

Order-theoretic Trees: Monadic Second-order Descriptions and Regularity

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Abstract. An *order-theoretic forest* is a countable partial order such that the set of elements larger than any element is linearly ordered. It is an *order-theoretic tree* if any two elements have an upper-bound. The order type of a *branch* (a maximal linearly ordered subset) can be any countable linear order. Such generalized infinite trees yield convenient definitions of the rank-width and the modular decomposition of countable graphs.

We define an algebra based on only four operations that generate up to isomorphism and *via* infinite terms these order-theoretic trees and forests. We prove that the associated *regular* objects, *i.e.*, those defined by regular terms, are exactly the ones that are the unique models of monadic second-order sentences.

We adapt some tools that we have used in a previous article for proving a similar result for *join-trees*, *i.e.*, for order-theoretic trees such that any two nodes have a least upper-bound.

Keywords: Order-theoretic tree, algebra, regular term, monadic second-order logic.

1. Introduction

Countably infinite trees have been studied in Fundamental Computer Science for many purposes: semantics of programs [2, 8], theory of games [3, 18, 20] and distributed computing [1, 14], just to name a few topics. To be usable for obtaining algorithms, these trees must have finitary descriptions: several notions have been developed for this purpose, *e.g.*, automata of various types, logical descriptions and equation systems [4, 7].

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In the logical setting of the *Theory of Relations* [15], a more general notion of tree is used : an *order-theoretic forest* (an *O-forest* in short) is a countable partial order whose elements are called *nodes* and such that the set of nodes larger than any node is linearly ordered; it is an *order-theoretic tree* (an *O-tree*) if any two nodes have an upper-bound and a *join-tree* if any two nodes have a least upper-bound. The order type of a linearly ordered subset can be any, possibly dense, linear order. Join-trees yield convenient definitions of the rank-width and the modular decomposition of countable graphs [9, 12].

Finitary descriptions of these objects can be given by logical sentences. For an example, the ordered set of rational numbers is, up to isomorphism, the unique linearly ordered countable set that is dense without maximal and minimal elements. This characterization is first-order (FO) expressible. First-order logic is actually not powerful enough for characterizing the structures we are interested in. Monadic second-order (MSO) logic is much more expressive, and furthermore, its decidability on infinite trees is a fundamental result due to Rabin (see, *e.g.* [18]). This fundamental result is based on an equivalence between certain finite automata and MSO descriptions. Such an equivalence was proved for finite words by Trakhtenbrot [19] and, independently, by others for finite trees. MSO formulas can also express transformations of structures that are very useful in proofs.

The rooted trees used in semantics are, in many cases, infinite terms (like are formal power series) built with finite sets of function symbols. We call them *F-terms*, where *F* is the set of function symbols: they can be conveniently handled as labelled rooted trees, and we will discuss them by using notions concerning trees: nodes, root, ancestor etc.

An *F-term* is *regular* if it has finitely many different subterms, equivalently if it is a component of the unique solution of a finite equation system of a certain type. Such systems yield finitary descriptions of regular *F-terms*. More complex equation systems yield wider notions of *F-terms* [4, 7] whose MSO-theory is decidable, so that some of our results are applicable to them.

The existence of equivalent characterizations of a same class of objects indicates a certain *robustness* as this class does not depend on a choice of definitions that may sound arbitrary. In particular, an *F-term* is regular if and only if it is the unique model of an MSO sentence. (This characterization uses a description of *F-terms* by logical structures).

Our objective is to obtain finitary descriptions of certain O-trees and O-forests, in particular by regular terms. For this purpose, we define an algebra structure on the class of O-forests that uses only three operations¹. These operations can generate all O-trees up to isomorphism *via* infinite terms. The *regular O-trees* are those defined by *regular F-terms*. Our main theorem states that an O-tree is regular if and only if it is *monadic second-order definable*, *i.e.*, is the unique model (up to isomorphism) of an MSO sentence. In this way, we extend the corresponding result obtained for join-trees in [11]. The proof uses the corresponding result for *arrangements* [17]. Arrangements are labelled linear orders, hence are generalized words whose ordered sets of positions may be dense.

A main technical tool is the notion of *structuring* of an O-forest : it is a partition in convex linearly ordered subsets that form a kind of tree. A regular O-forest has a structuring consisting of regular, whence MSO-definable, arrangements; furthermore, the tree of these arrangements is in some sense regular. These notions are collected in a finitary *regular description scheme*. A regular O-tree is

¹An additional constant denotes the empty O-forest.

definable, equivalently, by a regular F -term, by a regular description scheme and by an MSO sentence that can use a finiteness set predicate. (Such a use is necessary).

All constructions are effective (although intractable) and so, the isomorphism of two regular O-trees is decidable.

In Section 2 we review definitions and results concerning partial orders, rooted trees, F -terms, arrangements and monadic second-order logic. In Section 3, we define O-forests and O-trees, their structurings and descriptions schemes. In Section 4 we define the algebra of structured O-forests. We get the notion of a regular O-forest and we prove the main theorem.

2. Definitions and a few lemmas

In the present article, all ordered sets, trees and logical structures are *countable*, which means finite or countably infinite. We denote by $X \uplus Y$ the union of sets X and Y if we want to insist that they are disjoint. Isomorphism of ordered sets, trees and other logical structures is denoted by \simeq . The restriction of a relation R or a function f defined on a set V to a subset W is denoted by $R \upharpoonright W$ or $f \upharpoonright W$ respectively.

2.1. Orders

Notation and definitions 2.1

For partial orders \leq, \sqsubseteq, \dots we denote respectively by $<, \sqsubset, \dots$ the corresponding strict partial orders. In many cases a partial order \leq can be defined in a short way from the corresponding strict partial order $<$ so that $x \leq y$ holds if and only if $x = y$ or $x < y$.

Let (V, \leq) be a partial order. For subsets X, Y of V , $X < Y$ means that $x < y$ for every $x \in X$ and $y \in Y$. We write $X < y$ instead of $X < \{y\}$ and similarly for $x < Y$. The least upper bound of x and y is denoted by $x \sqcup y$ if it exists and is called their *join*. The notation $x \perp y$ means that x and y are incomparable. A subset Y of V is *convex* if $y \in Y$ whenever $x, z \in Y$ and $x < y < z$. A *line*² is a convex subset of V that is linearly ordered. Particular notations for convex sets (that are not necessarily linearly ordered) are $[x, y]$ denoting $\{z \mid x \leq z \leq y\}$, $] - \infty, x[$ denoting $\{y \mid y < x\}$ (even if V is finite), $[x, +\infty[$ denoting $\{y \mid x \leq y\}$. In a linearly ordered set, a line is called an *interval*.

Let (N, \leq) and (N', \le') be partial orders. An *embedding* $j : (N, \leq) \rightarrow (N', \le')$ is an injective order-preserving map such that $x \leq y$ if and only if $j(x) \le' j(y)$; in this case, (N, \leq) is isomorphic by j to $(j(N), \le' \upharpoonright j(N))$, a suborder of (N', \le') . We say that j is a *join-embedding* if, furthermore, $j(x \sqcup y) = j(x) \sqcup j(y)$ whenever $x \sqcup y$ is defined.

The first infinite ordinal and the linear order (\mathbb{N}, \leq) are denoted by ω .

Definitions 2.2 : Cuts in linear orders.

(a) A (Dedekind) *cut* of a linear order (U, \leq) is a pair (U_1, U_2) of non-empty intervals such that $U = U_1 \uplus U_2$ and $U_1 < U_2$. For an example, an element $x \in U$ that is not maximal defines the cut $(] - \infty, x],]x, +\infty[)$.

²In [9] we called *line* a linearly ordered subset, without imposing the convexity property. The present definition is from [11]

(b) If K is a set of cuts, we extend \leq into a linear order on $U \uplus K$, also denoted by \leq , such that:

$$\begin{aligned} x &< (U_1, U_2) \text{ if } x \in U_1 \text{ (equivalently } x < U_2), \\ (U_1, U_2) &< x \text{ if } x \in U_2 \text{ (equivalently } x > U_1), \\ (U_1, U_2) &< (U'_1, U'_2) \text{ if } U_1 \subset U'_1 \text{ (equivalently } U'_2 \subset U_2). \end{aligned}$$

We denote $U_K := (U \uplus K, \leq)$.

2.2. Functional trees

Functional trees are the infinite terms written with function symbols of fixed arity that are used in semantics of program ([7, 8]) and also (see below Section 2.3) for defining labelled linear orders called *arrangements*. These trees can be formalized in different ways. In order to avoid useless technicalities, we limit definitions to function symbols of arity at most 2, as we will only use this case. We call them F -terms to stress their algebraic nature.

Definitions 2.3 : *Functional trees, equivalently, infinite terms.*

(a) A *binary tree-domain* is a set of words $D \subseteq \{1, 2\}^*$ (called *Dewey words*) that is *prefix-closed*, which means that $u \in D$ if $uv \in D$, and is such that $u1 \in D$ if $u2 \in D$. We consider (D, \leq) as a rooted tree, where the *ancestor relation* \leq reverses the prefix order on words (we get $uv \leq u$). Its root is the empty word ε .

(b) Let F be a finite *binary functional signature*, i.e., a finite set of symbols equipped with an *arity mapping* $\rho : F \rightarrow \{0, 1, 2\}$. A *functional tree over F* , called an F -term for short is a pair $t = (D_t, \text{lab}_t)$ consisting of a binary tree-domain D_t and a mapping $\text{lab}_t : D_t \rightarrow F$ such that, for every u in D_t , $\rho(\text{lab}_t(u))$ is the number of integers $i \in \{1, 2\}$ such that $ui \in D_t$. (See Examples 2.5). The set of F -terms is denoted by $T^\infty(F)$ and the subset of finite ones by $T(F)$. We call $u \in D_t$ a *position* in t (or a *node* if we think of t as a tree); it is an *occurrence* of the function symbol $\text{lab}_t(u)$. We denote also by $\text{Pos}(t)$ the set of positions of t .

The ancestor relation on $\text{Pos}(t)$ is denoted by \leq_t .

(c) If $t = (D_t, \text{lab}_t)$ and $u \in D_t$, then t/u is the F -term with set of positions $D_t/u := \{w \in \{1, 2\}^* \mid uw \in D_t\}$ and labelling function such that $\text{lab}_{t/u}(w) := \text{lab}_t(uw)$. We call t/u the *subterm issued from u* . If $u \neq v$ and the F -terms t/u and t/v are equal, then the corresponding subtrees³ of the labelled rooted tree t are isomorphic but not equal, because their sets of nodes are $\{uw \in \{1, 2\}^* \mid uw \in D_t\}$ and $\{vw \in \{1, 2\}^* \mid vw \in D_t\}$.

(d) We recall that the *lexicographic order* on $\{1, 2\}^*$ is defined as follows:

$$u \leq_{\text{lex}} v \text{ if and only if } v = uv' \text{ or } u = w1u' \text{ and } v = w2v' \text{ for some words } w, u' \text{ and } v'.$$

It is a linear order. Hence, it defines a linear order on every tree-domain. Another order will be defined in Definition 2.17. \square

³In the sense of graph theory.

Definitions 2.4 : *Operations on F -terms.*

(a) Let $f \in F$ be of arity 2. Let t_1 and $t_2 \in T^\infty(F)$. Then $t := f(t_1, t_2)$ is defined as follows:

$$D_t := \{\varepsilon\} \uplus 1D_{t_1} \uplus 2D_{t_2},$$

$$\text{lab}_t(\varepsilon) := f, \text{lab}_t(1u) := \text{lab}_{t_1}(u), \text{lab}_t(2u) := \text{lab}_{t_2}(u).$$

If f has arity 1 and $t_1 \in T^\infty(F)$, then, $t := f(t_1)$ is defined as follows:

$$D_t := \{\varepsilon\} \uplus 1D_{t_1},$$

$$\text{lab}_t(\varepsilon) := f \text{ and } \text{lab}_t(1u) := \text{lab}_{t_1}(u).$$

If $t = f$, a nullary symbol, then $D_t := \{\varepsilon\}$ and $\text{lab}_t(\varepsilon) := f$. \square

The link between trees and terms is best illustrated by examples.

Examples 2.5 : Let $F := \{f, g, a, b\}$ where these symbols have respective arities 2,1,0 and 0.

