}(A)$

$$x^\beta \succeq x^\alpha \Rightarrow \mathrm{lm}(x^\gamma x^\beta x^\lambda) \succeq \mathrm{lm}(x^\gamma x^\alpha x^\lambda).$$

(ii) $x^\alpha \succeq 1$, for every $x^\alpha \in \mathrm{Mon}(A)$.

(iii) $\succeq$ is degree compatible, i.e., $|\beta| \geq |\alpha| \Rightarrow x^\beta \succeq x^\alpha$.

Monomial orders are also called *admissible orders*. It is worth noting that every monomial order on $\mathrm{Mon}(A)$ is a well order. Thus, there are not infinite decreasing chains in $\mathrm{Mon}(A)$. From now on we will assume that $\mathrm{Mon}(A)$ is endowed with some monomial order.

Definition 2.3. Let $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, we say that x^α divides x^β, denoted by $x^\alpha | x^\beta$, if there exists $x^\gamma, x^\lambda \in \mathrm{Mon}(A)$ such that $x^\beta = \mathrm{lm}(x^\gamma x^\alpha x^\lambda)$. We will say also that any monomial $x^\alpha \in \mathrm{Mon}(A)$ divides the polynomial zero.

The condition (iii) of Definition 2.2 is needed in the proof of the following proposition (see [3], Proposition 13.1.4), and this one will be used in right Division Algorithm (Theorem 3.8).

Proposition 2.4. Let A be a bijective skew PBW extension and $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ and $f, g \in A - \{0\}$. Then,

(a) $\operatorname{lm}(x^\alpha g) = \operatorname{lm}(x^\alpha \operatorname{lm}(g)) = x^{\alpha+\exp(\operatorname{lm}(g))}$, i.e., $\exp(\operatorname{lm}(x^\alpha g)) = \alpha + \exp(\operatorname{lm}(g)$. In particular,

$$\operatorname{lm}(\operatorname{lm}(f)\operatorname{lm}(g)) = x^{\exp(\operatorname{lm}(f))+\exp(\operatorname{lm}(g))}, \text{ i.e.,}$$
$$\exp(\operatorname{lm}(\operatorname{lm}(f)\operatorname{lm}(g))) = \exp(\operatorname{lm}(f)) + \exp(\operatorname{lm}(g))$$

and

$$\operatorname{lm}(x^\alpha x^\beta) = x^{\alpha+\beta}, \ i.e., \ \exp(\operatorname{lm}(x^\alpha x^\beta)) = \alpha + \beta. \tag{7}$$

(b) The following conditions are equivalent:

(i) $x^\alpha | x^\beta$.

(ii) There exists a unique $x^\theta \in \operatorname{Mon}(A)$ such that $x^\beta = \operatorname{lm}(x^\theta x^\alpha) = x^{\theta+\alpha}$ and hence $\beta = \theta + \alpha$.

(iii) There exists a unique $x^\theta \in \operatorname{Mon}(A)$ such that $x^\beta = \operatorname{lm}(x^\alpha x^\theta) = x^{\alpha+\theta}$ and hence $\beta = \alpha + \theta$.

(iv) $\beta_i \geq \alpha_i$ for $1 \leq i \leq n$, with $\beta := (\beta_1, \ldots, \beta_n)$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$.

Proof:
Apply [3; Proposition 13.1.4]. □

Remark 2.5.

(i) Let $\succeq$ be the monomial order on $\operatorname{Mon}(A)$. If there exists $f = x^{\gamma_1}c_1 + \cdots + x^{\gamma_t}c_t \in A - \{0\}$ such that $x^\beta = x^\alpha f$ or $x^\beta = f x^\alpha$, then by Proposition 2.4, $x^\beta = x^{\alpha+\gamma_1}$, i.e., $x^\alpha | x^\beta$.

(ii) We note that there exists a least common multiple of two elements of $\operatorname{Mon}(A)$: in fact, let $x^\alpha, x^\beta \in \operatorname{Mon}(A)$, then $\operatorname{lcm}(x^\alpha, x^\beta) = x^\gamma \in \operatorname{Mon}(A)$, where $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i := \max\{\alpha_i, \beta_i\}$ for each $1 \leq i \leq n$.

2.2. Monomial orders on $\operatorname{Mon}(A^m)$

We will often represent the elements of A^m also as row vectors, in case when if this does not causa confusion. We recall that the canonical basis of the free A-module A^m is

$$\boldsymbol{e}_1 = (1, 0, \ldots, 0), \boldsymbol{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \boldsymbol{e}_m = (0, 0, \ldots, 1).$$

Definition 2.6. A monomial in A^m is a vector $\boldsymbol{X} = X\boldsymbol{e}_i$, where $X = x^\alpha \in \operatorname{Mon}(A)$ and $1 \leq i \leq m$, i.e.,

$$\boldsymbol{X} = X\boldsymbol{e}_i = (0, \ldots, X, \ldots, 0),$$

where X is in the i-th position, named the index of $\boldsymbol{X}$, $\operatorname{ind}(\boldsymbol{X}) := i$. A term is a vector $c\boldsymbol{X}$, where $c \in R$. The set of monomials of A^m will be denoted by $\operatorname{Mon}(A^m)$. Let $\boldsymbol{Y} = Y\boldsymbol{e}_j \in \operatorname{Mon}(A^m)$, we say that $\boldsymbol{X}$ divides $\boldsymbol{Y}$ if $i = j$ and X divides Y. We will say that any monomial $\boldsymbol{X} \in \operatorname{Mon}(A^m)$ divides the null vector $\boldsymbol{0}$. The least common multiple of $\boldsymbol{X}$ and $\boldsymbol{Y}$, denoted by $\operatorname{lcm}(\boldsymbol{X}, \boldsymbol{Y})$, is $\boldsymbol{0}$ if $i \neq j$, and $U\boldsymbol{e}_i$, where $U = \operatorname{lcm}(X, Y)$, if $i = j$. Finally, we define

$$\exp(\boldsymbol{X}) := \exp(X) = \alpha \text{ and } \deg(\boldsymbol{X}) := \deg(X) = |\alpha|.$$

We now define monomial orders on $\mathrm{Mon}(A^m)$.

Definition 2.7. A monomial order on $\mathrm{Mon}(A^m)$ is a total order $\succeq$ satisfying the following three conditions:

(i) $\mathrm{lm}(x^\beta x^\alpha)\boldsymbol{e}_i \succeq x^\alpha \boldsymbol{e}_i$, for every monomial $\boldsymbol{X} = x^\alpha \boldsymbol{e}_i \in \mathrm{Mon}(A^m)$ and any monomial x^β in $\mathrm{Mon}(A)$.

(ii) If $\boldsymbol{Y} = x^\beta \boldsymbol{e}_j \succeq \boldsymbol{X} = x^\alpha \boldsymbol{e}_i$, then $\mathrm{lm}(x^\gamma x^\beta)\boldsymbol{e}_j \succeq \mathrm{lm}(x^\gamma x^\alpha)\boldsymbol{e}_i$ for every monomial $x^\gamma \in \mathrm{Mon}(A)$.

(iii) $\succeq$ is degree compatible, i.e., $\deg(\boldsymbol{X}) \geq \deg(\boldsymbol{Y}) \Rightarrow \boldsymbol{X} \succeq \boldsymbol{Y}$.

If $\boldsymbol{X} \succeq \boldsymbol{Y}$ and $\boldsymbol{X} \neq \boldsymbol{Y}$ we write $\boldsymbol{X} \succ \boldsymbol{Y}$. $\boldsymbol{Y} \preceq \boldsymbol{X}$ means that $\boldsymbol{X} \succeq \boldsymbol{Y}$.

Definition 2.7 implies that every monomial order on $\mathrm{Mon}(A^m)$ is a well order. Next we give a monomial order $\succeq$ on $\mathrm{Mon}(A)$, we can define two natural orders on $\mathrm{Mon}(A^m)$.

Definition 2.8. Let $\boldsymbol{X} = X\boldsymbol{e}_i$ and $\boldsymbol{Y} = Y\boldsymbol{e}_j \in \mathrm{Mon}(A^m)$.

(i) The TOP (term over position) order is defined by $$\boldsymbol{X} \succeq \boldsymbol{Y} \Longleftrightarrow \begin{cases} X \succeq Y \\ \text{or} \\ X = Y \text{and} \quad i > j. \end{cases}$$

(ii) The TOPREV order is defined by $$\boldsymbol{X} \succeq \boldsymbol{Y} \Longleftrightarrow \begin{cases} X \succeq Y \\ \text{or} \\ X = Y \text{and} \quad i < j. \end{cases}$$

Remark 2.9.

(i) Note that with TOP we have $\boldsymbol{e}_m \succ \boldsymbol{e}_{m-1} \succ \cdots \succ \boldsymbol{e}_1$ and $\boldsymbol{e}_1 \succ \boldsymbol{e}_2 \succ \cdots \succ \boldsymbol{e}_m$ for TOPREV.

(ii) The POT (position over term) and POTREV orders defined in [10] and [8] for modules over classical polynomial commutative rings are not degree compatible.

We fix a monomial order on $\mathrm{Mon}(A)$ and a non-zero vector $\boldsymbol{f} \in A^m$. Then we write $\boldsymbol{f}$ as a sum of terms in the following form
$$\boldsymbol{f} = c_1\boldsymbol{X}_1 + \cdots + c_t\boldsymbol{X}_t,$$
where $c_1, \ldots, c_t \in R - 0$ and $\boldsymbol{X}_1 \succ \boldsymbol{X}_2 \succ \cdots \succ \boldsymbol{X}_t$ are monomials of $\mathrm{Mon}(A^m)$.

Definition 2.10. Let $\boldsymbol{f} := c_1\boldsymbol{X}_1 + \cdots + c_t\boldsymbol{X}_t \in A^m$ where $c_1, \ldots, c_t \in R - 0$, $\boldsymbol{X}_1 \succ \boldsymbol{X}_2 \succ \cdots \succ \boldsymbol{X}_t$ monomials of $\mathrm{Mon}(A^m)$ and $\boldsymbol{X}_i := x^{\gamma_i}\boldsymbol{e}_{j_i}$ with $\gamma_i \in \mathbb{N}^n$. Given $g \in A$, we define
$$\boldsymbol{f}g := c_1 x^{\gamma_1} g \boldsymbol{e}_{j_1} + \cdots + c_t x^{\gamma_t} g \boldsymbol{e}_{j_t}$$
and we view fg as an element of A^m.

Remark 2.11. In the notation of Definition 2.10, we have $\exp(\mathrm{lm}(\boldsymbol{f}x^{\alpha})) = \exp(\mathrm{lm}(\boldsymbol{f})) + \alpha$. In fact, as $\succ$ is monomial order on $\mathrm{Mon}(A^m)$, then $\mathrm{lm}(x^{\gamma_1}x^{\alpha})\boldsymbol{e}_{j_1} \succ \mathrm{lm}(x^{\gamma_k}x^{\alpha})\boldsymbol{e}_{j_k}$ for each $2 \leq k \leq t$, thus, $\mathrm{lm}(\boldsymbol{f}x^{\alpha}) = \mathrm{lm}(x^{\gamma_1}x^{\alpha})\boldsymbol{e}_{j_1}$ so, $\exp(\mathrm{lm}(\boldsymbol{f}x^{\alpha})) = \gamma_1 + \alpha = \exp(\mathrm{lm}(\boldsymbol{f})) + \alpha$. Hence, $\mathrm{lc}(\boldsymbol{f}x^{\alpha}) = c_1c_{\gamma_1,\alpha} = \mathrm{lc}(\boldsymbol{f})c_{\gamma_1,\alpha}$.

Definition 2.12. Under the notation introduced earlier, say that:

(i) $\mathrm{lt}(\boldsymbol{f}) := c_1\boldsymbol{X}_1$ is the leading term of $\boldsymbol{f}$,

(ii) $\mathrm{lc}(\boldsymbol{f}) := c_1$ is the leading coefficient of $\boldsymbol{f}$,

(iii) $\mathrm{lm}(\boldsymbol{f}) := \boldsymbol{X}_1$ is the leading monomial of $\boldsymbol{f}$.

For $\mathbf{f} = \boldsymbol{0}$ we define $\mathrm{lm}(\boldsymbol{0}) = \boldsymbol{0}, \mathrm{lc}(\boldsymbol{0}) = 0, \mathrm{lt}(\boldsymbol{0}) = \boldsymbol{0}$, and if $\succeq$ is a monomial order on $\mathrm{Mon}(A^m)$, then we define $\boldsymbol{X} \succ \boldsymbol{0}$ for any $\boldsymbol{X} \in \mathrm{Mon}(A^m)$. So, we extend $\succeq$ to $\mathrm{Mon}(A^m)\bigcup\{\boldsymbol{0}\}$.

3. Right reduction in A^m

In this section we present the fundamental topics of reduction theory for right submodules of the free A-module A^m when A is a bijective skew PBW extension. This theory was studied in the bijective general case for left modules. Here we adapt the ideas used in [3].

Throughout are assume that R satisfies some natural computational conditions.

Definition 3.1. A ring R is right Gröbner soluble (RGS) if the following conditions hold:

(i) R is right Noetherian.

(ii) Given $a, r_1, \ldots, r_m \in R$ there exists an algorithm which decides whether a is in the right ideal $r_1R + \cdots + r_mR$, and if so, find $b_1, \ldots, b_m \in R$ such that $a = r_1b_1 + \cdots + r_mb_m$.

(iii) Given $r_1, \ldots, r_m \in R$ there exists an algorithm which finds a finite set of generators of the right R-module

$$\mathrm{Syz}_R^r[r_1 \ \cdots \ r_m] := \{(b_1, \ldots, b_m) \in R^m | r_1b_1 + \cdots + r_mb_m = 0\}.$$

Remark 3.2. The three conditions (i) - (iii)imposed on R are needed in order to guarantee a right Gröbner theory in the rings of coefficients, in particular, to have an effective solution of the membership problem in R (see (ii) in Definition 3.3 below). From now on in this paper we will assume that $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a skew PBW extension of R, where R is a RGS ring and $\mathrm{Mon}(A)$ is endowed with some monomial order.

The reduction process in A^m is defined as follows.

Definition 3.3. Let F be a finite set of non-zero vectors of A^m, and let $\boldsymbol{f}, \boldsymbol{h} \in A^m$, we say that $\boldsymbol{f}$ reduces to $\boldsymbol{h}$ by F in one step, denoted $\boldsymbol{f} \xrightarrow{F} \boldsymbol{h}$, if there exist elements $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t \in F$ and $r_1, \ldots, r_t \in R$ such that

(i) $\mathrm{lm}(\boldsymbol{f}_i) | \mathrm{lm}(\boldsymbol{f})$, $1 \leq i \leq t$, i.e., $\mathrm{ind}(\mathrm{lm}(\boldsymbol{f}_i)) = \mathrm{ind}(\mathrm{lm}(\boldsymbol{f}))$ and there exists $x^{\alpha_i} \in \mathrm{Mon}(A)$ such that $\beta_i + \alpha_i = \exp(\mathrm{lm}(\boldsymbol{f}))$ with $\beta_i := \exp(\mathrm{lm}(\boldsymbol{f}_i))$.

(ii) $\text{lc}(\boldsymbol{f}) = \text{lc}(\boldsymbol{f}_1)\sigma^{\beta_1}(r_1)c_{\boldsymbol{f}_1,\alpha_1} + \cdots + \text{lc}(\boldsymbol{f}_t)\sigma^{\alpha_t}(r_t)c_{\boldsymbol{f}_t,\alpha_t}$, where $c_{\boldsymbol{f}_i,\alpha_i} := c_{\beta_i,\alpha_i}$.

(iii) $\boldsymbol{h} = \boldsymbol{f} - \sum_{i=1}^{t} \boldsymbol{f}_i r_i x^{\alpha_i}$.

We say that $\boldsymbol{f}$ reduces to $\boldsymbol{h}$ by F, denoted $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$, if and only if there exist vectors $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{t-1} \in A^m$ such that

$$\boldsymbol{f} \xrightarrow{F} \boldsymbol{h}_1 \xrightarrow{F} \boldsymbol{h}_2 \xrightarrow{F} \cdots \xrightarrow{F} \boldsymbol{h}_{t-1} \xrightarrow{F} \boldsymbol{h}.$$

$\boldsymbol{f}$ is reduced (also called minimal) with respect to F if $\boldsymbol{f} = \boldsymbol{0}$ or there is no one step reduction of $\boldsymbol{f}$ by F, i.e., one of the first two conditions of Definition 3.3 fails. Otherwise, we will say that $\boldsymbol{f}$ is reducible with respect to F. If $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$ and $\boldsymbol{h}$ is reduced with respect to F, then we say that $\boldsymbol{h}$ is a remainder for $\boldsymbol{f}$ with respect to F.

Remark 3.4. Related to the previous definition we have the following remarks:

(i) By Theorem 1.6, the coefficients $c_{\boldsymbol{f}_i,\alpha_i}$ in the previous definition are unique and satisfy

$$x^{\exp(\text{lm}(\boldsymbol{f}_i))}x^{\alpha_i} = c_{\boldsymbol{f}_i,\alpha_i}x^{\exp(\text{lm}(\boldsymbol{f}_i))+\alpha_i} + p_{\boldsymbol{f}_i,\alpha_i},$$

where $p_{\boldsymbol{f}_i,\alpha_i} = 0$ or $\deg(\text{lm}(p_{\boldsymbol{f}_i,\alpha_i})) < |\exp(\text{lm}(\boldsymbol{f}_i)) + \alpha_i|$, $1 \leq i \leq t$.

(ii) $\text{lm}(\boldsymbol{f}) \succ \text{lm}(\boldsymbol{h})$ and $\boldsymbol{f} - \boldsymbol{h} \in \langle F \rangle$, where $\langle F \rangle$ is the right submodule of A^m generated by F.

(iii) The remainder of $\boldsymbol{f}$ is not unique.

(vi) By definition we will assume that $\mathbf{0} \xrightarrow{F} \mathbf{0}$.

(v)

$$\text{lt}(\boldsymbol{f}) = \sum_{i=1}^{t} \text{lt}(\text{lt}(\boldsymbol{f}_i)r_i x^{\alpha_i}),$$

From the reduction relation we obtain the following interesting properties.

Proposition 3.5. Assume that A is a bijective skew PBW extension. Let $\boldsymbol{f}, \boldsymbol{h} \in A^m$, $\theta \in \mathbb{N}^n$ and $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ be a finite set of non-zero vectors of A^m.

(i) If $\boldsymbol{f} \xrightarrow{F} \boldsymbol{h}$, then there exists $\boldsymbol{p} \in A^m$ with $\boldsymbol{p} = \boldsymbol{0}$ or $\text{lm}(\boldsymbol{f}x^\theta) \succ \text{lm}(\boldsymbol{p})$ such that $\boldsymbol{f}x^\theta + \boldsymbol{p} \xrightarrow{F} \boldsymbol{h}x^\theta$.

(ii) If $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$ and $\boldsymbol{p} \in A$ is such that $\boldsymbol{p} = \boldsymbol{0}$ or $\text{lm}(\boldsymbol{h}) \succ \text{lm}(\boldsymbol{p})$, then $\boldsymbol{f} + \boldsymbol{p} \xrightarrow{F}_{+} \boldsymbol{h} + \boldsymbol{p}$.

(iii) If $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{h}$, then there exists $\boldsymbol{p} \in A^m$ with $\boldsymbol{p} = \boldsymbol{0}$ or $\text{lm}(\boldsymbol{f}x^\theta) \succ \text{lm}(\boldsymbol{p})$ such that $\boldsymbol{f}x^\theta + \boldsymbol{p} \xrightarrow{F}_{+} x^\theta\boldsymbol{h}$.

(iv) If $\boldsymbol{f} \xrightarrow{F}_{+} \boldsymbol{0}$, then there exists $\boldsymbol{p} \in A^m$ with $\boldsymbol{p} = \boldsymbol{0}$ or $\text{lm}(\boldsymbol{f}x^\theta) \succ \text{lm}(\boldsymbol{p})$ such that $\boldsymbol{f}x^\theta + \boldsymbol{p} \xrightarrow{F}_{+} \boldsymbol{0}$.

Proof:

(i) If $\boldsymbol{f} = \boldsymbol{0}$, then $\boldsymbol{h} = \boldsymbol{0} = \boldsymbol{p}$. Let $\boldsymbol{f} \neq \boldsymbol{0}$ and $\mathrm{lm}(\boldsymbol{f}) := x^{\lambda}$; then there exist $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t \in F$ and $r_1 \ldots, r_t \in R$ such that $\mathrm{lm}(\boldsymbol{f}_i) | \mathrm{lm}(\boldsymbol{f})$, for $1 \leq i \leq t$, i.e., $\mathrm{ind}(\mathrm{lm}(\boldsymbol{f}_i)) = \mathrm{ind}(\mathrm{lm}(\boldsymbol{f}))$ and there exists $x^{\alpha_i} \in \mathrm{Mon}(A)$ such that $\lambda = \alpha_i + \exp(\mathrm{lm}(\boldsymbol{f}_i))$. Moreover,

$$\mathrm{lc}(\boldsymbol{f}) = \mathrm{lc}(\boldsymbol{f}_1)\sigma^{\beta_1}(r_1)c_{\beta_1,\alpha_1} + \cdots + \mathrm{lc}(\boldsymbol{f}_t)\sigma^{\beta_t}(r_t)c_{\beta_t,\alpha_t}$$

with $\beta_i := \exp(\mathrm{lm}(\boldsymbol{f}_i))$ and $\boldsymbol{h} = \boldsymbol{f} - \sum_{i=1}^{t} \boldsymbol{f}_i r_i x^{\alpha_i}$. We note that $\mathrm{ind}(\mathrm{lm}(\boldsymbol{f})) = \mathrm{ind}(\mathrm{lm}(\boldsymbol{f}x^{\theta}))$ and $\exp(\boldsymbol{f}x^{\theta}) = \theta + \lambda$, so

$$\mathrm{lm}(\boldsymbol{f}_i) | \mathrm{lm}(\boldsymbol{f}x^{\theta}), \text{ with } \theta + \lambda = (\theta + \alpha_i) + \beta_i;$$

we observe that

$$\mathrm{lc}(\boldsymbol{f}x^{\theta}) = \mathrm{lc}(\boldsymbol{f})c_{\lambda,\theta} = \sum_{i=1}^{t} \mathrm{lc}(\boldsymbol{f}_i)\sigma^{\beta_i}(r_i)c_{\beta_i,\alpha_i}c_{\lambda,\theta}.$$