(1) The term $t = f(g(a), f(a, b))$ has domain $D_t = \{\varepsilon, 1, 2, 11, 21, 22\}$ and labelling function lab_t such that :

$$\varepsilon \mapsto f, 1 \mapsto g, 2 \mapsto f, 11 \mapsto a, 21 \mapsto a, 22 \mapsto b.$$

(2) The infinite term $s = f(a, f(g(b), f(a, f(g(b), \dots))))$ has domain $D_s := 2^* \uplus (22)^*1 \uplus 2(22)^*1 \uplus 2(22)^*11$ and labelling function lab_s such that :

$$2^i \mapsto f, (22)^i1 \mapsto a, 2(22)^i1 \mapsto g, 2(22)^i11 \mapsto b \text{ for all } i \geq 0. \quad \square$$

Definition 2.6 : *Regular terms.*

An F -term t is *regular* if the set of its subtrees is finite. Every finite F -term is regular.

Proposition 2.10 states equivalent characterizations of regular terms, for which we review some definitions.

Definition 2.7 : *Automata.*

(a) A *finite automaton* over F (as in Definition 2.3(b)) is (in the present article) a tuple $\mathcal{B} = (F, S, \tau, s_{init})$ where S is the finite set of *states*, s_{init} is the *initial state* and τ is the *transition function*, a total mapping :

$$S \rightarrow F \times ((S \times S) \uplus S \uplus \{\varepsilon\})$$

such that, for each state s , $\tau(s)$ is either (f, s_1, s_2) if f has arity 2, or (f, s_1) if f has arity 1 or (f, ε) if f has arity 0, for some $s_1, s_2 \in S$ and $f \in F$.

(b) Let $t \in T^\infty(F)$. The *run* of \mathcal{B} on t is the (unique) mapping $\text{run}_{\mathcal{B}} : \text{Pos}(t) \rightarrow S$ such that $\text{run}_{\mathcal{B}}(\varepsilon) = s_{init}$ and, for every $u \in \text{Pos}(t)$, if u is an occurrence of f and $\text{run}_{\mathcal{B}}(u) = s$, we have the following cases:

if $\rho(f) = 2$, then $\tau(s) = (f, \text{run}_{\mathcal{B}}(u1), \text{run}_{\mathcal{B}}(u2))$,
 if $\rho(f) = 1$, then $\tau(s) = (f, \text{run}_{\mathcal{B}}(u1))$, and
 if $\rho(f) = 0$, then $\tau(s) = (f, \varepsilon)$.

Hence, a state s at a position u defines (via τ) the symbol at u and the states at the positions $u1$ and $u2$ when they do exist. There is at most one run $\text{run}_{\mathcal{B}}$. It is defined, deterministically, in a top-down way.

(c) An F -term t is *accepted* by \mathcal{B} if \mathcal{B} has a run on t . As automata are top-down deterministic, each of them accepts exactly one F -term.

Definition 2.8 : *First-order (FO) and monadic second-order (MSO) logic*

Let \mathcal{R} be a finite set of relation symbols $\{R_1, \dots, R_k\}$, each being given with a positive integer $\rho(R_i)$ called its *arity*.

(a) A \mathcal{R} -relational structure is a tuple $S = (D_S, R_{1S}, \dots, R_{kS})$ where D_S is the domain and R_{iS} is a $\rho(R_i)$ -ary relation on the domain. Its properties will be expressed by first-order (FO) or monadic second-order (MSO) formulas or sentences. A *sentence* is a formula without free variables.

(b) A partial order is by definition a relational structure. For expressing properties of an F -term t , we will use the relational structure:

$$\begin{aligned} \lfloor t \rfloor &:= (D_t, \text{son}_1, \text{son}_2, (\text{lab}_f)_{f \in F}) \text{ where } D_t \text{ is the domain,} \\ \text{son}_i(u, v) &:\iff v = ui, \\ \text{lab}_f(u) &:\iff u \text{ is an occurrence of } f. \end{aligned}$$

(c) The finiteness of a set X is not expressible in MSO logic. Hence, we will use MSO formulas written with a finiteness set predicate $\text{Fin}(X)$, and denote by MSO_{fin} the corresponding extension of MSO logic. However, if $X \subseteq V$, V is partially ordered and X linearly ordered, then its finiteness is MSO-expressible in terms of the order relation of V [11].

Definition 2.9: *Regular equation systems*

Let t_1, \dots, t_n be unknowns denoting F -terms. A *regular equation system* is an n -tuple $(t_i = s_i; i = 1, \dots, n)$ where each s_i is of the form $f(t_j, t_k)$ or $f(t_j)$ or f , the symbol f has arity respectively 2, 1 or 0, and $j, k \in \{1, \dots, n\}$.

A *solution* is an n -tuple of F -terms that satisfy the equations. Every such system has a unique solution (see [7]). \square

Proposition 2.10 : An F -term t is regular if and only if

- (i) it is a component of the unique solution of a regular equation system,
- (ii) it is accepted by a finite automaton,
- (iii) for each $f \in F$, the set of occurrences of f in t is a regular language (included in $\{1, 2\}^*$).
- (iv) the structure $\lfloor t \rfloor$ is the unique model *u.t.i.* (that means *up to isomorphism*) of an MSO sentence.

Proof:

Characterizations (i) and (ii) are clear from the definitions. For (iii) see [7]. For (iv) see [18] \square

Examples 2.11 : (1) The F -term s of Example 2.5(2) is regular, as it is defined by the equation system:

$$\{s = f(r_a, u), u = f(w, s), w = g(r_b), r_a = a, r_b = b\}.$$

The different subterms are s, u, w, r_a and r_b . An automaton accepting it has states s, u, w, r_a, r_b , initial state s and transitions $s \mapsto (f, r_a, u), u \mapsto (f, w, s), w \mapsto (g, r_b), r_a \mapsto (a, \varepsilon), r_b \mapsto (b, \varepsilon)$.

Other examples will be given in Examples 2.16.

(2) Let t be regular and accepted by an automaton $\mathcal{B} = (F, S, \tau, s_{init})$. We let $t_{\mathcal{B}}$ be obtained by replacing a symbol f at position u by the symbol $(f, run_{\mathcal{B}}(u))$. Then $t_{\mathcal{B}}$ is a regular $(F \times S)$ -term defined by an automaton having the same states as \mathcal{B} . The arity of (f, s) is that of f . \square

Definition 2.12 : *MSO definable sets of positions of an F -term.*

Let t be an F -term. Let $\varphi(X_1, \dots, X_n)$ be an MSO formula such that there is a unique tuple X_1, \dots, X_n of sets of positions of t such that $[t] \models \varphi(X_1, \dots, X_n)$. We say that (X_1, \dots, X_n) is *MSO-definable* in t .

Let now $F' := F \times \mathcal{P}([n])$ and t_{φ} the F' -term obtained from t by replacing a symbol f at an occurrence u by the symbol (f, I) of same arity where I is the set of indices i such that $u \in X_i$, where (X_1, \dots, X_n) is defined by φ . We have the following.

Lemma 2.13 : Let t be regular and F' and φ be as above.

(1) The F' -term t_{φ} is regular.

(2) In the case where $n = 1$, the cardinality of the unique set X_1 such that $[t] \models \varphi(X_1)$ can be computed.

Proof:

Easy consequences of Proposition 2.10 together with routine logical manipulations. \square

2.3. Arrangements and labelled sets

We review a notion introduced in [6] and further studied in [16, 17].

Definitions 2.14 : *Arrangements*

(a) Let X be a countable set. A linear order (V, \leq) equipped with a labelling mapping $lab : V \rightarrow X$ is called an *arrangement over X* . We denote by $\mathcal{A}(X)$ the set of arrangements over X . A linear order (V, \leq) is identified with the arrangement (V, \leq, Id) such that $Id(v) := v$ for each $v \in V$.

An arrangement over a finite set X can be considered as a generalized word over alphabet X .

(b) An *isomorphism of arrangements* $i : (V, \leq, lab) \rightarrow (V', \leq', lab')$ is an order preserving bijection $i : V \rightarrow V'$ such that $lab' \circ i = lab$. Isomorphism is denoted by \simeq (as for all structures).

(c) The concatenation of linear orders (denoted by the noncommutative operation $+$) yields a concatenation of arrangements denoted by \bullet . We denote by Ω the empty arrangement and by a the one reduced to a single occurrence of $a \in X$. Clearly, $w \bullet \Omega = \Omega \bullet w = w$ for every $w \in \mathcal{A}(X)$. The

infinite word $w = a^\omega$ is the arrangement over $\{a\}$ with underlying linear order ω ; it is described by the equation $w = a \bullet w$. Similarly, the arrangement $w = a^\eta$ over $\{a\}$ with underlying linear order (\mathbb{Q}, \leq) (that of rational numbers) is described by the equation $w = w \bullet (a \bullet w)$.

Definition 2.15 : *Terms defining arrangements.*

(a) Let X be a set of nullary symbols and $t \in T^\infty(\{\bullet, \Omega\} \uplus X)$. The set $Pos(t)$ of positions of t is the tree-domain $D_t \subseteq \{1, 2\}^*$. The *value* of t is the arrangement $val(t) := (Occ(t, X), \leq_{lex}, lab)$ where $Occ(t, X)$ is the set of positions of the elements of X and $lab(u)$ is the symbol of X occurring at position u . We say that t *denotes* an arrangement w if w is isomorphic to $val(t)$.

(b) An arrangement is *regular* if it is denoted by a regular term. \square

Examples 2.16 : (a) $t_0 := \bullet(a, \bullet(b, \bullet(a, \bullet(b, \bullet(\dots\dots))))))$ denotes the infinite word $abab\dots$. Its value is defined from the set of words $Occ(t_0, \{a, b\}) = 2^*1$, lexicographically ordered⁴ and the labelling function such that $lab(2^i1) := a$ if i is even and $lab(2^i1) := b$ if i is odd. The term t_0 is regular.

(b) The arrangements a^ω and a^η (whose underlying orders are, respectively, the natural and rational numbers) are denoted respectively by t_1 and t_2 that are the unique solutions in $T^\infty(\{\bullet, \Omega, a\})$ of the equations $t_1 = a \bullet t_1$ and $t_2 = t_2 \bullet (a \bullet t_2)$. These two arrangements are regular.

The term t_2 is defined from the two equations $t_2 = t_2 \bullet s$ and $s = a \bullet t_2$. The tree-domains of t_2 and s satisfy the equalities :

$$D_{t_2} = \varepsilon \cup 1D_{t_2} \cup 2D_s \text{ and } D_s = \varepsilon \cup 1 \cup 2D_{t_2},$$

hence D_{t_2} is defined by the regular expression $(1 \cup 22)^*(\varepsilon \cup 2 \cup 21)$. The positions in $(1 \cup 22)^*21$ are the occurrences of a . \square

An arrangement is regular⁵ if and only if it is the value of a *regular expression* in the sense of [16]. The later characterization implies that regularity is preserved under alphabetic homomorphisms : if $w = (V, \leq, lab) \in \mathcal{A}(X)$ is regular and $r : X \rightarrow Y$ is a partial mapping, then $(V', \leq, r \circ lab)$ is a regular arrangement over Y where V' is the set of $x \in V$ such that $r(lab(x))$ is defined⁶.

We will also use the result of [17] that an arrangement over a finite alphabet is regular if and only if it is MSO-definable, where we represent an arrangement $w = (V, \leq, lab)$ over X by the relational structure $[w] := (V, \leq, (lab_a)_{a \in X})$ such that $lab_a(u)$ is true if and only if $lab(u) = a$.

Definition 2.17 : *The inorder on words over $\{1, 2\}$.*

We define as follows a strict linear order $<_{in}$ on $W := \{1, 2\}^*$:

$$x <_{in} y \iff \text{either } x = y1x', \text{ or } y = x2y', \text{ or } x = z1x' \text{ and } y = z2y' \text{ for some words } x', y', z.$$

In the last two cases, we have $x <_{lex} y$. \square

⁴We have $1 < 21 < 221 < \dots$

⁵Equivalently is a component of the *initial solution of a regular equation system* over $\{\bullet, \Omega\} \uplus X$ (X is finite, cf. [6]). We will not use this characterization.

⁶A letter a is erased if $r(a)$ is undefined.

The verification that $<_{in}$ is indeed a strict linear order is easy from definitions. It generalizes to infinite binary trees the *inorder* on the nodes of finite ones.

Proposition 2.18 : Let F be a finite set of function symbols of arity at most 2. Let $t \in T^\infty(F)$ be a regular term.

- (1) The arrangement $\widehat{t} := (Pos(t), \leq_{in}, lab_t)$ over F is regular.
- (2) If $X \subseteq Pos(t)$ is MSO-definable in t , then the arrangement $\widehat{t}[X] := (X, \leq_{in}, lab_t)$ is regular.

Proof sketch: (1) Let $t = t_1$ be defined by equations $t_i = s_i$ where $i = 1, \dots, n$. We obtain as follows a finite equation systems defining the arrangements \widehat{t}_i for $i = 1, \dots, n$.

If $t_i = f(t_j, t_k)$ then $\widehat{t}_i = (\widehat{t}_j \bullet f) \bullet \widehat{t}_k$.

If $t_i = f(t_j)$ then $\widehat{t}_i = \widehat{t}_j \bullet f$.

If $t_i = f$ then $\widehat{t}_i = f$.

(2) By Lemma 2.13 and the fact that the regularity of arrangements is preserved by erasing letters in a deterministic way. \square

Definition 2.19 : *Labelled sets, or commutative arrangements*

(a) An X -labelled set is a pair $m = (V, lab)$ where V is a set and lab is a mapping $: V \rightarrow X$, equivalently, if X is finite, a relational structure $(V, (lab_a)_{a \in X})$ where each element of V belongs to a unique set lab_a . We denote by $\mathcal{S}(X)$ the set of X -labelled sets.

(b) We denote by $set(w)$ the X -labelled set obtained by forgetting the linear order of an arrangement w over X . A term in $t \in T^\infty(\{\bullet, \Omega\} \cup X)$ defines the X -labelled set $set(val(t))$. Over labelled sets, the operation \bullet is commutative.

Up to isomorphism, an X -labelled set m is defined by the cardinalities in $\mathbb{N} \cup \{\omega\}$ of the sets lab_a , hence is a countable *multiset of elements of X* : a number in $\mathbb{N} \cup \{\omega\}$ is associated with each $a \in X$ and represents its number of occurrences in m .

If X is finite, each X -labelled set is MSO_{fin}-definable, *i.e.*, is the unique countable model, *u.t.i.*, of a sentence of *monadic second-order logic* extended with a set predicate $Fin(U)$ expressing that a set U is finite. It is also *regular*, *i.e.*, is $set(val(t))$ for some regular term in $T^\infty(\{\bullet, \Omega\} \cup X)$. The notion of regularity is thus trivial for X -labelled sets when X is finite.

3. Order-theoretic trees and forests

In order to have a simple terminology, we will use the prefix *O-* to mean *order-theoretic* and to distinguish these generalized trees from the ordered ones in [11]. An order-theoretic forest is called simply a *tree* by Fraïssé in [15]. We will distinguish carefully trees, forests, O-trees and O-forests.

Definition 3.1 : *Order-theoretic forests and trees.*

- (a) An *O-forest* is a pair $J = (N, \leq)$ such that:

- 1) N is a countable⁷ set called the set of *nodes*,

⁷Countable means finite or countably infinite.

2) \leq is a partial order on N such that, for every node x , the set $[x, +\infty[$ (the set of nodes $y \geq x$) is linearly ordered.

It is called an *O-tree* if furthermore:

3) every two nodes x and y have an upper-bound.

An O-forest (N, \leq) is the disjoint union of O-trees whose sets of nodes are the connected components of the comparability graph of \leq . More precisely, two nodes are in a same component, *i.e.* in the same composing O-tree, if and only if they have a (common) upper bound.

(b) A minimal node is a *leaf*. If N has a largest element r (that is $x \leq r$ for all $x \in N$) then J is a *rooted* O-tree and r is its *root*. In an O-tree, the set of strict upper-bounds of a nonempty set $X \subseteq N$ is an upwards closed line⁸ L .

(c) An O-tree is a *join-tree*⁹ if every two nodes x and y have a least upper-bound denoted by $x \sqcup y$ and called their *join* (cf. Section 1). It is a *join-forest* if every two nodes having an upper-bound actually have a join. \square

If T is a finite rooted tree, then (N_T, \leq_T) is a join-tree (\leq_T is the ancestor relation) and every finite O-tree is a join-tree of this form.

3.1. Structurings

O-forests will be partitioned into lines, *i.e.*, into convex linearly ordered subsets.

Definition 3.2 : *Covering between lines.*

Let $J = (N, \leq)$ be an O-forest. If U and W are two lines, we say that W *covers* U , denoted by $U \prec W$, if $U < w$ for some w in W and, for all $x \in N$ and w in W , if $U < x < w$, then $x \in W$. Hence, there is nothing between U and W : if $U < y$ and $y \in N$, then there is $w \in W$ such that $U < w \leq y$. The covering relation is not transitive, hence \prec is not a strict partial order.

Definitions 3.3 : *Structurings of O-forests*

(a) Let $J = (N, \leq)$ be an O-forest. A *structuring* of J is a set \mathcal{U} of nonempty lines that forms a partition of N and satisfies the following condition:

For each x in N , we have¹⁰ $[x, +\infty[= I_k \uplus I_{k-1} \uplus \dots \uplus I_0$ for nonempty intervals I_0, \dots, I_k of $[x, +\infty[$ such that:

$$I_k < I_{k-1} < \dots < I_0,$$

for each j , we have $I_j \subseteq U$ for some unique line U in \mathcal{U} , denoted by U_j ,

each interval I_j is upwards closed in (U_j, \leq) , which implies that $U_j \neq U_{j'}$ if $j \neq j'$.

⁸See Section 2.1 for the notion of line. That a set A is *upwards closed* means that $[u, +\infty[\subseteq A$ for all $u \in A$.

⁹An *ordered tree* is a rooted tree such that the set of sons of each x is linearly ordered by an order depending on x . This notion is extended in [11] to join-trees. Ordered join-trees should not be confused with order-theoretic trees.

¹⁰The set $[x, +\infty[$ may have a greatest element that this notation does not specify.

It follows that $U_k \prec U_{k-1} \prec \dots \prec U_0$ and that I_0 and U_0 are upwards closed in J .

Then $J = (N, \leq, \mathcal{U})$ is a *structured O-forest*, an *SO-forest* in short.

A structuring of an O-forest is a union of structurings of the O-trees composing it.

(b) The sequence I_0, I_1, \dots, I_k is uniquely defined for each x , and k is called the *depth* of x . If $x \in N$, then $U(x)$ denotes the line of \mathcal{U} containing x . We define $\beta(x) := [x, +\infty[-U(x) = I_{k-1} \uplus \dots \uplus I_0$ (hence, $x \notin \beta(x)$).

(c) If $J = (N, \leq, \mathcal{U})$ is an *SO-forest* and $X \subseteq N$, then the set of nonempty sets $X \cap U$ for $U \in \mathcal{U}$ is a structuring of the O-forest $J[X]$. However, the depth of an element of X may be smaller in $J[X]$ than in J . \square

Example 3.4 : Figure 1 shows a structuring $\{A, B, C, D, E\}$ of an O-tree. The line A is upwards closed (we will call it the *axis*). The depth of x is 0, the depths of y and z are 2. We have $\beta(x) = \emptyset$ and $\beta(u) = I$ where u is any node in $B \cup D$. The lines $\{A, B, C, D, E\}$ form a kind of tree. \square

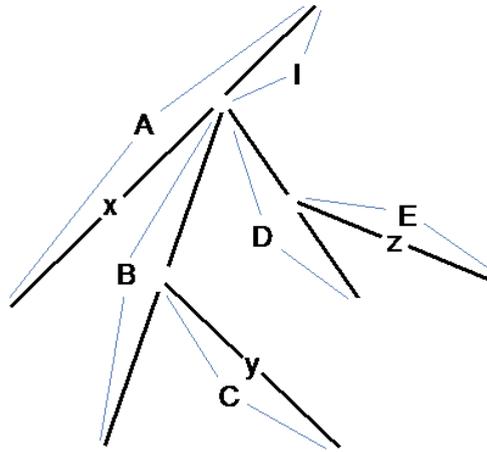


Figure 1. A structuring of an O-tree.

Proposition 3.5 : Every O-forest has a structuring.

Proof:

The proof is similar to that of [11] establishing that every join-tree has a structuring. We give it for completeness.

We first consider an O-tree $J = (N, \leq)$. We choose an enumeration $x_0, x_1, \dots, x_n, \dots$ of N and a maximal¹¹ line B_0 ; it is upwards closed. For each $i > 0$, we choose a maximal line B_i containing the first node not in $B_{i-1} \cup \dots \cup B_0$. We define $U_0 := B_0$ and, for $i > 0$, $U_i := B_i - (U_{i-1} \uplus \dots \uplus U_0) = B_i - (B_{i-1} \cup \dots \cup B_0)$. We define \mathcal{U} as the set of lines U_i . It is a structuring of J .

An O-forest has a structuring that is the union of structurings of the O-trees composing it. \square

¹¹Maximal for set inclusion.

Definition 3.6 : *Axis*

(a) An *axis* of an SO-forest $J = (N, \leq, \mathcal{U})$ is a distinguished upwards closed line $A \in \mathcal{U}$. It is an axis of one of the SO-trees composing J . An *SO-forest with axis*, an *SOA-forest* in short, is thus a 4-tuple $J = (N, \leq, \mathcal{U}, A)$. If A empty, we say that J has no axis; it can be defined as (N, \leq, \mathcal{U}) .

An nonempty SO-tree $J = (N, \leq, \mathcal{U})$ has a unique axis A , defined from \mathcal{U} , that we denote by $Ax\text{is}(J)$. Hence it is an SOA-tree in a unique way. However, as for SO-forests, we may decide that it has no axis.

(b) The operation *fg forgets* the axis of an SOA-forest (or of an SOA-tree):

$$fg(N, \leq, \mathcal{U}, A) := (N, \leq, \mathcal{U}) \text{ identified to } (N, \leq, \mathcal{U}, \emptyset).$$

(c) The union of pairwise disjoint SO-trees (without axes) is an SO-forest, and conversely. If J is an SO-forest, we denote by $Ax\text{es}(J)$ the set of axes of the SOA-trees composing it according to (a).

(d) If $J = (N, \leq, \mathcal{U})$ is an SO-forest and $U \in \mathcal{U}$, we denote by $J \downarrow U$ the SOA-tree $J[W]$ with axis U , where W is the union of the sets $] - \infty, x]$ for all $x \in U$. \square

Definition 3.7 : *Cuts defined from a structuring.*

Let $J = (N, \leq, \mathcal{U})$ be an SO-forest.

(a) If $U \in \mathcal{U}$, we say that a node $x \in N$ *defines a cut* (U_1, U_2) of the linear order (U, \leq) if $x \notin U$, $U_2 = U \cap [x, +\infty[\neq \emptyset$ and $x \perp U_1$ (i.e., x is incomparable with each element of the nonempty set U_1). This cut is denoted by $\kappa(U, x)$.

(b) We let $Cuts(U)$ be the set of cuts $\kappa(U, x)$ of U , and we denote by $U_{Cuts(U)}$ the linearly ordered set $(U \uplus Cuts(U), \leq)$ (cf. Definition 2.2). It is countable as N is.

(c) We denote by \mathcal{K} the union of the sets $Cuts(U)$ for all $U \in \mathcal{U}$. This set is countable because cuts are defined by nodes and each node x of depth $k > 0$ defines finitely many cuts in lines of smaller depth.

(d) If κ is a cut of U of depth $k \geq 0$, we denote by $Def(\kappa)$ the SO-forest (without axis) induced by the nodes x such that $\kappa(U, x) = \kappa$. Then $Ax\text{es}(Def(\kappa))$ is the set of lines W of $Def(\kappa)$ that are at depth $k + 1$ (in J), hence such that $W \prec U$. Each of them defines an SOA-tree $J \downarrow W$ and $Def(\kappa)$ is the union of the SO-trees $fg(J \downarrow W)$ for all $W \in Ax\text{es}(Def(\kappa))$.