Hence Remark 1.7 yields:

$$\begin{aligned}
\mathrm{lc}(\boldsymbol{f}x^{\theta}) &= \sum_{i=1}^{t} \mathrm{lc}(\boldsymbol{f}_i)\sigma^{\beta_i}(r_i)c_{\beta_i,\alpha_i}c_{\alpha_i+\beta_i,\theta} \\
&= \sum_{i=1}^{t} lc(\boldsymbol{f}_i)\sigma^{\beta_i}(r_i)\sigma^{\beta_i}(c_{\alpha_i,\theta})c_{\beta_i,\alpha_i+\theta} \\
&= \sum_{i=1}^{t} lc(\boldsymbol{f}_i)\sigma^{\beta_i}(r_i c_{\alpha_i,\theta})c_{\beta_i,\alpha_i+\theta} \\
&= \sum_{i=1}^{t} lc(\boldsymbol{f}_i)\sigma^{\beta_i}(r_i')c_{\beta_i,\alpha_i+\theta},
\end{aligned}$$

where $r_i' := r_i c_{\alpha_i,\theta}$. Moreover,

$$\begin{aligned}
\boldsymbol{h}x^{\theta} &= \boldsymbol{f}x^{\theta} - \sum_{i=1}^{t} \boldsymbol{f}_i r_i x^{\alpha_i} x^{\theta} \\
&= \boldsymbol{f}x^{\theta} - \sum_{i=1}^{t} \boldsymbol{f}_i r_i c_{\alpha_i,\theta} x^{\alpha_i+\theta} + \boldsymbol{p} \\
&= \boldsymbol{f}x^{\theta} + \boldsymbol{p} - \sum_{i=1}^{t} \boldsymbol{f}_i r_i' x^{\alpha_i+\theta}
\end{aligned}$$

where $\boldsymbol{p} := \sum_{i=1}^{t}(-\boldsymbol{f}_i) r_i p_{\alpha_i,\theta}$; note that $\boldsymbol{p} = \boldsymbol{0}$ or $\deg(\boldsymbol{p}) < |\theta + \alpha_i + \beta_i| = |\theta + \lambda| = deg(\boldsymbol{f}x^{\theta})$, so $\mathrm{lm}(\boldsymbol{f}x^{\theta}) \succ \mathrm{lm}(\boldsymbol{p})$. Moreover, $\mathrm{lm}(\boldsymbol{f}x^{\theta} + \boldsymbol{p}) = \mathrm{lm}(\boldsymbol{f}x^{\theta})$ and $\mathrm{lc}(\boldsymbol{f}x^{\theta} + \boldsymbol{p}) = \mathrm{lc}(\boldsymbol{f}x^{\theta})$, so by the previous discussion $x^{\theta}\boldsymbol{f} + \boldsymbol{p} \xrightarrow{F} x^{\theta}\boldsymbol{h}$.

(ii) Let

$$\boldsymbol{f} \xrightarrow{F} \boldsymbol{h}_1 \xrightarrow{F} \boldsymbol{h}_2 \xrightarrow{F} \cdots \xrightarrow{F} \boldsymbol{h}_{t-1} \xrightarrow{F} \boldsymbol{h}_t := \boldsymbol{h}. \tag{8}$$

We start with $f \xrightarrow{F} \boldsymbol{h}_1$. If $\boldsymbol{f} = \boldsymbol{0}$, then $\boldsymbol{h}_1 = \boldsymbol{0} = \boldsymbol{p}$ and there is nothing to prove. Let $\boldsymbol{f} \neq \boldsymbol{0}$. If $\boldsymbol{h}_1 = \boldsymbol{0}$ then $\boldsymbol{p} = \boldsymbol{0}$ and hence $\operatorname{lm}(\boldsymbol{f}) \succ \operatorname{lm}(\boldsymbol{p})$; if $\boldsymbol{h}_1 \neq \boldsymbol{0}$, then $\operatorname{lm}(\boldsymbol{f}) \succ \operatorname{lm}(\boldsymbol{h}_1) \succ \operatorname{lm}(\boldsymbol{p})$, and hence $\operatorname{lm}(\boldsymbol{f}+\boldsymbol{p}) = \operatorname{lm}(\boldsymbol{f})$, $\operatorname{lc}(\boldsymbol{f}+\boldsymbol{p}) = \operatorname{lc}(\boldsymbol{f})$. Now, as in the proof of the first part of (i), we obtain $\boldsymbol{h}_1 + \boldsymbol{p} = \boldsymbol{f} + \boldsymbol{p} - \sum_{i=1}^{t} \boldsymbol{f}_i r_i x^{\alpha_i}$. Since $\operatorname{lm}(\boldsymbol{f}+\boldsymbol{p}) = \operatorname{lm}(\boldsymbol{f})$ and $\operatorname{lc}(\boldsymbol{f}+\boldsymbol{p}) = \operatorname{lc}(\boldsymbol{f})$, then $\boldsymbol{f} + \boldsymbol{p} \xrightarrow{F} \boldsymbol{h}_1 + \boldsymbol{p}$. Since $\operatorname{lm}(\boldsymbol{h}_i) \succ \operatorname{lm}(\boldsymbol{p})$ we can repeat this procedure for $\boldsymbol{h}_i \xrightarrow{F} \boldsymbol{h}_{i+1}$ with $1 \leq i \leq t-1$. This completes the proof of (ii).

(iii) By (i) and (8), there exists $\boldsymbol{p}_1 \in A^m$ with $\boldsymbol{p}_1 = \boldsymbol{0}$ or $\operatorname{lm}(\boldsymbol{f}x^\theta) \succ \operatorname{lm}(\boldsymbol{p}_1)$ such that $\boldsymbol{f}x^\theta + \boldsymbol{p}_1 \xrightarrow{F} \boldsymbol{h}_1 x^\theta$. Moreover there exists $\boldsymbol{p}_2 \in A^m$ with $\boldsymbol{p}_2 = 0$ or $\operatorname{lm}(\boldsymbol{h}_1 x^\theta) \succ \operatorname{lm}(\boldsymbol{p}_2)$ such that $\boldsymbol{h}_1 x^\theta + \boldsymbol{p}_2 \xrightarrow{F} \boldsymbol{h}_2 x^\theta$. Hence, in view of (ii), we obtain $\boldsymbol{f}x^\theta + \boldsymbol{p}_1 + \boldsymbol{p}_2 \xrightarrow{F} \boldsymbol{h}_1 x^\theta + \boldsymbol{p}_2 \xrightarrow{F} \boldsymbol{h}_2 x^\theta$, so the element $\boldsymbol{p}'' := \boldsymbol{p}_1 + \boldsymbol{p}_2 \in A^m$ is such that

$$\boldsymbol{f}x^\theta + \boldsymbol{p}'' \xrightarrow{F}_{+} \boldsymbol{h}_2 x^\theta,$$

with $\boldsymbol{p}'' = 0$ or $\operatorname{lm}(\boldsymbol{f}x^\theta) \succ \operatorname{lm}(\boldsymbol{p}'')$, because we have $\operatorname{lm}(\boldsymbol{f}x^\theta) \succ \operatorname{lm}(\boldsymbol{p}_1)$ and $\operatorname{lm}\boldsymbol{f}(x^\theta) \succ \operatorname{lm}(\boldsymbol{p}_2)$. By induction on t we find $\boldsymbol{p}' \in A^m$ such that

$$\boldsymbol{f}x^\theta + \boldsymbol{p}' \xrightarrow{F}_{+} h_{t-1} x^\theta,$$

with $\boldsymbol{p}' = 0$ or $\operatorname{lm}(\boldsymbol{f}x^\theta) \succ \operatorname{lm}(\boldsymbol{p}')$. By (i) there exists $\boldsymbol{p}_t \in A^m$ such that $\boldsymbol{h}_{t-1} x^\theta + \boldsymbol{p}_t \xrightarrow{F} hx^\theta$, with $\boldsymbol{p}_t = \boldsymbol{0}$ or $\operatorname{lm}(\boldsymbol{h}_{t-1} x^\theta) \succ \operatorname{lm}(\boldsymbol{p}_t)$. By (ii), $\boldsymbol{f}x^\theta + \boldsymbol{p}' + \boldsymbol{p}_t \xrightarrow{F}_{+} \boldsymbol{h}_{t-1} x^\theta + \boldsymbol{p}_t \xrightarrow{F} \boldsymbol{h}x^\theta$. Thus,

$$\boldsymbol{f}x^\theta + \boldsymbol{p} \xrightarrow{F}_{+} \boldsymbol{h}x^\theta,$$

with $\boldsymbol{p} := \boldsymbol{p}' + \boldsymbol{p}_t = 0$ or $\operatorname{lm}(\boldsymbol{f}x^\theta) \succ \operatorname{lm}(\boldsymbol{p})$ since $\operatorname{lm}(\boldsymbol{f}x^\theta) \succ \operatorname{lm}(\boldsymbol{p}')$ and $\operatorname{lm}(\boldsymbol{f}x^\theta) \succ \operatorname{lm}(\boldsymbol{p}_t)$.

(iv) This is a direct consequence of (iii) taking $\boldsymbol{h} = \boldsymbol{0}$. □

Definition 3.6. Let $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$ a bijective skew PBW extension. Let $\theta_1, \theta_2 \in \mathbb{N}^n$. We define the following automorphism over R, $\psi_{\theta_1,\theta_2} : R \to R$ that assigns to each $r \in R$.

$$\psi_{\theta_1,\theta_2}(r) := \sigma^{\theta_1+\theta_2}(\sigma^{-\theta_2}(r)).$$

Remark 3.7.

(i) The inverse function of ψ is given by $\psi^{-1}_{\theta_1,\theta_2}(r) = \sigma^{\theta_2}\sigma^{-(\theta_1+\theta_2)}(r)$.

(ii) Let $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$ a bijective skew PBW extension. For $\alpha, \beta, \gamma \in \mathbb{N}^n$ and $r \in R$, using the identities of Remark 1.7, we obtain

$$\sigma^\beta(r)c_{\beta,\alpha} = c_{\beta,\alpha}\psi_{\beta,\alpha}(r) \tag{9}$$

$$c_{\beta,\alpha} r = \sigma^\beta(\psi^{-1}_{\beta,\alpha}(r))c_{\beta,\alpha}. \tag{10}$$

Moreover, we have

$$\begin{aligned} c_{\beta,\alpha} r c_{\beta+\alpha,\gamma} &= \sigma^{\beta}(\psi^{-1}_{\beta,\alpha}(r)) c_{\beta,\alpha} c_{\alpha+\beta,\gamma} \\ &= \sigma^{\beta}(\psi^{-1}_{\beta,\alpha}(r)) \sigma^{\beta}(c_{\alpha,\gamma}) c_{\beta,\alpha+\gamma} \\ &= \sigma^{\beta}(\psi^{-1}_{\beta,\alpha}(r) c_{\alpha,\gamma}) c_{\beta,\alpha+\gamma} \\ &= c_{\beta,\alpha+\gamma} \psi_{\beta,\alpha+\gamma}(\psi^{-1}_{\beta,\alpha}(r) c_{\alpha,\gamma}). \end{aligned} \tag{11}$$

(iii) Under the notation used in proof of Proposition 3.5 (i); $s_1, \ldots, s_t$ are solutions of the equation

$$\operatorname{lc}(\boldsymbol{h}) = \sum_{i=1}^{t} \operatorname{lc}(\boldsymbol{f}_i) c_{\beta_i,\alpha_i} s_i,$$

if and only if, $r_i = \psi^{-1}_{\alpha_i,\beta_i}(s_i)$ for $i = 1, \ldots, t$, are solutions of the equation

$$\operatorname{lc}(\boldsymbol{h}) = \sum_{i=1}^{t} \operatorname{lc}(\boldsymbol{f}_i) c_{\beta_i,\alpha_i} \psi_{\beta_i,\alpha_i}(r_i).$$

The following theorem is a theoretical support of the right Division Algorithm (Algorithm 1) for bijective skew PBW extensions.

Algorithm 1: Right division algorithm in A^m

Input: $\boldsymbol{f}, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_t \in A^m$ with $\boldsymbol{f}_j \neq \boldsymbol{0}$ $(1 \leq j \leq t)$
Output: $q_1, \ldots, q_t \in A$, $\boldsymbol{h} \in A^m$ with $\boldsymbol{f} = \boldsymbol{f}_1 q_1 + \cdots + \boldsymbol{f}_t q_t + \boldsymbol{h}$, $\boldsymbol{h}$ reduce with respect to $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ and $\operatorname{lm}(\boldsymbol{f}) = \max\{\operatorname{lm}(\operatorname{lm}(\boldsymbol{f}_1)\operatorname{lm}(q_1)), \ldots, \operatorname{lm}(\operatorname{lm}(\boldsymbol{f}_t)\operatorname{lm}(q_t)), \operatorname{lm}(\boldsymbol{h})\}$
Initialization: $q_1 \leftarrow 0, q_2 \leftarrow 0, \ldots, q_t \leftarrow 0, \boldsymbol{h} \leftarrow \boldsymbol{f}$;
while $\mathbf{h} \neq \mathbf{0}$ *and there exists* j *such that* $\operatorname{lm}(\mathbf{f}_j)$ *divides* $\operatorname{lm}(\mathbf{h})$ **do**
    $J \leftarrow \{j \mid \operatorname{lm}(\boldsymbol{f}_j)$ divides $\operatorname{lm}(\boldsymbol{h})\}$;
    for $i \in J$ **do**
        $\beta_j \leftarrow \exp(\operatorname{lm}(\boldsymbol{f}_j))$;
        $\alpha_j \leftarrow \exp(\operatorname{lm}(\boldsymbol{h})) - \beta_j$;
    end
    if *the equation* $\operatorname{lc}(\mathbf{h}) = \sum_{j \in J} \operatorname{lc}(\mathbf{f}_j) c_{\mathbf{f}_j,\alpha_j} s_j$ *is soluble* **then**
        Calculate one solution $(s_j)_{j \in J}$;
        for $j \in J$ **do**
            $r_j \leftarrow \psi^{-1}_{\beta_j,\alpha_j}(s_j)$;
            $q_j \leftarrow q_j + r_j x^{\alpha_j}$;
            $\boldsymbol{h} \leftarrow \boldsymbol{h} - \boldsymbol{f}_j r_j x^{\alpha_j}$;
        end
    else
        Break;
    end
end

Theorem 3.8. Let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$ be a set of non-zero vectors of A^m and $\boldsymbol{f} \in A^m$, then the right division algorithm (Algorithm 1) produces polynomials $q_1, \ldots, q_t \in A$ and a reduced vector $\boldsymbol{h} \in A^m$ with respect to F such that $\boldsymbol{f} \xmapsto{F}_+ \boldsymbol{h}$ and

$$\boldsymbol{f} = \boldsymbol{f}_1 q_1 + \cdots + \boldsymbol{f}_t q_t + \boldsymbol{h}$$

with

$$\operatorname{lm}(\boldsymbol{f}) = \max\{\operatorname{lm}(\operatorname{lm}(\boldsymbol{f}_1)\operatorname{lm}(q_1)), \ldots, \operatorname{lm}(\operatorname{lm}(\boldsymbol{f}_t)\operatorname{lm}(q_t)), \operatorname{lm}(\boldsymbol{h})\}.$$

Proof:
We first note that Algorithm 1 is the iteration of the reduction process. If $\boldsymbol{f}$ is reduced with respect to $F := \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\}$, then $\boldsymbol{h} = \boldsymbol{f}, q_1 = \cdots = q_t = 0$ and $\operatorname{lm}(\boldsymbol{f}) = \operatorname{lm}(\boldsymbol{h})$. If $\boldsymbol{f}$ is not reduced, then we make the first reduction, $\boldsymbol{f} \xmapsto{F} \boldsymbol{h}_1$, with $\boldsymbol{f} = \sum_{j\in J_1} \boldsymbol{f}_j r_{j1} x^{\alpha_j} + \boldsymbol{h}_1$, with $J_1 := \{j \mid \operatorname{lm}(\boldsymbol{f}_j)$ divides $\operatorname{lm}(\boldsymbol{f})\}$ and $r_{j1} \in R$. If $\boldsymbol{h}_1$ is reduced with respect to F, then the cycle **While** ends and we obtain $q_j = r_{j1}x^{\alpha_j}$ for $j \in J_1$ and $q_j = 0$ for $j \notin J_1$. Moreover, $\operatorname{lm}(\boldsymbol{f}) \succ \operatorname{lm}(\boldsymbol{h}_1)$ and $\operatorname{lm}(\boldsymbol{f}) = \operatorname{lm}(\operatorname{lm}(\boldsymbol{f}_j)\operatorname{lm}(q_j))$ for $j \in J_1$ such that $r_{j1} \neq 0$, hence, $\operatorname{lm}(\boldsymbol{f}) = \max_{1\leq j\leq t}\{\operatorname{lm}(\operatorname{lm}(\boldsymbol{f}_j))\operatorname{lm}(q_j), \operatorname{lm}(\boldsymbol{h}_1)\}$. If $\boldsymbol{h}_1$ is not reduced, so we make the second reduction with respect to F, $\boldsymbol{h}_1 \xmapsto{F} \boldsymbol{h}_2$, with $\boldsymbol{h}_1 = \sum_{j\in J_2} \boldsymbol{f}_j r_{j2} x^{\alpha_j} + \boldsymbol{h}_2$, $J_2 := \{j \mid \operatorname{lm}(\boldsymbol{f}_j)$ divides $\operatorname{lm}(\boldsymbol{h}_1)\}$ and $r_{j2} \in R$. We have

$$\boldsymbol{f} = \textstyle\sum_{j\in J_1} \boldsymbol{f}_j r_{j1} x^{\alpha_j} + \sum_{j\in J_2} \boldsymbol{f}_j r_{j2} x^{\alpha_j} + \boldsymbol{h}_2$$

If $\boldsymbol{h}_2$ is reduced with respect to F the procedure ends and we get $q_j = q_j$ for $j \notin J_2$ and $q_j = q_j + r_{j2}x^{\alpha_j}$ for $j \in J_2$. Since $\operatorname{lm}(\boldsymbol{f}) \succ \operatorname{lm}(\boldsymbol{h}_1) \succ \operatorname{lm}(\boldsymbol{h}_2)$, then the algorithm produces polynomials q_j with monomials ordered according to the monomial order fixed, and again we have $\operatorname{lm}(\boldsymbol{f}) = \max_{1\leq j\leq t}\{\operatorname{lm}(\operatorname{lm}(q_j)\operatorname{lm}(\boldsymbol{f}_j)), \operatorname{lm}(\boldsymbol{h}_2)\}$. If we continue this way, the algorithm ends since $\operatorname{Mon}(A^m)$ is well ordered. □

4. Gröbner bases for right submodules of A^m

In this section we present the general theory of Gröbner bases for right submodules of A^m, $m \geq 1$, where $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a bijective skew PBW extension of R, with R a RGS ring (see Definition 3.1) and $\operatorname{Mon}(A)$ endowed with some monomial order (see Definition 2.2). A^m is the right free A-module of column vectors of length $m \geq 1$; since A is a right Noetherian ring, then A is an IBN ring (Invariant Basis Number, see [9]), and hence, all bases of the free module A^m have m elements. Note moreover that A^m is right Noetherian, and hence, any submodule of A^m is finitely generated.

The plan is to define and calculate Gröbner bases for right submodules of A^m, we will present some equivalent conditions in order to define right Gröbner bases, and finally, we will compute right Gröbner bases using a procedure similar to right Buchberger's algorithm over bijective skew PBW extensions. This theory was studied in the bijective general case for left modules. Here we adapt the ideas and technique used in [3].

Our next purpose is to define Gröbner bases for right submodules of A^m.

Definition 4.1. Let $M \neq 0$ be a right submodule of A^m and let G be a non-empty finite subset of non-zero vectors of M, we say that G is a Gröbner basis for M if each element $0 \neq \boldsymbol{f} \in M$ is reducible with respect to G. We will say that $\{\boldsymbol{0}\}$ is a Gröbner basis for $M = 0$.

Theorem 4.2. Let $M \neq 0$ be a right submodule of the free A-module A^m and let G be a finite subset of non-zero vectors of M. Then the following conditions are equivalent:

(i) G is a Gröbner basis for M.

(ii) For any vector $\boldsymbol{f} \in A^m$,

$$\boldsymbol{f} \in M \text{ if and only if } \boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{0}.$$

(iii) For any $\boldsymbol{0} \neq \boldsymbol{f} \in M$ there exist $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t \in G$ such that $\operatorname{lm}(\boldsymbol{g}_j) | \operatorname{lm}(\boldsymbol{f})$, $1 \leq j \leq t$, (i.e., $\operatorname{ind}(\operatorname{lm}(\boldsymbol{g}_j)) = \operatorname{ind}(\operatorname{lm}(\boldsymbol{f}))$ and there exist $\alpha_j \in \mathbb{N}^n$ such that $\exp(\operatorname{lm}(\boldsymbol{g}_j)) + \alpha_j = \exp(\operatorname{lm}(\boldsymbol{f}))$) and

$$\operatorname{lc}(\boldsymbol{f}) \in \{\operatorname{lc}(\boldsymbol{g}_1)c_{\boldsymbol{g}_1,\alpha_1}, \dots, \operatorname{lc}(\boldsymbol{g}_t)c_{\boldsymbol{g}_t,\alpha_t}\rangle.$$

(iv) For $\alpha \in \mathbb{N}^n$ and $1 \leq v \leq m$, let $\{\alpha, I\rangle_v$ be the right ideal of R defined by

$$\{\alpha, M\rangle_v := \{\{\operatorname{lc}(\boldsymbol{f}) \mid \boldsymbol{f} \in M, \operatorname{ind}(\operatorname{lm}\boldsymbol{f}) = v, \exp(\operatorname{lm}(\boldsymbol{f})) = \alpha\}\rangle.$$

Then, $\{\alpha, I\rangle_v = J_v$, with

$$J_v := \{\{\operatorname{lc}(\boldsymbol{g})c_{\boldsymbol{g},\beta} \mid \boldsymbol{g} \in G, \operatorname{ind}(\operatorname{lm}\boldsymbol{g}) = v \text{ and } \exp(\operatorname{lm}(\boldsymbol{g})) + \beta = \alpha\}\rangle.$$

Proof:
(i) $\Rightarrow$ (ii): Let $\boldsymbol{f} \in M$, if $\boldsymbol{f} = \boldsymbol{0}$, then by definition $\boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{0}$. If $\boldsymbol{f} \neq \boldsymbol{0}$, then there exists $\boldsymbol{h}_1 \in A^m$ such that $\boldsymbol{f} \xrightarrow{G} \boldsymbol{h}_1$, with $\operatorname{lm}(\boldsymbol{f}) \succ \operatorname{lm}(\boldsymbol{h}_1)$ and $\boldsymbol{f} - \boldsymbol{h}_1 \in \{G\rangle \subseteq M$, hence $\boldsymbol{h}_1 \in M$; if $\boldsymbol{h}_1 = \boldsymbol{0}$, so we end. If $\boldsymbol{h}_1 \neq \boldsymbol{0}$, then we can repeat this reasoning for $\boldsymbol{h}_1$, and since $\operatorname{Mon}(A^m)$ is well ordered, therefore $\boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{0}$.

Conversely, if $f \xrightarrow{G}_+ 0$, then by the Theorem 3.8, there exist $\boldsymbol{g}_1, \dots, \boldsymbol{g}_t \in G$ and $q_1, \dots, q_t \in A$ such that $\boldsymbol{f} = \boldsymbol{g}_1 q_1 + \cdots + \boldsymbol{g}_t q_t$, i.e., $\boldsymbol{f} \in M$.

(ii) $\Rightarrow$ (i): evident.