We denote by \mathcal{L} the set of O-forests $Def(\kappa)$ for all $\kappa \in \mathcal{K}$. It is in bijection with \mathcal{K} . \square

In Example 3.4 and Figure 1, we let $\kappa := \kappa(A, y) = \kappa(A, z)$. Then $Def(\kappa)$ consists of the two trees $B \cup C$ and $D \cup E$, $Ax\text{es}(Def(\kappa)) = \{B, D\}$ and $J \downarrow B$ is the SOA-tree $B \cup C$ with axis B .

Lemma 3.8 : Let $J = (N, \leq, \mathcal{U})$ be an SO-forest.

- (1) Each node x at depth $k > 0$ defines a cut of some $U \in \mathcal{U}$ of depth $k - 1$.
- (2) Each cut of U of depth $k \geq 0$ is $\kappa(U, x)$ for some node x at depth $k + 1$.

Proof: Clear from definitions. \square

3.2. Description schemes of SOA-forests

As observed above, a structuring of an O-tree can be seen as a kind of "tree of lines". In a *regular SO-tree* (a notion to be defined), the structuring consists of finitely many lines *up to isomorphism*, and they are regular arrangements. By defining these arrangements by monadic second-order sentences, we will obtain finitary descriptions. The formal definitions are more involved.

Definitions and notation 3.9 : Concerning SO-forests

Let $J = (N, \leq, \mathcal{U})$ be an SO-forest. It is a disjoint union of SO-trees.

(a) As observed in Definition 3.6(a), each of these SO-trees $L = (N_L, \leq, \mathcal{U}_L)$ has a unique axis $A \in \mathcal{U}_L \subseteq \mathcal{U}$ denoted by $Axis(L)$. We denote by $Axes(J)$ the set of lines $Axis(L)$ for all these SO-trees L .

(b) Let $U \in \mathcal{U}$. Its *tail*, denoted by $Tail(U)$, is the SO-forest induced on $\{x \in N \mid x < U\}$. It may be empty¹². We let \mathcal{T} be the set of nonempty SO-forests $Tail(U)$ for $U \in \mathcal{U}$. For each $U \in \mathcal{U}$, we let τ_U be a new symbol, standing for $Tail(U)$ and located so as to mark that $Tail(U) < U$, as shown by the next definition. Similarly, a cut κ of U marks the position of $Def(\kappa)$ among the nodes of U .

(c) We recall (Definition 2.2) that $U_{Cuts(U)}$ is the arrangement extending U by inserting its cuts at their natural places. We let U^+ be the arrangement $\tau_U \bullet U_{Cuts(U)}$ defined by prefixing $U_{Cuts(U)}$ with τ_U . It is countable.

Definition 3.10 : Substitutions of SO-forests in linear orders.

Let $H = (N, \leq_H)$ be a linear order, $X \subseteq N$ and for each $x \in X$, let $F_x = (M_x, \leq_x, \mathcal{U}_x)$ be an SO-forest such that the sets M_x are pairwise disjoint and disjoint from N . Then $H[x \leftarrow F_x; x \in X]$ is the SOA-forest $(M, \leq, \mathcal{U}, A)$ such that :

M is the union of $N - X$ and the sets M_x ,
 $u \leq v$ if and only if $u, v \in N - X$ and $u \leq_H v$,
or $u, v \in M_x$ and $u \leq_x v$ for some $x \in X$,
or $v \in N - X$ and $u \in M_x$ for some $x \in X$ and $x <_H v$.
 $A := N - X$,
 \mathcal{U} consists of A and the sets in the \mathcal{U}_x 's for all $x \in X$. \square

With these definition and notation, we have :

Lemma 3.11 : Let $J = (N, \leq, \mathcal{U})$ be an SO-forest and $U \in \mathcal{U}$. Then we have:

$J \downarrow U = U^+[\kappa \leftarrow Def(\kappa); \kappa \in Cuts(U), \tau_U \leftarrow Tail(U)]$
if $Tail(U)$ is not empty. Otherwise,
 $J \downarrow U = U^+[\kappa \leftarrow Def(\kappa); \kappa \in Cuts(U)]$.

¹²If we define \mathcal{U} by means of the construction of Proposition 3.5, all tails are empty.

Proof:

Straightforward from the definitions. We have equalities, not just isomorphisms. \square

Labellings of SO-forests.

Let $J = (N, \leq, \mathcal{U})$ be an SO-forest, as in Definitions 3.7 and 3.9. Then \mathcal{K} will denote the set of all cuts, \mathcal{L} the set of associated SO-forests $Def(\kappa)$ for all $\kappa \in \mathcal{K}$ and \mathcal{T} the set of all tails. Hence, \mathcal{L} and \mathcal{T} are disjoint sets of SO-forests.

Our objective is to use labelling functions $r : \mathcal{U} \rightarrow D$ and $s : \mathcal{L} \uplus \mathcal{T} \rightarrow Q$, into disjoint sets D and Q , in such a way that :

$$\begin{aligned} r(U) = r(U') \text{ implies } J \downarrow U \simeq J \downarrow U' \text{ and} \\ s(L) = s(L') \text{ implies } L \simeq L'. \end{aligned}$$

Actually, stronger conditions will be useful.

Definitions 3.12 : *Good labelling of an SO-forest J .*

A good labelling of $J = (N, \leq, \mathcal{U})$ is defined from the following items:

$$\begin{aligned} &\text{disjoint sets } D, Q_{cut} \text{ and } Q_{tail}, \text{ with } Q := Q_{cut} \uplus Q_{tail}, \\ &\text{mappings } r : \mathcal{U} \rightarrow D \text{ and } s : \mathcal{L} \uplus \mathcal{T} \rightarrow Q \text{ such that:} \\ &s(\mathcal{L}) \subseteq Q_{cut}, s(\mathcal{T}) \subseteq Q_{tail}, \\ &r(U) = r(U') \text{ implies } s \triangleright U^+ \simeq s \triangleright U'^+ \text{ and} \\ &s(L) = s(L') \text{ implies } r\{Axes(L)\} = r\{Axes(L')\}, \end{aligned}$$

where we use the following notation:

$s \triangleright U^+$ is the arrangement over $\{*\} \uplus Q$, obtained by replacing τ_U by $s(Tail(U))$, each κ in $Cuts(U)$ by $s(Def(\kappa))$ and each $u \in U$ by $*$,

$r\{Axes(L)\}$ is the multiset of elements of D (cf. Definition 2.19) consisting of the labels $r(Axis(W))$ for all W in $Axes(L)$, hence $r\{Axes(L)\} \in \mathcal{S}(D)$.

If r and s define a good labelling, then $r(U) = r(U')$ implies $s \triangleright U^+ \simeq s \triangleright U'^+$ whence also $J \downarrow U \simeq J \downarrow U'$ (by Lemma 3.11), and $s(L) = s(L')$ implies $r\{Axes(L)\} = r\{Axes(L')\}$ whence $L \simeq L'$.

In what follows, we focus on O-trees. Extending the results to O-forests will be straightforward as they are disjoint unions of O-trees.

Definition 3.13 : *Description schemes.*

(a) A *description scheme* is a tuple $\Delta = (D, Q, d_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$ where D and Q are disjoint sets, $d_{Ax} \in D$, $m_q \in \mathcal{S}(D)$ for each $q \in Q$ and each w_d is an arrangement over $\{*\} \uplus Q$.

(b) It describes an SOA-tree $J = (N, \leq, \mathcal{U}, A)$ if J has a good labelling with mappings $r : \mathcal{U} \rightarrow D$ and $s : \mathcal{L} \uplus \mathcal{T} \rightarrow Q$ such that :

$$\begin{aligned}
 r(A) &= d_{Ax}, \\
 \text{if } r(U) &= d, \text{ then } w_d \simeq s \triangleright U^+, \\
 \text{if } s(L) &= q, \text{ then } m_q = r\{Axes(L)\}.
 \end{aligned}$$

In that case, we say that Δ describes as well the SO-tree (N, \leq, \mathcal{U}) and the O-tree (N, \leq) .

(c) A description scheme $\Delta = (D, Q, d_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$ is *regular* if $D \uplus Q$ is finite and each arrangement w_d is regular. The finiteness of D implies that the D -labelled sets m_q are regular (cf. Section 2.3).

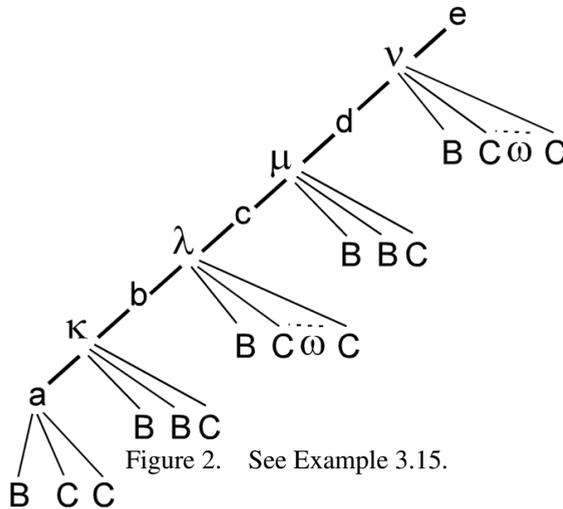
Lemma 3.14 : (1) Every SOA-tree is described by some description scheme.
 (2) Every description scheme describes a unique SOA-tree, *u.t.i.*

Proof:

- (1) One can take $D := \mathcal{U}, Q := \mathcal{L} \uplus \mathcal{T}$ and identity mappings r and s .
- (2) Similar to the proof of Proposition 3.24 in [11]. □

Examples 3.15 : (1) If all components D, Q, m_q and w_d of a description scheme Δ are finite, then, the defined SOA-tree is a regular, possibly infinite, rooted tree (with an axis).

For example, $\Delta := (\{d\}, \{q\}, d, m_q, w_d)$ where $m_q = \{d\}$ and $w_d = q^*$ defines the SOA-tree $J := (\mathbb{N}, \leq_J, \mathcal{U}, \{0\})$ such that $n \leq_J m$ if and only if $m \leq n$ and \mathcal{U} is the set of singletons $\{n\}$.



(2) Figure 2 sketches a description scheme of a structured O-tree whose lines have depth 0 or 1. The axis A is $\{a, b, c, d, e\}$, it has four cuts named $\kappa, \lambda, \mu, \nu$. The other lines of the structuring have isomorphism types B or C . The SO-forests $Def(\kappa)$ and $Def(\mu)$ are both isomorphic to the multiset $\{B, B, C\}$: we mean that the SO-forest $Def(\kappa)$ has three components respectively isomorphic to B, B and C . Similarly, $Def(\lambda)$ and $Def(\nu)$ are isomorphic to $\{B, C, \dots, C\}$ with ω times C . The axis has a tail isomorphic to $\{B, C, C\}$. We let p label κ and μ , q label λ and ν , and z label τ_A . Then $s \triangleright A^+ = z * p * q * p * q^*$ (cf. Definition 3.12).

A corresponding description scheme has $D := \{\alpha, \beta, \gamma\}$, $Q := \{z, p, q\}$, $d_{Ax} := \alpha$, $w_\alpha := s \triangleright A^+$, $w_\beta := s \triangleright B$, $w_\gamma := s \triangleright C$, $m_z := \{\beta, \gamma, \gamma\}$, $m_p := \{\beta, \beta, \gamma\}$, $m_q := \{\beta, \gamma, \dots, \gamma\}$ with ω times γ . As B and C have neither cuts nor tails, $B^+ = B$, $C^+ = C$ and the arrangements $s \triangleright B$ and $s \triangleright C$ are over $\{*\}$.

The elements of D name the isomorphism types of the lines that form the structuring. The elements of Q name the isomorphism types of the axes of the SOA-trees composing the SO-forests defined by the tail of A and by its cuts. More than the isomorphism type of the axis A , the element α of D designates the arrangement $s \triangleright A^+$ that incorporates descriptions of the cuts of A (and the SO-forests they define) and of its tail. \square

3.3. Monadic second-order description of SO-forests

In view of our use of monadic second-order logic, we give a description of SO-forests by relational structures. An SO-forest $J = (N, \leq, \mathcal{U})$ is not *per se* a relational structure because \mathcal{U} is a partition of N in an unbounded number of sets. We will encode \mathcal{U} by a bipartition of N , by using a tool introduced in Definition 3.6 of [11].

Definition 3.16 : *SO-forests represented by relational structures.*

If $J = (N, \leq, \mathcal{U})$ is an SO-forest, we define $S(J)$ as the relational structure (N, \leq, N_0, N_1) such that N_0 is the set of nodes at even depth and $N_1 := N - N_0$ is the set of those at odd depth; N_0 and N_1 are sets but we consider them also as unary relations.

Proposition 3.17 : (1) There is an MSO formula $\varphi(N_0, N_1)$ expressing that a relational structure (N, \leq, N_0, N_1) is $S(J)$ for some SO-forest $J = (N, \leq, \mathcal{U})$.

(2) There exist MSO formulas $\theta_1(N_0, N_1, U)$ and $\theta_2(N_0, N_1, U, W)$ expressing respectively, in a structure $(N, \leq, N_0, N_1) = S((N, \leq, \mathcal{U}))$, the properties " $U \in \mathcal{U}$ " and " $U, W \in \mathcal{U}$ and $U \prec W$ " (cf. Definition 3.2 for \prec).

Proof:

Easy extension of Proposition 3.7 of [11] \square

We recall from Definition 3.6 that if (N, \leq, N_0, N_1) is $S(J)$ for some SO-tree $J = (N, \leq, \mathcal{U})$, then its only possible axis is the unique set A in \mathcal{U} such that $A \subseteq N_0$ and there is no $x \in N_1$ such that $A < x$.

One of our key results is the following theorem.

Theorem 3.18 : Let Δ be a regular description scheme. There exists an MSO_{fin} sentence that characterizes *u.t.i.* the SOA-trees described by Δ .

Its proof will be given after some more notation and definitions.

Definitions 3.19 : *Defining nodes.*

Let $J = (N, \leq, \mathcal{U}, A)$ be an SOA-tree. We use the notation of Definitions 3.7, 3.9, 3.12 and 3.13.

(a) We say that $x \in N$ defines¹³ $W \in \mathcal{U}$ if $W = U(x)$.

We say that $x \in N$ defines the tail of $W \in \mathcal{U}$ if $U(x) < W$ and $U(x) \prec W$, which implies that $Tail(W)$ is not empty and $U(x) \subseteq Tail(W)$. Furthermore, $U(x) \in Axes(Tail(W))$.

We say that $x \in N$ defines the cut $(W_1, W_2) = \kappa$ of $W \in \mathcal{U}$ if $U(x) < W_2$, $U(x) \perp W_1$ and $U(x) \prec W$, which implies that $U(x) \subseteq Def(\kappa)$ and $U(x) \in Axes(Def(\kappa))$ (cf. Definition 3.9(b)).

(b) A triple of sets $(N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}})$ is well-defining (for J) if :

$N_{\mathcal{U}}$ contains exactly one element defining each $W \in \mathcal{U}$,

$N_{\mathcal{T}}$ contains exactly one element defining a nonempty tail $Tail(W)$ for all $W \in \mathcal{U}$,

$N_{\mathcal{K}}$ contains exactly one element defining each cut in \mathcal{K} , the set of all cuts of all lines.

We have $N_{\mathcal{T}} \cap N_{\mathcal{K}} = \emptyset$.

(c) Let p, p' be positive integers. A mapping $r : \mathcal{U} \rightarrow [p]$ is described by a partition (R_1, \dots, R_p) of $N_{\mathcal{U}}$, such that $r(U) = i$ if and only if there is some $x \in U \cap N_{\mathcal{U}} \cap R_i$. A mapping $s : \mathcal{T} \uplus \mathcal{L} \rightarrow [p']$ is described similarly by a partition $(S_1, \dots, S_{p'})$ of $N_{\mathcal{T}} \uplus N_{\mathcal{K}}$. (We recall that Def defines a bijection $\mathcal{K} \rightarrow \mathcal{L}$).

(d) Let N_0, N_1 be as in Definition 3.16. A pair (r, s) that defines a good labelling (cf. Definition 3.12) where $D = [p]$ and $Q = [p']$ can be described by a tuple of sets $(N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}}, R_1, \dots, R_p, S_1, \dots, S_{p'})$ that satisfies $\varphi(N_0, N_1)$ and, thanks to Proposition 3.17(2), the conditions of (b) and (c).

We denote by $\psi(N_0, N_1, N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}}, R_1, \dots, R_p, S_1, \dots, S_{p'})$ the MSO-formula expressing the conjunction of these conditions (including $\varphi(N_0, N_1)$). \square

Proof of Theorem 3.18 : Let $\Delta = (Q, D, d_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$ be a regular description scheme.

For each $d \in D$, there is an MSO-formula ψ_d that characterizes, *u.t.i.*, the regular arrangement w_d , cf. Section 2.3. It is written with the relation symbol \leq and unary relation symbols lab_q, lab_* used for encoding the labelling in $Q \uplus \{*\}$.

For each $q \in Q$, there is an MSO_{fin}-formula η_q that characterizes, *u.t.i.*, the labelled set (the multiset) m_q . It is written with the unary symbols lab_d for encoding the labelling in D . The finiteness predicates are necessary to distinguish the infinite sets from the finite ones.

Without loss of generality, we assume that $D = [p]$ and $Q = [p']$.

Let (N, \leq) be an O-tree. (An FO sentence can check that (N, \leq) is actually an O-tree). Let $N_0, N_1, N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}}, R_1, \dots, R_p, S_1, \dots, S_{p'}$ be $5 + p + p'$ sets of nodes. The MSO formula $\psi(N_0, N_1, N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}}, R_1, \dots, R_p, S_1, \dots, S_{p'})$ expresses that they define a structuring \mathcal{U} of the O-tree (N, \leq) , that the sets $N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}}$ and the mappings r and s on the sets \mathcal{U} and $\mathcal{T} \cup \mathcal{K}$ are as in Definition 3.19(b,c).

In order to build an MSO_{fin} formula that checks the conditions of Definition 3.13(b), we use the follows steps.

¹³A different notion of representation of a line by an element (actually a position in a term defining J) will be used in Section 4.

(a) For each $x \in R_d$, $1 \leq d \leq p$, the arrangement $s \triangleright U(x)^+$ has a domain consisting of $U(x)$ whose elements are labelled by $*$, together with the nodes y of $N_{\mathcal{K}}$ that define cuts¹⁴ of $U(x)$; each such y is labelled by $s(Def(V, W)) \in Q_{cut}$ where (V, W) is the cut it defines. Furthermore, if the tail of $U(x)$ is not empty, then $s \triangleright U(x)^+$ has a first element z in $N_{\mathcal{T}}$ marking its place, and that is labelled by $s(Tail(U(x)))$. The linear ordering of $U(x)^+$ following from Definition 2.2 is MSO-definable, as for each $y \in N_{\mathcal{K}}$, the corresponding pair (V, W) can be MSO-defined.

It remains to express that $s \triangleright U(x)^+$ satisfies $\psi_{r(x)}$ which can be done by an MSO formula, constructed from $\psi_{r(x)}$ by *relativization* to the domain of $U(x)^+$ described above.

In this way, we express that the arrangements U^+ associated with the lines U of \mathcal{U} satisfy Definition 3.13(b).

(b) Next we consider the cuts, defined by nodes $y \in S_q \cap N_{\mathcal{K}}$. Let y define a cut κ . We recall that $Axes(Def(\kappa))$ is the set of axes of the SOA-trees that form the SO-forest $Def(\kappa)$. Each of these axes A has a defining node $\partial(A)$ in $N_{\mathcal{U}}$, and each $\partial(A)$ has an image in D by r (specified by (R_1, \dots, R_p)). We obtain a labelled set $r\{Axes(Def(\kappa))\}$. By means of the MSO_{fin} formula $\eta_{s(y)}$, one can express that it is isomorphic to m_q .

The case of tails, defined by the nodes in the sets $S_q \cap N_{\mathcal{T}}$ can be treated similarly.

(c) We also express that the axis of the structuring of (N, \leq) defined by N_0, N_1 contains a node in $R_{r(d_{Ax})}$.

Hence, we get an MSO_{fin} formula $\Theta_{\Delta}(N_0, N_1, N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}}, R_1, \dots, R_p, S_1, \dots, S_{p'})$ (it implies $\psi(N_0, N_1, N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}}, R_1, \dots, R_p, S_1, \dots, S_{p'})$), that expresses the conditions of Definition 3.13(b) and 3.19(b,c).

It follows that an O-tree (N, \leq) has a structuring defined by Δ if and only if it satisfies the MSO_{fin} sentence:

$$\exists N_0, N_1, N_{\mathcal{U}}, N_{\mathcal{T}}, N_{\mathcal{K}}, R_1, \dots, R_p, S_1, \dots, S_{p'}. \Theta_{\Delta}(N_0, \dots, S_{p'}) . \quad \square$$

Our aim is now to establish the converse to Theorem 3.18 yielding Theorem 4.19 that is our main theorem. We will use for that an algebra of SO-forests of possible independent interest.

4. The algebra of SO-forests

We will actually use SO-forests enriched with distinguished axes. Their algebra is similar to that of [11] for structured join-trees, but it uses less operations and a single sort instead of two.

Definition 4.1 : *The algebra of SOA-forests.*

SOA-forests are defined in Definition 3.6. We define the following operations.

(a) *Concatenation of SOA-forests along axes.*

¹⁴We replace in the definition of U^+ (Definition 3.9(c)) a cut by the node in $N_{\mathcal{K}}$ that defines it.

The *concatenation* $J \bullet J'$ of disjoint SOA-forests $J = (N, \leq, \mathcal{U}, A)$ and $J' = (N', \leq', \mathcal{U}', A')$ is defined as follows:

$$J \bullet J' := (N \uplus N', \leq'', \mathcal{U}'', A \uplus A') \text{ where :}$$

$$x \leq'' y : \iff x \leq y \vee x \leq' y \vee (x \in N \wedge y \in A'),$$

$$\mathcal{U}'' := (\mathcal{U} - \{A\}) \uplus (\mathcal{U}' - \{A'\}) \uplus \{A \uplus A'\}.$$

This operation is associative. If $A = A' = \emptyset$, then $J \bullet J'$ is the union of J and J' , and then $J \bullet J' = J' \bullet J$.

If J and J' are not disjoint, we replace one of them by an isomorphic copy disjoint from the other. The result is well-defined *u.t.i.* (we recall *up to isomorphism*) as different isomorphic copies can be chosen. (This is a standard technique. See [13]).

If J and J' are linear orders with $A = N, \mathcal{U} = \{A\}, A' = N'$ and $\mathcal{U}' = \{A'\}$ then $J \bullet J'$ is their concatenation.

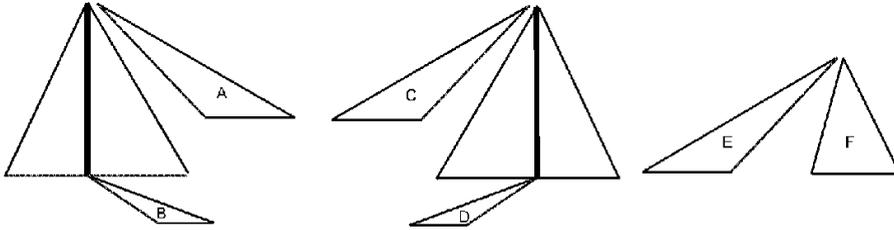


Figure 3. From left to right O-forests J, J' and K . Thick lines show the axes. See Figure 4 for concatenations.

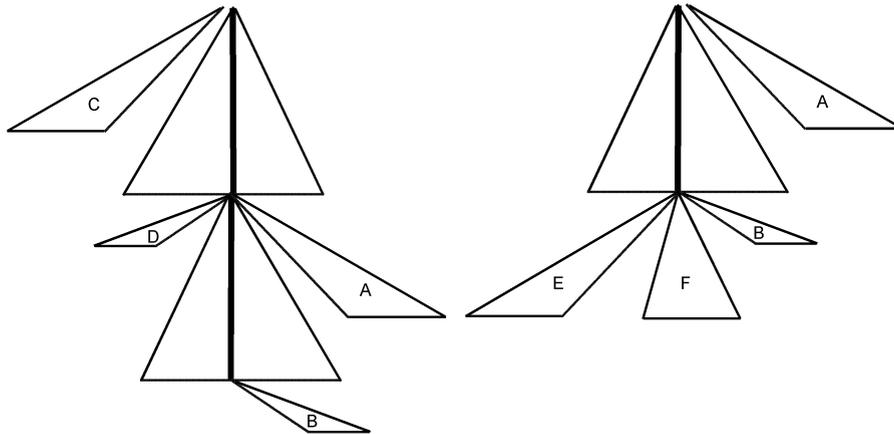


Figure 4. The concatenations $J \bullet J'$ and $K \bullet J$.