(i) $\Leftrightarrow$ (iii): this is a direct consequence of Definition 3.3 and the equation (9).

(iii) $\Rightarrow$ (iv) Since R is a right Noetherian ring, there exist $r_1, \dots, r_s \in R, \boldsymbol{f}_1, \dots, \boldsymbol{f}_n \in M$ such that $\{\alpha, M\rangle_v = \{r_1, \dots, r_s\rangle$, $\operatorname{ind}(\operatorname{lm}(\boldsymbol{f}_i)) = v$, $\operatorname{lm}(f_i) = x^\alpha$, $1 \leq i \leq n$, with $\{r_1, \dots, r_s\rangle \subseteq \{\operatorname{lc}(f_1), \dots, \operatorname{lc}(f_n)\rangle$, then $\{\operatorname{lc}(\boldsymbol{f}_1), \dots, \operatorname{lc}(\boldsymbol{f}_n)\rangle = \{\alpha, M\rangle_v$. Let $r \in \{\alpha, M\rangle_v$, there exist $a_1, \dots, a_n \in R$ such that $r = \operatorname{lc}(\boldsymbol{f}_1)a_1 + \cdots + \operatorname{lc}(\boldsymbol{f}_n)a_n$; by (iii), for each i there exist $\boldsymbol{g}_{1i}, \dots, \boldsymbol{g}_{t_i i} \in G$ and $b_{ji} \in R$ such that

$$\operatorname{lc}(\boldsymbol{f}_i) = \operatorname{lc}(\boldsymbol{g}_{1i})c_{\boldsymbol{g}_{1i},\alpha_{1i}}b_{1i} + \cdots + \operatorname{lc}(\boldsymbol{g}_{t_i i})c_{\boldsymbol{g}_{t_i i},\alpha_{t_i i}}b_{t_i i},$$

with $v = \operatorname{ind}(\operatorname{lm}\boldsymbol{f}_i) = \operatorname{ind}(\operatorname{lm}(\boldsymbol{g}_{ji}))$ and $\alpha = \exp(\operatorname{lm}(\boldsymbol{f}_i)) = \alpha_{ji} + \exp(\operatorname{lm}(\boldsymbol{g}_{ji}))$, thus $\{\alpha, M\rangle_v \subseteq J_v$.

Conversely, if $r \in J_v$, then $r = \mathrm{lc}(\boldsymbol{g}_1)c_{\boldsymbol{g}_1,\beta_1}b_1 + \cdots + \mathrm{lc}(\boldsymbol{g}_t)c_{\boldsymbol{g}_t,\beta_t}b_t$, with $b_i \in R$, $\beta_i \in \mathbb{N}^n$, $\boldsymbol{g}_i \in G$ such that $\mathrm{ind}(\mathrm{lm}(\boldsymbol{g}_i)) = v$, $\beta_i + \exp(\mathrm{lm}(\boldsymbol{g}_i)) = \alpha$ for any $1 \leq i \leq t$.

Note that $\boldsymbol{g}_i x^{\beta_i} \in M$, $\mathrm{ind}(\boldsymbol{g}_i x^{\beta_i}) = v$, $\exp(\mathrm{lm}(\boldsymbol{g}_i x^{\beta_i})) = \alpha$, $\mathrm{lc}(\boldsymbol{g}_i x^{\beta_i}) = \mathrm{lc}(\boldsymbol{g}_i)c_{\boldsymbol{g}_i,\beta_i}$, for $1 \leq i \leq t$, thus $r = \mathrm{lc}(\boldsymbol{g}_1 x^{\beta_1})b_1 + \cdots + \mathrm{lc}(\boldsymbol{g}_t x^{\beta_t})b_t$, i.e., $r \in \{\alpha, M\rangle_v$.

(iv)$\Rightarrow$ (iii): let $\boldsymbol{0} \neq \boldsymbol{f} \in M$ and let $\alpha = \exp(\mathrm{lm}(\boldsymbol{f}))$ and $v = \mathrm{ind}(\mathrm{lm}(\boldsymbol{f}))$, then $\mathrm{lc}(\boldsymbol{f}) \in \{\alpha, M\rangle_v$; by (iv) $\mathrm{lc}(\boldsymbol{f}) = \mathrm{lc}(\boldsymbol{g}_1)c_{\boldsymbol{g}_1,\beta_1}b_1 + \cdots + \mathrm{lc}(\boldsymbol{g}_t)c_{\boldsymbol{g}_t,\beta_t}b_t$, with $b_i \in R$, $\beta_i \in \mathbb{N}^n$, $\boldsymbol{g}_i \in G$ such that $= \mathrm{ind}(\mathrm{lm}(\boldsymbol{g}_i)) = v$ and $\beta_i + \exp(\mathrm{lm}(\boldsymbol{g}_i)) = \alpha$ for any $1 \leq i \leq t$. From this we conclude that, $\mathrm{lm}(\boldsymbol{g}_i) \mid \mathrm{lm}(\boldsymbol{f})$. □

Some useful consequences of Theorem 4.2 are the following results.

Corollary 4.3. Let $M \neq 0$ be a right submodule of A^m. Then,

(i) If G is a Gröbner basis for M, then $M = \langle G \rangle$.

(ii) Let G be a Gröbner basis for M. If $\boldsymbol{f} \in M$ and $\boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{h}$, with $\boldsymbol{h}$ reduced, then $\boldsymbol{h} = \boldsymbol{0}$.

(iii) Let $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ be a set of non-zero vectors of M with $\mathrm{lc}(\boldsymbol{g}_i) = 1$ for each $1 \leq i \leq t$. If given $\boldsymbol{0} \neq \boldsymbol{r} \in M$ there exists i such that $\mathrm{lm}(\boldsymbol{g}_i) \mid \mathrm{lm}(\boldsymbol{r})$, then G is a Gröbner basis of M.

Proof:

(i) Apply Theorem 4.2.

(ii) Assume that $\boldsymbol{f} \in M$ and $\boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{h}$, with $\boldsymbol{h}$ reduced. Since $\boldsymbol{f} - \boldsymbol{h} \in \langle G \rangle = M$, then $\boldsymbol{h} \in M$; if $\boldsymbol{h} \neq \boldsymbol{0}$ then $\boldsymbol{h}$ can be reduced by G, but this is not possible since $\boldsymbol{h}$ is reduced.

(iii) Assume that $\boldsymbol{f} \in A^m$. By Theorem 3.8 there exists $\boldsymbol{r}$ reduced such that $\boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{r}$. If $\boldsymbol{f} \in M$ then $\boldsymbol{r} \in M$; if $\boldsymbol{r} \neq \mathbf{0}$, then by hypothesis there exists $\boldsymbol{g}_i \in G$ such that $lm(\boldsymbol{g}_i)$ divides $lm(\boldsymbol{r})$, thus, since $lc(\boldsymbol{g}_i) = 1$, then $\boldsymbol{r}$ is reducible, which is a contradiction and therefore, $\boldsymbol{f} \xrightarrow{G}_+ \mathbf{0}$. On the other hand, if $\boldsymbol{f} \xrightarrow{G}_+ \mathbf{0}$, then $\boldsymbol{f} \in M$. Now Theorem 4.2 (ii) implies that G is Gröbner basis of M. □

Corollary 4.4. Let G be a Gröbner basis for a right submodule M of A^m. Given $\boldsymbol{g} \in G$, if $\boldsymbol{g}$ is reducible with respect to $G' = G - \{\boldsymbol{g}\}$, then G' is a Gröbner basis for M.

Proof:
According to Theorem 4.2, it is enough to show that every $\boldsymbol{f} \in M$ is reducible with respect to G'. Let $\boldsymbol{f}$ be a nonzero vector in M; since G is a Gröbner basis for M, $\boldsymbol{f}$ is reducible with respect to G and there exist elements $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t \in G$ satisfying the conditions (i), (ii) and (iii) in the Definition 3.3. If $\boldsymbol{g} \neq \boldsymbol{g}_i$ for each $1 \leq i \leq t$, then we finished. Suppose that $\boldsymbol{g} = \boldsymbol{g}_j$ for some $j \in \{1, \ldots, t\}$ and let $\beta_i = \exp(\mathrm{lm}(\boldsymbol{g}_i))$ for $i \neq j$, $\beta = \exp(\mathrm{lm}(\boldsymbol{g}))$, and $\alpha_i, \alpha \in \mathbb{N}^n$ such that $\alpha_i + \beta_i = \exp(\mathrm{lm}(\boldsymbol{f})) = \alpha + \beta$. Thus,

$$\mathrm{lc}(\boldsymbol{f}) = \mathrm{lc}(\boldsymbol{g}_1)c_{\beta_1,\alpha_1}r_1 + \cdots + \mathrm{lc}(\boldsymbol{g})c_{\beta,\alpha}r_j + \cdots + \mathrm{lc}(\boldsymbol{g}_t)c_{\beta_t,\alpha_t}r_t.$$

On the other hand, since $\boldsymbol{g}$ is reducible with respect to G', there exist $\boldsymbol{g}'_1,\dots,\boldsymbol{g}'_s \in G'$ such that $\operatorname{lm}(\boldsymbol{g}'_l) \mid \operatorname{lm}(\boldsymbol{g})$ and $\operatorname{lc}(\boldsymbol{g}) = \sum_{k=1}^{s} \operatorname{lc}(\boldsymbol{g}'_k)c_{\beta'_k,\alpha'_k}r'_k$, where $\beta'_k = \exp(\operatorname{lm}(\boldsymbol{g}'_k))$, $\alpha'_k \in \mathbb{N}^n$ and $\alpha'_k+\beta'_k = \exp(\operatorname{lm}(\boldsymbol{g})) = \beta$. Thus, $\operatorname{lm}(\boldsymbol{g}'_k) \mid \operatorname{lm}(\boldsymbol{f})$ for $1 \le i \le s$; moreover, using the equation (11) of Remark 3.7, we have

$$c_{\beta'_k,\alpha'_k}r'_k c_{\beta,\alpha} = c_{\beta'_k,\alpha'_k}r'_k c_{\beta'_k+\alpha'_k,\alpha} = c_{\beta'_k,\alpha'_k+\alpha}r''_k,$$

where $r''_k = \psi_{\beta'_k,\alpha'_k+\alpha}(\psi^{-1}_{\beta'_k,\alpha'_k}(r)c_{\alpha'_k,\alpha})$. Therefore,

$$\operatorname{lc}(\boldsymbol{g})c_{\beta,\alpha} = \sum_{k=1}^{s} \operatorname{lc}(\boldsymbol{g}'_k)c_{\beta'_k,\alpha'_k}r'_k c_{\beta,\alpha} = \sum_{k=1}^{s} \operatorname{lc}(\boldsymbol{g}'_k)c_{\beta'_k,\alpha'_k+\alpha}r''_k.$$

Since $\alpha+\beta = \exp(\operatorname{lm}(\boldsymbol{f}))$, then $\alpha+\alpha'_k+\beta'_k = \exp(\operatorname{lm}(\boldsymbol{f}))$. Further, if there exists $\boldsymbol{g}_w \in \{\boldsymbol{g}_1,\dots,\boldsymbol{g}_t\}$ such that $\boldsymbol{g}_w = \boldsymbol{g}'_z$ for some $z \in \{1,\dots,s\}$, then $\beta_w = \beta'_z$ and $\alpha+\alpha'_z = \alpha_w$; therefore, in the representation of $\operatorname{lc}(\boldsymbol{f})$ would appear the term $\operatorname{lc}(\boldsymbol{g}_w)c_{\beta_w,\alpha_w}(r_w + r''_z r_j)$. Hence we conclude that $\boldsymbol{f}$ is reducible with respect to G' and consequently G' is a Gröbner basis for M. □

5. Buchberger's algorithm for right modules

Recall that we are assuming that A is a bijective skew PBW extension, we will prove in the present section that every submodule M of A^m has a Gröbner basis, and also we will construct the Buchberger's algorithm for computing such bases.