Figure 3 shows SOA-forests J, J' and K where J and J' have axes shown by thick lines and K has no axis. They all consist of two SO-trees. Figure 4 shows the concatenations $J \bullet J'$ and $K \bullet J$. Both consist of two O-trees.

(b) The empty SOA-forest is denoted by the nullary symbol Ω .

Clearly, $J \bullet \Omega = \Omega \bullet J = J$.

(c) Nullary symbols for nodes. For each u (intended to be a node), we denote by \underline{u} the SOA-tree $(\{u\}, =, \{\{u\}\}, \{u\})$. When considering SOA-trees and SOA-forests *u.t.i.*, we replace every \underline{u} by the unique symbol $*$ denoting any singleton SOA-tree.

(d) Forgetting the axis (cf. Definition 3.6).

To forget the axis, equivalently, to make it empty, we use the unary operation fg such that¹⁵ :

$$fg(N, \leq, \mathcal{U}, A) := (N, \leq, \mathcal{U}, \emptyset), \text{ equivalently, } (N, \leq, \mathcal{U}).$$

It is clear that:

$$fg(\Omega) = \Omega,$$

$$fg(fg(J)) = fg(J) \text{ and}$$

$$fg(J) \bullet fg(J') = fg(J') \bullet fg(J) = fg(J \bullet fg(J')),$$

where J and J' are disjoint or replaced by disjoint copies; in the latter case, equality is replaced by isomorphism.

(e) We let $F := \{\bullet, fg, \Omega, *\}$ and we obtain an F -algebra of SOA-forests *u.t.i.*, denoted by \mathbb{S} . \square

The value in \mathbb{S} of a term in $T(F)$, *i.e.*, of a finite term over F , follows immediately from the definition of the operations. It is defined *u.t.i.* because we need to take disjoint copies of the SO-forests that are arguments of concatenation: the simplest example is for the term $* \bullet *$. Alternatively, $\underline{u} \bullet \underline{v}$ denotes the SOA-tree $(\{u, v\}, \leq, \{\{u, v\}\}, \{u, v\})$ where $u < v$.

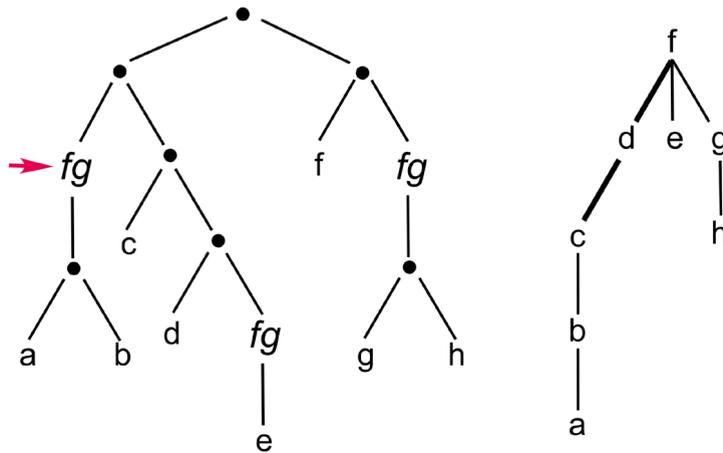


Figure 5. A term and the SOA-tree it defines, cf. Example 3.2.

¹⁵We could define an algebra with two sorts for O-forests with and without an axis, and more operations.

Example 4.2 : Figure 5 shows a finite term t (on the left) and the SOA-tree J it defines. The bold edges indicate the axis $\{c, d, f\}$. We have $\mathcal{U} = \{\{a, b\}, \{c, d, f\}, \{e\}, \{g, h\}\}$. If in this term we omit the operation fg at position 11 (we use Dewey notation for positions, cf. Definition 2.3), marked in Figure 5 by an arrow, we obtain an SOA-tree with axis $\{a, b, c, d, f\}$ and same ancestor relation. \square

Before giving a formal definition of the SOA-forest denoted by an infinite term t , *i.e.*, that will be its value in \mathbb{S} , we give three examples.

Examples 4.3 : *Some SOA-forests and terms that denote them.*

(1) The SOA-tree $(\mathbb{Q}, \leq, \{\mathbb{Q}\}, \mathbb{Q})$ that we will denote by \mathbb{Q} is the value of the regular term t_0 defined as the (unique) solution of the equation $t_0 = t_0 \bullet (* \bullet t_0)$, cf. Example 2.16(b) in Section 2.3 about arrangements.

(2) Let now $J := (\mathbb{Q}, \leq', \mathcal{U}, \mathbb{Q} - B)$ be the SOA-tree such that:

$$B := \{-n \mid n \in \mathbb{N}\},$$

$$x \leq' y \text{ if and only if } x \leq y \text{ and } y \notin B,$$

and its structuring \mathcal{U} consists of the axis $\mathbb{Q} - B$ and the sets $\{b\}$ for $b \in B$.

It is not a join-tree because $(-n) \sqcup x$ is undefined if $n \in \mathbb{N}$ and $x < -n$. We have $J \simeq J \bullet (fg(*) \bullet \mathbb{Q})$. It is the value of the term t_1 that is the unique solution of the equation $t_1 = t_1 \bullet (fg(*) \bullet t_0)$.

(3) Let H be an SOA-forest and $J := fg(H) \bullet \underline{u}$. Then J is an SOA-tree with root u and axis $\{u\}$; its subtrees below u are the components of the forest H whose axis has been "forgotten" by the operation fg . This construction is denoted by $ext_u(H)$ in [11]. \square

We will use concrete terms that define SOA-forests with explicit nodes, as opposed to SOA-forests up to isomorphism. For this purpose, if M is a set of potential nodes, then \underline{M} is the set of nullary symbols \underline{u} for $u \in M$. We let $F_{\underline{M}}$ be the set $\{\bullet, fg, \Omega\} \uplus \underline{M}$. A *concrete term* is a term in $T^\infty(F_{\underline{M}})$ such that each \underline{u} has at most one occurrence (because the arguments of \bullet must define disjoint forests). Its value is an SOA-forest with nodes in M . The following definition is similar to Definition 3.28 of [11].

Definition 4.4 : *The value of a term in $T^\infty(F_{\underline{M}})$ or in $T^\infty(F)$.*

Positions of terms are defined as words over $\{1, 2\}$, cf. Definition 2.3.

(a) If u, v are positions of a term t and v is an ancestor of u , we write $u <_{t_1} v$ if $u = v1u'$ for some word u' (*i.e.*, u is the left son of v or is below it), and similarly, $u <_{t_2} v$ if $u = v2u'$. We will also use the lexicographic order \leq_{lex} on words.

(b) Let t be a term in $T^\infty(F_{\underline{M}})$ or in $T^\infty(F)$, and $w, w' \in Pos(t)$. We write $w \approx w'$ if and only if $w = w'$ or $w \neq w'$ and there is no occurrence of the operation fg on the path in t (considered as a rooted tree) between its nodes w and w' . However, w and/or w' may be occurrences of fg (cf. Example 4.10(2) below).

If w and w' are occurrences respectively of \underline{u} and \underline{u}' , we write $u \approx u'$ if and only if $w \approx w'$. We denote by $occ(t, u)$ the occurrence of \underline{u} in t . We will frequently replace $occ(t, u)$ by u .

(c) *The value of a concrete term.* Let $t \in T^\infty(F_{\underline{M}})$. We let N be the set of $u \in M$ such that \underline{u} has one occurrence in t (hence a unique one). It is empty if, for example, t is Ω or $\Omega \bullet \Omega$.

The *value* $val(t)$ is the tuple $(N, \leq, \mathcal{U}, A)$ defined as follows:

(i) Let $u, v \in N$. We define:

$u < v$ if and only if $occ(t, u) <_{lex} occ(t, v)$ and $occ(t, v) \approx w$ where $w := occ(t, u) \sqcup_t occ(t, v)$ (the join of $occ(t, u)$ and $occ(t, v)$ in t).

Hence w is an occurrence of \bullet , $occ(t, u) <_{t_1} w$ and $occ(t, v) <_{t_2} w$.

(ii) The axis A is the set of nodes $u \in N$ such that $occ(t, u) \approx root_t$ provided $root_t$ is not an occurrence of fg ; otherwise, there is no axis.

(iii) The sets in \mathcal{U} are the nonempty sets $[x]_{\approx} \cap N$ for some position x in t ; each such set is $U(u)$ for some u in N . On a set $U(u)$, the order \leq is the lexicographic order.

We will see that $val(t)$ is an SOA-forest.

(d) *The value of a term in $T^\infty(F)$.*

We construct from $t \in T^\infty(F)$ an SOA-forest $J = (N, \leq, \mathcal{U}, A)$ whose isomorphism class is the value of t , also denoted by $val(t)$:

$N := Occ(t, *)$, the set of occurrences of the nullary symbol $*$;

the partial order \leq , the set \mathcal{U} and the axis A are defined as above in (c).

For comparing (c) and (d), we observe that if $t' \in T^\infty(F_{\underline{M}})$ and $t \in T^\infty(F)$ is obtained from t' by substituting $*$ for each \underline{u} , then $val(t') \simeq val(t)$.

(e) An SOA-forest J is *regular* if it is, up to isomorphism, the value of a regular term in $T^\infty(F)$. The SO-forest and the O-forest underlying J are said to be *regular*. We say that a term $t' \in T^\infty(F_{\underline{M}})$ is regular if t defined as in (d) is. \square

The logical structure $[t]$ representing a term t is defined in Definition 2.8.

Proposition 4.5 : If $t \in T^\infty(F_{\underline{M}})$, then $val(t) = (N, \leq, \mathcal{U}, A)$ is an SOA-forest. The order \leq and the set \mathcal{U} are MSO-definable in $[t]$.

Proof sketch: The following claims are easily proved from the definitions and yield the stated facts. The pair (N, \leq) is an O-forest. The sets in \mathcal{U} are linearly ordered and form a partition of N . If $u < v < w$ and $u \approx w$, then $u \approx v \approx w$: it follows that (N, \leq, \mathcal{U}) is an SO-forest. Finally, $A \in \mathcal{U}$ and is an axis. Definition 4.4(b) yields a monadic second-order definition in $[t]$ of the structuring of $val(t)$ because the equivalence relation \approx is MSO definable. \square

Proposition 4.6 : (1) If $t \in T^\infty(F_{\underline{M}})$, $val(t) = (N, \leq, \mathcal{U}, A)$ and $X \subseteq N$, then the induced SOA-forest $J[X]$ (cf. Definition 3.3(c)) is defined by the term t' obtained from t by replacing, for each $u \notin X$, the nullary symbol \underline{u} by Ω .

(2) Let J_i , $i = 1, \dots$ be countably many pairwise disjoint SOA-forests, defined respectively by concrete terms t_1, t_2, \dots . The term $fg(t_1) \bullet (fg(t_2) \bullet \dots (fg(t_i) \bullet (\dots)))$ defines an SO-forest without axis that is their union.

Proof:

Clear from the definitions. □

The following proposition justifies Definition 4.4.

Proposition 4.7 : Let $t \bullet t' \in T^\infty(F_M)$. Then $val(t \bullet t') = val(t) \bullet val(t')$. We also have $val(fg(t)) = fg(val(t))$, $val(\Omega) = \Omega$ and $val(\underline{u}) = u$.

Proof:

Routine proofs from definitions. □

Hence, for a finite term t , the value $val(t)$ from Definition 4.4 is the same as the one defined by induction on its structure. We also get that every finite SOA-forest with set of nodes N is the value of a term in $T(F_N)$.

Proposition 4.8 : Every SOA-forest J is the value of a term in $T^\infty(F_N)$ where N is its set of nodes.

Proof:

Similar to Proposition 3.19 in [11]. □

Representations of lines, tails and cuts by positions.

In the following Lemmas 4.9, 4.11, 4.13 and 4.14, a term $t \in T^\infty(F_M)$ defines a nonempty SOA-tree $J = (N, \leq, \mathcal{U}, A)$. The ancestor relation in t is denoted by \leq_t . As positions in t are words, we will also use on $Pos(t)$ the linear orders \leq_{lex} and \leq_{in} . Furthermore, we will identify a node u in N with $occ(t, u)$, the position in t that defines it by being the unique occurrence of \underline{u} . Hence, if $u, v \in N$, the notations $u \approx v$, $[u]_\approx$, $u \leq_t v$, $u \sqcup_t v$, $u \leq_{lex} v$ and $u \leq_{in} v$ will stand respectively for $occ(t, u) \approx occ(t, v)$, $[occ(t, u)]_\approx$, $occ(t, u) \leq_t occ(t, v)$, $occ(t, u) \sqcup_t occ(t, v)$, $occ(t, u) \leq_{lex} occ(t, v)$ and $occ(t, u) \leq_{in} occ(t, v)$. The notation $u \leq v$ refers to the order of J defined from t : here, u and v are nodes.

Lemma 4.9 : (1) For each $U = U(u)$ where $u \in N$, the set of positions $[u]_\approx$ has a \leq_t -maximal element x that is an occurrence of \bullet or of \underline{u} . We denote it by $Rep(U)$.

(2) For all U, U' in \mathcal{U} , if $t/Rep(U) \simeq t/Rep(U')$, then $(U, \leq) \simeq (U', \leq)$ and $Tail(U) \simeq Tail(U')$. □

Because of the second assertion, we say that position $Rep(U)$ is *representative* of U and, also, of $Tail(U)$. This notion differs from that of a defining node in Definition 3.19.

Proof:

(1) Consider $u \in N$ and $U := U(u)$. The set $[u]_\approx$ has a unique maximal element x with respect to \leq_t . It cannot have two because any two nodes have a join that is on the path linking them.

This position x is an occurrence of a nullary symbol \underline{w} for some w in N if and only if $w = u$ and $U = \{u\}$. It can also be an occurrence of \bullet . In these two cases, we define $Rep(U) := x$.

Otherwise, x is an occurrence of fg whose unique son w must be an occurrence of \bullet or of \underline{u} (because if w is also an occurrence of fg , then x is not \approx -equivalent to u). Then we define $Rep(U) := w$.

(2) The occurrences of the nodes in U are below $Rep(U)$ or equal to it. Furthermore, the order \leq (of $val(t)$) restricted to U is \leq_{lex} . It follows from Definition 4.4(a,b) that (U, \leq) is fully defined from $t/Rep(U)$. Hence $(U, \leq) \simeq (U', \leq)$ if $t/Rep(U) \simeq t/Rep(U')$.

Note that t is a concrete term with nullary symbols denoting the nodes of $val(t)$. Hence, our hypothesis is $t/Rep(U) \simeq t/Rep(U')$. If t' is obtained from t by replacing each \underline{u} by $*$ and $t'/Rep(U) = t'/Rep(U')$, then we have also $(U, \leq) \simeq (U', \leq)$. Similarly, all positions of t that define $Tail(U)$ (we mean those that define the nodes of $Tail(U)$ and those that are on the paths in t between any two nodes) are below $Rep(U)$ as one can check easily. (See Example 4.10(1)). Hence $Tail(U) \simeq Tail(U')$ if $t/Rep(U) \simeq t/Rep(U')$. \square

If $val(t)$ has a (nonempty) axis A , then $Rep(A)$ is the root of t .

Examples 4.10: (1) Consider the term $t_0 := fg(\underline{w}) \bullet (\underline{u} \bullet \underline{v})$. The set $U := \{u, v\}$ is the axis of $val(t_0)$ and $Rep(U)$ is the root of t_0 (the position of the first occurrence of \bullet). We are in the third case of the proof of Lemma 4.9(1). Note that $u \sqcup_t v$ (the position of the second occurrence of \bullet), that is the join of (the positions of) two elements of U , is not the maximal element of $[u]_{\approx}$.

The tail of U is $\{w\}$ (precisely $(\{w\}, =, \{\{w\}\})$). The maximal element of $[w]_{\approx}$ is the position of the unique occurrence of fg ; let x be its son. We have $Rep(\{w\}) = x$. This tail is defined by $fg(\underline{w})$, below $Rep(U)$.

(2) The term $t_1 = fg(\Omega \bullet (\underline{a} \bullet (\underline{b} \bullet fg(\Omega))))$ defines $J_1 := (\{a, b\}, \leq, \{\{a, b\}\}, \emptyset)$ where $a < b$. The position of the first occurrence of \bullet in t is $Rep(\{a, b\})$. The two occurrences of fg are equivalent by \approx .

(3) Let $J_2 := (\{a, b, c, d, e, f\}, \leq, \{\{a, b\}, \{c, d\}, \{e\}, \{f\}\}, \{a, b\})$ where $<$ is defined from $a < b, e < d < c < b$ and $f < d$ by transitivity. It is the value of the term $\underline{a} \bullet [fg(\underline{d} \bullet [fg(\underline{e}) \bullet fg(\underline{f})]) \bullet \underline{c}] \bullet \underline{b}$. The underlined occurrence of \bullet , call it x , is representative of the line $\{c, d\}$ of \mathcal{U} ; the tail of this line is $(\{e, f\}, =, \{\{e\}, \{f\}\})$ defined by the subterm $fg(\underline{e}) \bullet fg(\underline{f})$ that is actually below x . \square

We now define representative positions of cuts. If $U \in \mathcal{U}$ and $\kappa(U, x) = (U_1, U_2)$, we recall that $U_1 \perp x$ and $x < U_2$. The same cut is defined by each $z \in N$ such that $U_1 \perp z$ and $z < U_2$.

Lemma 4.11 : Let $\kappa(U, x) = (U_1, U_2)$ be a cut of U . Let K be the set of joins $u \sqcup_t v$ for all $u \in U_1$ and $v \in U_2$.

(1) If $u \in U_1$ and $v \in U_2$, we have $x <_t u \sqcup_t v$. The set K is linearly ordered with respect to $<_t$. It has a minimal element that we denote by $Rep(\kappa)$ and is an occurrence of \bullet . This position is \approx -equivalent to u and v .

(2) The subterm $t/Rep(\kappa)$ of t defines an SOA-tree whose axis is the interval of (U, \leq_{lex}) consisting of the elements w of U such that $w <_t Rep(\kappa)$.

(3) The SO forest $Def(\kappa)$ is $val(t/Rep(\kappa))[L]$ where L is the set of $x \in N$ such that $x <_t Rep(\kappa)$, $U_1 \perp x$ and $x < U_2$.

(4) If κ is a cut of U and κ' is a cut of U' such that $t/Rep(\kappa) \simeq t/Rep(\kappa')$, then $Def(\kappa) \simeq Def(\kappa')$.

Proof:

(1) Let $u \in U_1$ and $v \in U_2$. The position $u \sqcup_t v$ is an occurrence of \bullet . As $u, v \in U$, we have $u \approx v$, and so, $u \approx u \sqcup_t v$.

We have $u < v$ and $x < v$, hence, by Definition 4.4, we have $u <_{t_1} u \sqcup_t v$ and $v <_{t_2} u \sqcup_t v$, and, similarly $x <_{t_1} x \sqcup_t v$ and $v <_{t_1} x \sqcup_t v$. Hence $u \sqcup_t v$ and $x \sqcup_t v$ are comparable (with respect to \leq_t).

If $u \sqcup_t v <_t x \sqcup_t v$, then $x <_{t_1} x \sqcup_t v$, $u <_t u \sqcup_t v <_{t_2} x \sqcup_t v$. We also have $x \sqcup_t v \approx v$, hence $x \sqcup_t v \approx u \sqcup_t v$ as we have $u \approx u \sqcup_t v \approx v$. Hence, we have $x < u$, because $x <_{t_1} x \sqcup_t v = x \sqcup_t u$, $u <_{t_2} x \sqcup_t u$ and $u \approx x \sqcup_t u$. This contradicts the assumption that $U_1 \perp x$. Hence $u \sqcup_t v \geq_t x \sqcup_t v$ and $u \sqcup_t v >_t x$.

It follows that the set K is linearly ordered in t with respect to the ancestor relation. It has a minimal element, say w , that we denote by $Rep(\kappa)$. We have $\{u, x, v\} <_t Rep(\kappa)$ and $Rep(\kappa) \approx y$ for each $y \in U$.

(2) We also have $Rep(\kappa) \approx y$ for each y in $[u]_{\approx}$ such that $u \in U$. The axis of $val(t/Rep(\kappa))$ consists of the elements u of U that are below $Rep(\kappa)$ in t .

(3) and (4) are clear. Some leaves of $t/Rep(\kappa)$ not in U may not belong to $Def(\kappa)$ and so, may not define the cut κ . See Example 4.12. □

We will say that the position $Rep(\kappa)$ is *representative* of the cut κ .

Example 4.12 : Let $t = [a \bullet ((fg(y) \bullet b) \bullet [fg(x) \bullet (c \bullet (fg(z) \bullet d))])] \bullet e$ and $J := val(t)$. See Figure 6. Its axis is $a < b < c < d < e$ and x defines its cut $\kappa = (\{a, b\}, \{c, d, e\})$. The representative position of κ is the underlined occurrence of \bullet marked by an arrow in Figure 6. The axis of $val(t/Rep(\kappa))$ consists of $\{b, c, d\}$ with the cut $(\{b\}, \{c, d\})$. We have $Def(\kappa) = (\{x\}, =, \{\{x\}\})$ (without axis). Nodes y and z are below $Rep(\kappa)$ in t but are not in $Def(\kappa)$. □

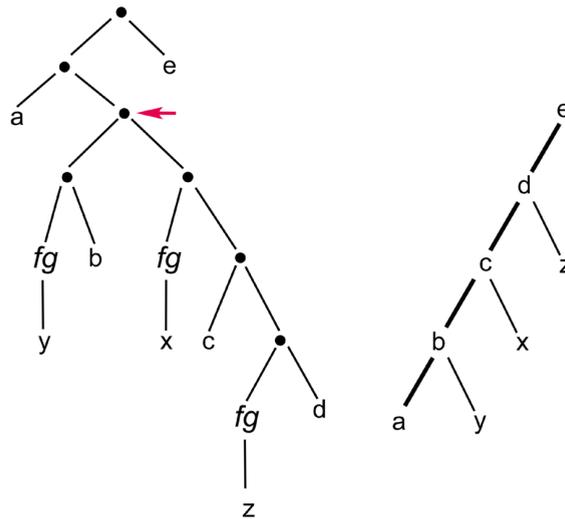


Figure 6. Example 4.12.

Let $U \in \mathcal{U}$ and R be $Rep(Cuts(U))$, defined as the set of representing positions of the cuts of U . The linear order $U_{Cuts(U)}$ on $U \uplus Cuts(U)$ is defined in Definition 2.2.

We let $U_R := (U \uplus R, \leq_{in})$. The following lemma shows that it is isomorphic to $U_{Cuts(U)}$. The in-order \leq_{in} is defined in Definition 2.17.

Lemma 4.13 : Let $U \in \mathcal{U}$. If $u, v \in U$ and $\kappa, \kappa' \in Cuts(U)$, then :

- (i) $u < v \iff u <_{in} v$,
- (ii) $u < \kappa \iff u <_{in} Rep(\kappa)$,
- (iii) $\kappa < u \iff Rep(\kappa) <_{in} u$,
- (iv) $\kappa < \kappa' \iff Rep(\kappa) <_{in} Rep(\kappa')$.

Proof:

(i) has been observed in Definition 4.4(b).