We start this section by fixing some notation and by proving a couple of preliminary results used later.

Definition 5.1. Let $F := \{\boldsymbol{g}_1,\dots,\boldsymbol{g}_s\} \subseteq A^m$, $\boldsymbol{X}_F$ the least common multiple of $\{\operatorname{lm}(\boldsymbol{g}_1),\dots,\operatorname{lm}(\boldsymbol{g}_s)\}$, $\theta \in \mathbb{N}^n$, $\beta_i := \exp(\operatorname{lm}(\boldsymbol{g}_i))$ and $\gamma_i \in \mathbb{N}^n$ such that $\gamma_i+\beta_i = \exp(\boldsymbol{X}_F)$, $1 \le i \le s$. $B_{F,\theta}$ will denote a finite set of generators in R^s of right R-module

$$S_{F,\theta} := \operatorname{Syz}^r_R[\operatorname{lc}(\boldsymbol{g}_1)c_{\beta_1,\gamma_1+\theta} \ \cdots \ \operatorname{lc}(\boldsymbol{g}_s)c_{\beta_s,\gamma_s+\theta}].$$

For $\theta = \boldsymbol{0} := (0,\dots,0)$, $S_{F,\theta}$ will be denoted by S_F and $B_{F,\theta}$ by B_F.

Remark 5.2. Let $(b_1,\dots,b_s) \in S_{F,\theta}$. Since A is bijective, then there exists an unique $(b'_1,\dots,b'_s) \in S_F$ such that

$$b_i = \psi_{\beta_i,\gamma_i+\theta}(\psi^{-1}_{\beta_i,\gamma_i}(b'_i)c_{\gamma_i,\theta}), \text{ for each } 1 \le i \le s. \tag{12}$$

In fact, the existence and uniqueness of $(b'_1,\dots,b'_s)$ follows from the bijectivity of A. Now, since $(b_1,\dots,b_s) \in S_{F,\theta}$, then $\sum_{i=1}^{s} lc(\boldsymbol{g}_i)c_{\beta_i,\gamma_i+\theta}b_i = 0$. Replacing b_i of (12) in the last equation, we obtain

$$\sum_{i=1}^{s} lc(\boldsymbol{g}_i)c_{\beta_i,\gamma_i+\theta}\psi_{\beta_i,\gamma_i+\theta}(\psi^{-1}_{\beta_i,\gamma_i}(b'_i)c_{\gamma_i,\theta}) = 0.$$

The equation (11) of Remark 3.7, yields

$$c_{\beta_i,\gamma_i+\theta}\psi_{\beta_i,\gamma_i+\theta}(\psi^{-1}_{\beta_i,\gamma_i}(b'_i)c_{\gamma_i,\theta}) = c_{\beta_i,\gamma_i}b'_i c_{\beta_i+\gamma_i,\theta}.$$

Thus, $\sum_{i=1}^{s} lc(\boldsymbol{g}_i)c_{\beta_i,\gamma_i}b'_i c_{\beta_i+\gamma_i,\theta} = 0$, and since $c_{\beta_i+\gamma_i,\theta} = c_{\boldsymbol{X}_F,\theta}$ is invertible, then $\sum_{i=1}^{s} lc(g_i)c_{\beta_i,\gamma_i}b'_i = 0$, i.e., $(b'_1,\ldots,b'_s) \in S_F$.

Lemma 5.3. Let $\boldsymbol{g}_1,\ldots,\boldsymbol{g}_s \in A$, $c_1,\ldots,c_s \in R-\{0\}$ and $\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_s \in \mathbb{N}^n$ such that, $\boldsymbol{X}_\delta := \mathrm{lm}(\mathrm{lm}(\boldsymbol{g}_i)x^{\alpha_i})$ and $\beta_i := \exp(\boldsymbol{g}_i)$, for each $1 \le i \le s$. If $\mathrm{lm}(\sum_{i=1}^{s}\boldsymbol{g}_i c_i x^{\alpha_i}) \prec \boldsymbol{X}_\delta$, then there exist $r_1,\ldots,r_k \in R$ and $z_1,\ldots,z_s \in A$ such that

$$\sum_{i=1}^{s}\boldsymbol{g}_i c_i x^{\alpha_i} = \sum_{j=1}^{k}\left(\sum_{i=1}^{s}\boldsymbol{g}_i\psi^{-1}_{\beta_i,\gamma_i}(b_{ji})x^{\gamma_i}\right)r_j x^{\delta-\exp(\boldsymbol{X}_F)} + \sum_{i=1}^{s}\boldsymbol{g}_i z_i,$$

where $\boldsymbol{X}_F$ is the least common multiple of $\mathrm{lm}(\boldsymbol{g}_1),\ldots,\mathrm{lm}(\boldsymbol{g}_s)$, $\gamma_i \in \mathbb{N}^n$ is such that $\gamma_i+\beta_i = \exp(\boldsymbol{X}_F)$, for each $1 \le i \le s$, and

$$B_F = \{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_k\} := \{(b_{11},\ldots,b_{1s}),\ldots,(b_{k1},\ldots,b_{ks})\}.$$

Moreover, $\mathrm{lm}(\sum_{i=1}^{s}\boldsymbol{g}_i\psi^{-1}_{\beta_i,\gamma_i}(b_{ji})x^{\gamma_i}r_j x^{\delta-\exp(\boldsymbol{X}_F)}) \prec \boldsymbol{X}_\delta$ for every $1 \le j \le k$, and $\mathrm{lm}(\boldsymbol{g}_i z_i) \prec \boldsymbol{X}_\delta$ for every $1 \le i \le s$.

Proof:
Since $\boldsymbol{X}_\delta = \mathrm{lm}(\mathrm{lm}(\boldsymbol{g}_i)x_i^{\alpha})$, then $\mathrm{lm}(\boldsymbol{g}_i) \mid \boldsymbol{X}_\delta$ and hence $\boldsymbol{X}_F \mid \boldsymbol{X}_\delta$, so there exists $\theta \in \mathbb{N}^n$ such that $\exp(\boldsymbol{X}_F)+\theta = \delta$. On the other hand, $\gamma_i+\beta_i = \exp(\boldsymbol{X}_F)$ and $\alpha_i+\beta_i = \delta$, so $\alpha_i = \gamma_i+\theta$ for each $1 \le i \le s$. Now, $\mathrm{lm}(\sum_{i=1}^{s}\boldsymbol{g}_i c_i x^{\alpha_i}) \prec \boldsymbol{X}_\delta$ implies that $\sum_{i=1}^{s}\mathrm{lc}(\boldsymbol{g}_i)\sigma^{\beta_i}(c_i)c_{\beta_i,\alpha_i} = 0$. The equation (9) of Remark 3.7, yields $\sum_{i=1}^{s}\mathrm{lc}(\boldsymbol{g}_i)c_{\beta_i,\alpha_i}d_i = 0$, with $d_i = \psi_{\beta_i,\alpha_i}(c_i)$, for each $1 \le i \le s$. So, $\sum_{i=1}^{s}\mathrm{lc}(\boldsymbol{g}_i)c_{\beta_i,\gamma_i+\theta}d_i = 0$. This implies that $(d_1,\ldots,d_s) \in S_{F,\theta}$. By Remark 5.2, there exists a unique $(d'_1,\ldots,d'_s) \in S_F$ such that $d_i = \psi_{\beta_i,\gamma_i+\theta}(\psi^{-1}_{\beta_i,\gamma_i}(d'_i)c_{\gamma_i,\theta})$. Then,

$$c_i = \psi^{-1}_{\beta_i,\alpha_i}(d_i) = \psi^{-1}_{\beta_i,\gamma_i+\theta}(d_i) = \psi^{-1}_{\beta_i,\gamma_i}(d'_i)c_{\gamma_i,\theta},$$

and therefore, we have

$$\begin{aligned}\sum_{i=1}^{s}\boldsymbol{g}_i c_i x^{\alpha_i} &= \sum_{i=1}^{s}\boldsymbol{g}_i\psi^{-1}_{\beta_i,\gamma_i}(d'_i)c_{\gamma_i,\theta}x^{\alpha_i}\\ &= \sum_{i=1}^{s}\boldsymbol{g}_i\psi^{-1}_{\beta_i,\gamma_i}(d'_i)c_{\gamma_i,\theta}x^{\gamma_i+\theta}\\ &= \sum_{i=1}^{s}\boldsymbol{g}_i\psi^{-1}_{\beta_i,\gamma_i}(d'_i)(x^{\gamma_i}x^{\theta}-p_{\gamma_i,\theta})\\ &= \sum_{i=1}^{s}\boldsymbol{g}_i\psi^{-1}_{\beta_i,\gamma_i}(d'_i)x^{\gamma_i}x^{\theta}+\sum_{i=1}^{s}\boldsymbol{g}_i p_i\end{aligned}$$

with $p_i = 0$ or $\mathrm{lm}(p_i) \prec x^{\theta+\gamma_i}$. Hence, $\boldsymbol{g}_i p_i = 0$ or $\mathrm{lm}(\boldsymbol{g}_i p_i) \prec x^{\theta+\gamma_i+\beta_i} = \boldsymbol{X}_\delta$. On the other hand, since $(d'_1,\ldots,d'_s) \in S_F$, then there exist $r'_1,\ldots,r'_k \in R$ such that $(d'_1,\ldots,d'_s) = \boldsymbol{b}_1 r'_1+\cdots+\boldsymbol{b}_k r'_k = (b_{11},\ldots,b_{1s})r'_1+\cdots+(b_{k1},\ldots,b_{ks})r'_k$, thus $d'_i = \sum_{j=1}^{k}b_{ji}r'_j$.

Therefore

$$\begin{aligned}\sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}(d_i')x^{\gamma_i}x^{\theta} &= \sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}\Big(\sum_{j=1}^{k}b_{ji}r_j'\Big)x^{\gamma_i}x^{\theta}\\ &= \sum_{i=1}^{s}\boldsymbol{g}_i\Big(\sum_{j=1}^{k}\psi_{\beta_i,\gamma_i}^{-1}(b_{ji})\psi_{\beta_i,\gamma_i}^{-1}(r_j')\Big)x^{\gamma_i}x^{\theta}\\ &= \sum_{i=1}^{s}\boldsymbol{g}_i\Big(\sum_{j=1}^{k}\psi_{\beta_i,\gamma_i}^{-1}(b_{ji})(x^{\gamma_i}\sigma^{-\gamma_i}(\psi_{\beta_i,\gamma_i}^{-1}(r_j'))+q_{ij}')\Big)x^{\theta},\end{aligned}$$

with $q_{ij}' = 0$ or $\operatorname{lm}(q_{ij}') \prec x^{\gamma_i}$. Since $\psi_{\beta_i,\gamma_i}^{-1}(r) = \sigma^{\gamma_i}(\sigma^{-(\gamma_i+\beta_i)}(r)) = \sigma^{\gamma_i}(\sigma^{-\exp(\boldsymbol{X}_F)}(r))$, then we obtain

$$\begin{aligned}\sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}(d_i')x^{\gamma_i}x^{\theta} &= \sum_{i=1}^{s}\boldsymbol{g}_i\Big(\sum_{j=1}^{k}\psi_{\beta_i,\gamma_i}^{-1}(b_{ji})(x^{\gamma_i}\sigma^{-\exp(\boldsymbol{X}_F)}(r_j')+q_{ij}')\Big)x^{\theta}\\ &= \sum_{j=1}^{k}\sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}(b_{ji})x^{\gamma_i}r_jx^{\theta} + \sum_{i=1}^{s}\sum_{j=1}^{k}\boldsymbol{g}_iq_{ij}x^{\theta}\\ &= \sum_{j=1}^{k}\Big(\sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}(b_{ji})x^{\gamma_i}\Big)r_jx^{\theta} + \sum_{i=1}^{s}\boldsymbol{g}_iq_i,\end{aligned}$$

where $q_{ij} := \psi_{\beta_i,\gamma_i}^{-1}(b_{ji})q_{ij}'$, $r_j := \sigma^{-\exp(\boldsymbol{X}_F)}(r_j')$ and so, $q_i := \sum_{j=1}^{k} q_{ij}x^{\theta} = 0$ or $\operatorname{lm}(q_i) \prec x^{\theta+\gamma_i}$. Finally we get,

$$\sum_{i=1}^{s}\boldsymbol{g}_ic_ix^{\alpha_i} = \sum_{j=1}^{k}\left(\sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}(b_{ji})x^{\gamma_i}\right)r_jx^{\delta-\exp(\boldsymbol{X}_F)} + \sum_{i=1}^{s}\boldsymbol{g}_iz_i,$$

with $z_i := p_i + q_i$ for $1 \le i \le s$. Is easy to see that $\operatorname{lm}(\sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}(b_{ji})x^{\gamma_i}r_jx^{\theta}) \prec \boldsymbol{X}_\delta$ since $\operatorname{lm}(\sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}(b_{ji})x^{\gamma_i}) \prec x^{\gamma_i+\beta_i}$, and $\operatorname{lm}(\boldsymbol{g}_iz_i) = \operatorname{lm}(\boldsymbol{g}_ip_i + \boldsymbol{g}_iq_i) \prec \boldsymbol{X}_\delta$. □

Under the notation used in Definition 5.1 and Lemma 5.3, we will prove the main result of the present section.

Theorem 5.4. Let $M \neq 0$ be a right submodule of A^m and let G be a finite subset of non-zero generators of M. Then the following conditions are equivalent.

(i) G is a Gröbner basis of M.

(ii) For all $F := \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_s\} \subseteq G$, with $\boldsymbol{X}_F \neq \boldsymbol{0}$ and for any $(b_1, \dots, b_s) \in B_F$, we have

$$\sum_{i=1}^{s}\boldsymbol{g}_i\psi_{\beta_i,\gamma_i}^{-1}(b_i)x^{\gamma_i} \xrightarrow{G}_{+} \boldsymbol{0}.$$

Proof:
(i) $\Rightarrow$ (ii): We observe that $\boldsymbol{f} := \sum_{i=1}^{s} \boldsymbol{g}_i \psi_{\beta_i,\gamma_i}^{-1}(b_i) x^{\gamma_i} \in M$. Then Theorem 4.2 yields $\boldsymbol{f} \xrightarrow{G}_+ \boldsymbol{0}$.
(ii) $\Rightarrow$ (i): Assume that $\boldsymbol{0} \neq \boldsymbol{f} \in M$. We will prove that the condition (iii) of Theorem 4.2 holds. If $G := \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$, then there exist $h_1, \ldots, h_t \in A$ such that $\boldsymbol{f} = \boldsymbol{g}_1 h_1 + \cdots + \boldsymbol{g}_t h_t$ and we can choose $\{h_i\}_{i=1}^{t}$ such that $\boldsymbol{X}_\delta := \max\{\operatorname{lm}(\operatorname{lm}(\boldsymbol{g}_i)\operatorname{lm}(h_i))\}_{i=1}^{t}$ is minimal. Let $\operatorname{lm}(h_i) := x^{\alpha_i}$, $c_i := \operatorname{lc}(h_i)$, $\operatorname{lm}(\boldsymbol{g}_i) := x^{\beta_i}$ for $1 \leq i \leq t$ and $F := \{\boldsymbol{g}_i \in G \mid \operatorname{lm}(\operatorname{lm}(\boldsymbol{g}_i)\operatorname{lm}(h_i)) = \boldsymbol{X}_\delta\}$. Up to a renumbering the elements of G, we can assume that $F = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_s\}$. We will consider two possible cases.

Case 1: $\operatorname{lm}(\boldsymbol{f}) = \boldsymbol{X}_\delta$. Then $\operatorname{lm}(\boldsymbol{g}_i) \mid \operatorname{lm}(\boldsymbol{f})$ for $1 \leq i \leq s$ and

$$\operatorname{lc}(\boldsymbol{f}) = \sum_{i=1}^{s} \operatorname{lc}(\boldsymbol{g}_i)\sigma^{\beta_i}(c_i)c_{\beta_i,\alpha_i} = \sum_{i=1}^{s} \operatorname{lc}(\boldsymbol{g}_i)c_{\beta_i,\alpha_i}\psi_{\beta_i,\alpha_i}(c_i).$$

i.e., the condition (iii) of Theorem 4.2 holds.

Case 2: $\operatorname{lm}(\boldsymbol{f}) \prec \boldsymbol{X}_\delta$. We will prove that this yields a contradiction. To begin, note that f can be written as

$$\boldsymbol{f} = \sum_{i=1}^{s} \boldsymbol{g}_i c_i x^{\alpha_i} + \sum_{i=1}^{s} \boldsymbol{g}_i (h_i - c_i x^{\alpha_i}) + \sum_{i=s+1}^{t} \boldsymbol{g}_i h_i. \tag{13}$$

We have $\operatorname{lm}(\boldsymbol{g}_i(h_i - c_i x^{\alpha_i})) \prec \boldsymbol{X}_\delta$ for each $1 \leq i \leq s$ and $\operatorname{lm}(\boldsymbol{g}_i h_i) \prec \boldsymbol{X}_\delta$ for each $s+1 \leq i \leq t$. Hence

$$\operatorname{lm}(\sum_{i=1}^{s} \boldsymbol{g}_i(h_i - c_i x^{\alpha_i})) \prec \boldsymbol{X}_\delta \text{ and } \operatorname{lm}(\sum_{i=s+1}^{t} \boldsymbol{g}_i h_i) \prec \boldsymbol{X}_\delta,$$

and $\operatorname{lm}(\sum_{i=1}^{s} \boldsymbol{g}_i c_i x^{\alpha_i}) \prec \boldsymbol{X}_\delta$. Under the notation used in Lemma 5.3 (and its notation), we have

$$\sum_{i=1}^{s} \boldsymbol{g}_i c_i x^{\alpha_i} = \sum_{j=1}^{k} \Big(\sum_{i=1}^{s} \boldsymbol{g}_i \psi_{\beta_i,\gamma_i}^{-1}(b_{ji}) x^{\gamma_i} \Big) r_j x^{\delta - \exp(\boldsymbol{X}_F)} + \sum_{i=1}^{s} \boldsymbol{g}_i z_i, \tag{14}$$

where $\operatorname{lm}(\sum_{i=1}^{s} \boldsymbol{g}_i \psi_{\beta_i,\gamma_i}^{-1}(b_{ji}) x^{\gamma_i} x^{\delta - \exp(\boldsymbol{X}_F)}) \prec \boldsymbol{X}_\delta$ for each $1 \leq j \leq k$ and $\operatorname{lm}(\boldsymbol{g}_i z_i) \prec \boldsymbol{X}_\delta$ for $1 \leq i \leq s$. By the hypothesis, $\sum_{i=1}^{s} \boldsymbol{g}_i \psi_{\beta_i,\gamma_i}^{-1}(b_{ji}) x^{\gamma_i} \xrightarrow{G}_+ \boldsymbol{0}$, whence, by Theorem 3.8, there exist $q_1, \ldots, q_t \in A$ such that

$$\sum_{i=1}^{s} \boldsymbol{g}_i \psi_{\beta_i,\gamma_i}^{-1}(b_{ji}) x^{\gamma_i} = \sum_{i=1}^{t} \boldsymbol{g}_i q_i,$$

with $\operatorname{lm}(\sum_{i=1}^{s} \boldsymbol{g}_i \psi_{\beta_i,\gamma_i}^{-1}(b_{ji}) x^{\gamma_i}) = \max\{\operatorname{lm}(\operatorname{lm}(\boldsymbol{g}_i)\operatorname{lm}(q_i))\}_{i=1}^{t}$. Since $(b_{j1}, \ldots, b_{js}) \in B_F$, then using the equation (10) of Remark 3.7, we get

$$\operatorname{lc}(\sum_{i=1}^{s} \boldsymbol{g}_i \psi_{\beta_i,\gamma_i}^{-1}(b_{ji}) x^{\gamma_i}) = \sum_{i=1}^{s} \operatorname{lc}(\boldsymbol{g}_i)\sigma^{\beta_i}(\psi_{\beta_i,\gamma_i}^{-1}(b_{ji}))c_{\beta_i,\gamma_i} = \sum_{i=1}^{s} \operatorname{lc}(\boldsymbol{g}_i)c_{\beta_i,\gamma_i} b_{ji} = 0.$$

Hence $\operatorname{lm}\Big(\sum_{i=1}^{s} \boldsymbol{g}_i \psi^{-1}_{\beta_i,\gamma_i}(b_{ji})x^{\gamma_i}\Big) \prec \boldsymbol{X}_F$ and $\operatorname{lm}(\operatorname{lm}(\boldsymbol{g}_i)\operatorname{lm}(q_i)) \prec \boldsymbol{X}_F$ for each $1 \leq i \leq t$. Therefore,

$$\begin{aligned}\sum_{j=1}^{k}\Big(\sum_{i=1}^{s} \boldsymbol{g}_i \psi^{-1}_{\beta_i,\gamma_i}(b_{ji})x^{\gamma_i}\Big) r_j x^{\delta-\exp(\boldsymbol{X}_F)} &= \sum_{j=1}^{k}\Big(\sum_{i=1}^{t} \boldsymbol{g}_i q_i\Big) r_j x^{\delta-\exp(\boldsymbol{X}_F)} \\ &= \sum_{i=1}^{t}\sum_{j=1}^{k} \boldsymbol{g}_i q_i r_j x^{\delta-\exp(\boldsymbol{X}_F)} \\ &= \sum_{i=1}^{t} \boldsymbol{g}_i \widetilde{q}_i,\end{aligned}$$

with $\widetilde{q}_i := \sum_{j=1}^{k} q_i r_j x^{\delta-\exp(\boldsymbol{X}_F)}$ and $\operatorname{lm}(\boldsymbol{g}_i\widetilde{q}_i) \prec \boldsymbol{X}_\delta$ for each $1 \leq i \leq t$. Substituting $\sum_{i=1}^{s} \boldsymbol{g}_i c_i x^{\alpha_i} = \sum_{i=1}^{t} \boldsymbol{g}_i \widetilde{q}_i + \sum_{i=1}^{s} \boldsymbol{g}_i z_i$ into equation (13), we obtain

$$f = \sum_{i=1}^{t} \boldsymbol{g}_i \widetilde{q}_i + \sum_{i=1}^{s} \boldsymbol{g}_i (h_i - c_i x^{\alpha_i}) + \sum_{i=1}^{s} \boldsymbol{g}_i z_i + \sum_{i=s+1}^{t} \boldsymbol{g}_i h_i,$$

and so we have expressed $\boldsymbol{f}$ as a combination of vectors $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t$, where each of its terms has leading monomial $\prec \boldsymbol{X}_\delta$. This contradicts the minimality of $\boldsymbol{X}_\delta$ and finishes the proof. □

Corollary 5.5. Let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\}$ be a set of non-zero vectors of A^m. The algorithm below (Algorithm 2) produces a Gröbner basis for the right submodule $\langle F \rangle$ of A^m, where $P(X)$ denotes the set of subsets of the set X.

Algorithm 2: Right Buchberger's algorithm in A^m

Input: $F := \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s\} \subseteq A^m, \boldsymbol{f}_i \neq \boldsymbol{0}, 1 \leq i \leq s$
Output: $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t\}$ a Gröbner basis for $\langle F \rangle$
Initialization: $G \leftarrow \emptyset, G' \leftarrow F$;
while $G' \neq G$ **do**
  $D \leftarrow P(G') - P(G)$;
  $G \leftarrow G'$;
  for *each* $S := \{\mathbf{g}_{i_1}, \ldots, \mathbf{g}_{i_k}\} \in D$ *with* $\mathbf{X}_S \neq \mathbf{0}$ **do**
    Compute B_S;
    for *each* $\mathbf{b} = (b_1, \ldots, b_k) \in B_S$ **do**
      Reduce $\sum_{j=1}^{k} \boldsymbol{g}_{i_j} \psi^{-1}_{\beta_{i_j},\gamma_j}(b_j)x^{\gamma_j} \xrightarrow{G'}_{+} \boldsymbol{r}$; # with r reduced with respect to G'; β_{i_j}, γ_j as in Def 5.1
      if $\mathbf{r} \neq \mathbf{0}$ **then**
        $G' \leftarrow G' \cup \{\boldsymbol{r}\}$;
      end
    end
  end
end

We finish this section by the following useful result.

Corollary 5.6. Every right submodule of the free A-module A^m has a Gröbner basis.

Proof:
Apply Theorem 5.4 and Corollary 5.5. □

6. Examples implemented in `SPBWE` library

The extensions skew PBW extensions were implemented in Maple with the development of the `SPBWE` library (see [1]), which allows to make important computations with this type of rings and can also provide answers to several homological problems such as the computation of syzygies; within the library are already developed the algorithms that we present in this paper: the right division algorithm and the right Buchberger algorithm, below here we will present only a brief view of its execution.

Example 6.1. Consider the diffusion algebra $A := \sigma(\mathbb{Q}[x_1, x_2])\langle D_1, D_2\rangle$ subject to relation:

$$D_2D_1 = 2D_1D_2 + x_2D_1 - x_1D_2.$$

Taking the following polynomials in A

$$f := x_1x_2^2D_1^2D_2 + x_1^2x_2D_2 : \quad f_1 := x_1^2x_2D_1D_2 : \quad f_2 := x_2D_1 : \quad f_3 := x_2D_1 :$$

We use the right division algorithm over these polynomials as follow and we get polynomials $g_1 = \frac{1}{2}x_2D_1 + \frac{1}{2}x_1x_2$, $g_2 = -\frac{x_1x_2^2D_1}{2}$ and $g_3 = x_1x_2$, such that

$$f = f_1g_1 + f_2g_2 + f_3g_3.$$

Therefore,

$$x_1x_2^2D_1^2D_2 + x_1^2x_2D_2 \in \langle x_1^2x_2D_1D_2, x_2D_1, x_2D_1\rangle_A$$

with

$$x_1x_2^2D_1^2D_2 + x_1^2x_2D_2 = x_1^2x_2D_1D_2(\tfrac{1}{2}x_2D_1 + \tfrac{1}{2}x_1x_2) + x_2D_1\left(-\tfrac{x_1x_2^2D_1}{2}\right) + x_2D_1(x_1x_2).$$

The following example is a non-trivial instance of applicability of the `SPBWE` library. In particular, it is possible to define iterated skew PBW extensions in the library and compute left or right Gröbner bases over theses.

Example 6.2. Let $A = \mathbb{C}[w, \varphi]$ the skew polynomial ring of endomorphism type with $\varphi(q) = \overline{q}$, for $q \in \mathbb{C}$. Using the `SPBWE`, we can to define the extension $C = \sigma(A)\langle x, y, z\rangle$, subjects to relations

$$yx = 2xy, \quad zx = 4xz - x, \quad zy = 4yz - y,$$

with A-endomorphisms $\sigma_i : \sigma_1(w) = 2w$, $\sigma_2(w) = 3w$, $\sigma_3(w) = w$, and σ_i-derivations $\delta_i = 0$ for $i = 1, 2, 3$.

Consider the right submodule $M := \langle f_1, f_2, f_3, f_4\rangle_C \subseteq C^4$, with

$$\begin{aligned}
\boldsymbol{f}_1 &= (-y^2, -wy+y, y, wx-xy),\\
\boldsymbol{f}_2 &= (-wy-y, xy+y^2-w, w^2, w^2+wx+wy),\\
\boldsymbol{f}_3 &= (-x+1, -wy+x^2+xy, w^2+wx+w, x),\\
\boldsymbol{f}_4 &= (y^2+x, wx+1, 0, w^2+wy-y^2).
\end{aligned}$$

Let $v := \boldsymbol{f}_1 p_1 + \boldsymbol{f}_2 p_2 + \boldsymbol{f}_3 p_3 + \boldsymbol{f}_4 p_4 \in M$, with $p_1 := 76x+95z$, $p_2 := xz$, $p_3 := w - iy$ and $p_4 := iz$, next, we will use the SPBWE library, in particular, the right division algorithm and the Buchberger algorithm over C, to verify that v lies in M.

First, we use the right division algorithm on V and M as follow

$$V := \begin{bmatrix} -304xy^2+(-2w-2)xyz+(-95+I)y^2z+Ixy+Ixz-2wx-Iy+w \\ 2x^2yz+4xy^2z-Ix^2y-Ixy^2+4wx^2+(-146w+152)xy+(-1-I)wxz-Iwy^2+(-95w+95)yz-3w^2y+Iz \\ (Iw+152)xy+w^2xz+95yz+2w^2x+(-Iw^2+Iw)y+w^3+w^2 \\ -152x^2y+wx^2z+(2w-95)xyz-Iy^2z+76wx^2-Ixy+(w^2+95w)xz-Iwyz+2wx+Iw^2z \end{bmatrix}:$$

$$M := \begin{bmatrix} -y^2 & -wy+y & y & wx-xy \\ -wy-y & xy+y^2-w & w^2 & w^2+wx+wy \\ -x+1 & -wy+x^2+xy & w^2+wx+w & x \\ y^2+x & wx+1 & 0 & w^2+wy-y^2 \end{bmatrix}:$$

```
>  DivisionAlgorithm(V, M, gradlexrev, TOP, C, right)
```

We obtain four polynomials $q_1 = 76x+95z$, $q_2 = xz - \frac{1}{2}Ix + Iy$, $q_3 = q_4 = 0$ and a vector

$$\boldsymbol{h} = \begin{bmatrix} Iy^2z + Iwxy + Ixz + (-Iw+I)y^2 - 2wx - Iy + w \\ -Iy^3 + 4wx^2 + 6wxy - Iwxz - Iwy^2 + \frac{1}{2}Iwx + (-3w^2 - Iw)y + Iz \\ Iwxy + \left(2+\frac{1}{2}I\right)w^2x + (-2Iw^2+Iw)y + w^3 + w^2 \\ -Iy^2z - \frac{1}{2}Iwx^2 - Ixy + Iwy^2 - Iwyz + \left(\frac{1}{2}Iw^2+2w\right)x - Iw^2y + Iw^2z \end{bmatrix} \in C^4$$

such that

$$\boldsymbol{f} = \boldsymbol{f}_1 q_1 + \boldsymbol{f}_2 q_2 + \boldsymbol{f}_3 q_3 + \boldsymbol{h}.$$

Since $\boldsymbol{h} \neq \boldsymbol{0}$, we have a second option. For this purpose, we use the following statement in Maple

```
>  G := BuchbergerAlgSkewPoly(M, gradlex, TOP, C, right)
```

We obtain a Gröbner basis of M, $G = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3, \boldsymbol{h}_4, \boldsymbol{h}_5, \boldsymbol{h}_6, \boldsymbol{h}_7, \boldsymbol{h}_8\}$, with

$$\boldsymbol{h}_1 = \boldsymbol{f}_1,\ \boldsymbol{h}_2 = \boldsymbol{f}_2,\ \boldsymbol{h}_3 = \boldsymbol{f}_3,\ \boldsymbol{h}_4 = \boldsymbol{f}_4,$$

$$\boldsymbol{h}_5 = \begin{bmatrix} -xy^2-y^3-\frac{1}{4}x^2 \\ -\frac{1}{4}wx^2+(-w+1)y^2-\frac{1}{4}x \\ y^2 \\ \frac{1}{2}wxy-\frac{1}{4}w^2x \end{bmatrix}, \ \boldsymbol{h}_6 = \begin{bmatrix} -2wxy+(2w+2)y^2-2y \\ -2y^3+2wy^2-wx+2wy \\ -2wxy+w^2x+(-4w^2-2w)y \\ wx^2-2xy-2wy^2+w^2x-2w^2y \end{bmatrix},$$

$$\boldsymbol{h}_7 = \begin{bmatrix} -\frac{1}{4}wx^2y+(-\frac{1}{2}w+\frac{1}{2})xy^2-\frac{1}{4}xy-y^2 \\ \frac{1}{2}wxy^2+wy^3-\frac{1}{16}wx^2-\frac{1}{4}wxy \\ -\frac{1}{4}wx^2y-wxy^2+\frac{1}{16}w^2x^2-\frac{1}{4}wxy+(-w^2-w)y^2 \\ \frac{1}{16}wx^3+(\frac{1}{2}w-\frac{1}{4})x^2y+(\frac{1}{2}w-1)xy^2+\frac{1}{16}w^2x^2+\frac{1}{4}w^2xy \end{bmatrix},$$

$$\boldsymbol{h}_8 = \begin{bmatrix} \frac{5}{4}wx^2y^2+(4w-2)xy^3+\frac{1}{2}xy^2+2y^3 \\ -wxy^3-2wy^4+(\frac{1}{16}w^2+\frac{1}{16}w)x^2y+(\frac{1}{2}w^2-\frac{1}{2}w+\frac{1}{2})xy^2 \\ \frac{1}{2}wx^2y^2+2wxy^3+(-\frac{1}{8}w^2-\frac{1}{16}w)x^2y+\frac{1}{2}xy^2+(2w^2+2w)y^3 \\ (-w+2)xy^3-\frac{1}{96}w^2x^3+(-\frac{7}{24}w^2+\frac{1}{4}w)x^2y-\frac{1}{2}w^2xy^2 \end{bmatrix}.$$

Finally, using the statement

> $>$ `DivisionAlgorithm(`V, G, `gradlex, TOP,` C, `right)`

we obtain eight polynomials $q_1 = 76x + 95z$, $q_2 = xz - 1/2Ix + Iy$, $q_3 = w$, $q_4 = iz$, $q_6 = 1/2i$ and $q_5 = q_7 = q_8 = 0$ such that

$$\boldsymbol{h} = \boldsymbol{h}_1q_1 + \boldsymbol{h}_2q_2 + \boldsymbol{h}_3q_3 + \boldsymbol{h}_4q_4 + \boldsymbol{h}_5q_5 + \boldsymbol{h}_6q_6 + \boldsymbol{h}_7q_7 + \boldsymbol{h}_8q_8.$$

Consequently, the vector h lies in M.

7. Future perspective

As consequence of the algorithms presented in this paper over a bijective skew PBW extension A, we can respond to problems of homological algebra such as: computation of the right module of syzygies of a right A-module M; computation of a right inverse of rectangular matrix on A; computation of the intersection and quotient for ideals or modules over A; computation of the $\text{Ext}^r_A(M, N)$, where M is a finitely generated left A-submodule of A^m and N is a finitely generated centralizing A-subbimodule of A^l; among another applications. Now is possible to complete the SPBWE library and provide support in areas of non-commutative algebra that have not yet implemented computationally.

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