(ii) and (iii). Let $u < \kappa = (U_1, U_2)$. Let $w \in U_1$ and $z \in U_2$ be such that $Rep(\kappa) = w \sqcup_t z$. We have $u \in U_1$ hence $u <_{in} z$ and so $u <_{t_1} u \sqcup_t z$ and $z <_{t_2} u \sqcup_t z$. We have $Rep(\kappa) \leq_t u \sqcup_t z$. If $Rep(\kappa) = u \sqcup_t z$, then $u <_{in} Rep(\kappa)$ since $u <_{t_1} u \sqcup_t z$. If $Rep(\kappa) <_t u \sqcup_t z$ then we have $Rep(\kappa) <_{t_2} u \sqcup_t z$ as we have $z <_{t_2} u \sqcup_t z$ and $z <_t Rep(\kappa)$. As $u <_{t_1} u \sqcup_t z$, we have $u <_{in} Rep(\kappa)$. Similarly, $\kappa < u$ implies $Rep(\kappa) <_{in} u$.

Assume now $u <_{in} Rep(\kappa)$. If $\kappa < u$, then $Rep(\kappa) <_{in} u$. Hence, we must have $x < \kappa$.

Similarly, $Rep(\kappa) <_{in} u$ implies $\kappa < u$.

(iv) If $\kappa = (U_1, U_2) < \kappa' = (U_3, U_4)$, then U_2 contains some u not in U_4 , hence $u < U_4$, and thus $(U_1, U_2) < u < (U_3, U_4)$. Hence, if $\kappa < \kappa'$, we have $Rep(\kappa) <_{in} x <_{in} Rep(\kappa')$.

If conversely $Rep(\kappa) <_{in} Rep(\kappa')$, we cannot have $\kappa > \kappa'$ by the previous proof, hence $\kappa < \kappa'$. \square

Lemma 4.14 : (i) There is an MSO formula $\chi_1(x, U, W)$ such that, if $t \in T^\infty(F_M)$, $val(t) = (N, \leq, \mathcal{U}, A)$, $U, W \subseteq N, x \in N$, then $\chi_1(x, U, W)$ holds in $[t]$ if and only if $U \in \mathcal{U}, x = Rep(U)$ and $W = Rep(Cuts(U))$.

(ii) Similarly, there is an MSO formula $\chi_2(x, y, U, Z)$ that holds in $[t]$ if and only if $U \in \mathcal{U}, x = Rep(U), y = Rep(\kappa)$ where κ is a cut of U , and Z is the set of positions $Rep(V)$ such that V is an axis of an SOA-tree composing $Def(\kappa)$.

(iii) There is an MSO formula $\chi_3(x, U, Z)$ that holds in $[t]$ if and only if $U \in \mathcal{U}, x = Rep(U)$ and Z is the nonempty set of representing positions $Rep(V)$ such that V is an axis of an SOA-tree composing $Tail(U)$.

Proof:

(i) The following facts are MSO-expressible by the corresponding definitions:

- $U \in \mathcal{U}$,
- $x = Rep(U)$,
- $y = Rep(U_1, U_2)$ where (U_1, U_2) is a cut of U ,
- and $W = Rep(Cuts(U))$.

From these observations, we can construct $\chi_1(x, U, W)$.

(ii) Similar to (i). An MSO formula $\alpha(U, y, X)$ can identify the set of nodes X of $Def(\kappa)$ where $y = Rep(\kappa)$ and κ is a cut of U .

The SO-trees composing the SO-forest $Def(\kappa)$ can be identified, and so can be their axes and thus the representing positions of these axes.

(iii) Similar to the previous cases. □

Construction of a regular description scheme from a regular term.

We consider a regular term $t \in T^\infty(F_M)$. It defines a nonempty SOA-tree $J = (N, \leq, \mathcal{U}, A)$. Without loss of generality, we assume that A is not empty. Its representative position $Rep(A)$ is the root.

Our aim is to construct for J a regular description scheme. This construction will be based on a finite automaton accepting t , cf. Section 1.

Let \mathcal{B} such an automaton with set of states S . We get a regular term $t_{\mathcal{B}}$ (cf. Example 2.11(2)) by attaching to each position of t , the state reached there by the unique run $run_{\mathcal{B}}$ of \mathcal{B} . Each position x in this term is labelled by a pair (s, q) where s is the symbol of F_M occurring at x and $q \in S$.

We will construct a description scheme for J of the form $\Delta = (D, Q, d_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$ where D and Q are finite sets built from S , $m_q \in \mathcal{S}(D)$ for each $q \in Q$, w_d is a regular arrangement over $Q \cup \{*\}$ and $d_{Ax} \in D$.

Here is a key idea. Every line U of the structuring of J has a representative position $Rep(U)$ in t . The state of \mathcal{B} at $Rep(U)$ is some d . If another line U' has representative position $Rep(U')$ with the same state d , then $t/Rep(U) \simeq t/Rep(U')$, hence $U \simeq U'$ and even $s \triangleright U^+ \simeq s \triangleright U'^+$ by Lemma 4.17 below. Furthermore $s \triangleright U^+$ is a regular arrangement. It depends only on d and is the desired w_d . The definition of m_q is similar by means of Lemma 4.16.

Construction 4.15 : *From a regular term t to a description scheme for $val(t)$.*

We let $J := (N, \leq, \mathcal{U}, A) = val(t)$ where t is regular and A not empty. We will build a regular description scheme $\Delta = (Q, D, d_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$ that defines J .

(i) We define the finite sets D and Q :

$D := run_{\mathcal{B}}(P_1) \subseteq S$ where P_1 is the set of positions $Rep(U)$ for all $U \in \mathcal{U}$.

$Q_{tail} := run_{\mathcal{B}}(P'_1) \times \{1\}$ where $P'_1 \subseteq P_1$ is the set of positions $Rep(U)$ such that $Tail(U)$ is not empty.

$Q_{cut} := run_{\mathcal{B}}(P_2) \times \{2\}$ where P_2 is the set of positions $Rep(\kappa)$ of all cuts $\kappa \in \mathcal{K}$.

The sets D and $Q := Q_{tail} \uplus Q_{cut}$ are finite and disjoint because $D \subseteq S$, $Q_{tail} \subseteq S \times \{1\}$ and $Q_{cut} \subseteq S \times \{2\}$.

The element d_{Ax} is $run_{\mathcal{B}}(root_t)$ as we have $Rep(A) = root_t$ since A is not empty.

The multisets m_q and the arrangements w_d will be defined in Lemmas 4.16 and 4.17.

(ii) We define a good labelling (Definition 3.12) :

$r(U) := run_{\mathcal{B}}(Rep(U))$ for $U \in \mathcal{U}$,

$s(\text{Tail}(U)) := (\text{run}_{\mathcal{B}}(\text{Rep}(U)), 1) \in Q_{\text{tail}}$ for each $U \in \mathcal{U}$ such that $\text{Tail}(U)$ is not empty; this is well defined because each tail $\text{Tail}(U)$ corresponds to a unique line U ,
 $s(\text{Def}(\kappa)) := (\text{run}_{\mathcal{B}}(\text{Rep}(\kappa)), 2) \in Q_{\text{cut}}$ for each cut $\kappa \in \mathcal{K}$; this is well defined because each O-forest $\text{Def}(\kappa)$ corresponds to a unique cut κ .

Furthermore, $s(\text{Def}(\kappa)) = s(\text{Def}(\kappa'))$ implies $\text{run}_{\mathcal{B}}(\text{Rep}(\kappa)) = \text{run}_{\mathcal{B}}(\text{Rep}(\kappa'))$ hence $t/\text{Rep}(\kappa) \simeq t/\text{Rep}(\kappa')$ and $\text{Def}(\kappa) \simeq \text{Def}(\kappa')$ by Lemma 4.11(4). Similarly, $r(U) = r(U')$ implies $\text{run}_{\mathcal{B}}(\text{Rep}(U)) = \text{run}_{\mathcal{B}}(\text{Rep}(U'))$ hence $t/\text{Rep}(U) \simeq t/\text{Rep}(U')$ and $U \simeq U'$ by Lemma 4.9(2). Also, $s(\text{Tail}(U)) = s(\text{Tail}(U'))$ implies $t/\text{Rep}(U) \simeq t/\text{Rep}(U')$ and $\text{Tail}(U) \simeq \text{Tail}(U')$.

Lemmas 4.16 and 4.17 will prove that we have actually defined a good labelling and a regular description scheme.

With the previous notation, we have the following.

Lemma 4.16 : (1) The sets D , Q_{tail} and Q_{cut} are computable.

(2) For each pair $(q, 1)$ in Q_{tail} , one can compute the multiset

$$m_{(q,1)} := r\{\text{Axes}(\text{Tail}(U))\} \in \mathcal{S}(D),$$

where U is any line such that $s(\text{Tail}(U)) = (q, 1)$.

(3) For each pair $(q, 2)$ in Q_{cut} , one can compute the multiset

$$m_{(q,2)} := r\{\text{Axes}(\text{Def}(\kappa))\} \in \mathcal{S}(D),$$

where κ is any cut such that $s(\text{Def}(\kappa)) = (q, 2)$.

Proof:

(1) Follows from Lemma 2.13 ($t_{\mathcal{B}}$ is regular) by using the MSO formulas of Lemma 4.14.

(2) Let $(q, 1) \in Q$. One can determine the position $\text{Rep}(U)$ of some $U \in \mathcal{U}$ such that $\text{run}_{\mathcal{B}}(\text{Rep}(U)) = q$.

The term $t/\text{Rep}(U)$ is regular. An MSO formula¹⁶ can identify the nodes x of $\text{val}(t)$ (they are defined at leaves) such that $x < U$. They form a set disjoint from U that induces the SO-forest $\text{Tail}(U)$. The lines L in $\text{Axes}(\text{Tail}(U))$ can be identified by an MSO formula. Their representing positions $\text{Rep}(L)$ are all in $t/\text{Rep}(U)$, and can be identified by an MSO formula. For each of them, $r(L) := \text{run}_{\mathcal{B}}(\text{Rep}(L))$ is a state p of \mathcal{B} that can be read in the symbol (\bullet, p) occurring at $\text{Rep}(L)$. (We know that an MSO formula determines the symbol of $t_{\mathcal{B}}$ at each position.) By Lemma 2.13(2), for each $d \in D$, the number of lines $L \in \text{Axes}(\text{Tail}(U))$ such that $r(L) = d$ can be computed. Hence, we can compute $m_{(q,1)}$.

(3) The proof is similar. □

¹⁶that does not depend on t .

The next task is to prove that for each U in \mathcal{U} , the arrangement $s \triangleright U^+$ depends only on $r(U) := \text{run}_{\mathcal{B}}(\text{Rep}(U))$ and is regular. We recall that U^+ is the arrangement (U, \leq) in which each cut $\kappa(U, x)$ is inserted at its natural place and, furthermore, that is prefixed by τ_U standing for the tail of U if this tail is not empty.

We will describe $s \triangleright U^+$ as an MSO definable arrangement inside $[t]$, whose linear order is \leq_{in} . For this purpose, without introducing additional notation, we replace in U^+ a cut κ by the position $\text{Rep}(\kappa)$ and we will use Lemma 4.13.

Lemma 4.17 : Let t be a regular term accepted by a finite automaton \mathcal{B} and $\text{val}(t) = (N, \leq, \mathcal{U}, A)$. Let $U \in \mathcal{U}$.

(1) The arrangement U^+ can be defined from $t/\text{Rep}(U)$.

(2) The arrangement $s \triangleright U^+$ can be defined from $t_{\mathcal{B}}/\text{Rep}(U)$. It is regular and its MSO description can be computed.

Proof:

Let $U \in \mathcal{U}$.

(1) This is clear from definitions and previous remarks.

(2) Furthermore, $s \triangleright U^+$ can be defined from $t_{\mathcal{B}}/\text{Rep}(U)$. Hence, for any other line $U' \in \mathcal{U}$, if $r(U) = r(U')$, we have $s \triangleright U^+ \simeq s \triangleright U'^+$.

The term $t_{\mathcal{B}}/\text{Rep}(U)$ is regular as it is a subterm of the regular term $t_{\mathcal{B}}$. We first assume that $\text{Tail}(U)$ is empty. MSO formulas can identify the leaves belonging to U and the nodes $\text{Rep}(\kappa)$ for the cuts κ of U . Let X be the set of all these nodes : we label by $*$ the leaves belonging to U and a node $\text{Rep}(\kappa)$ by $(\text{run}_{\mathcal{B}}(\text{Rep}(\kappa)), 2)$. Then $s \triangleright U^+ \simeq (X, \leq_{in}, \text{lab})$. This arrangement is regular by Proposition 2.18(2). It is w_d where $d = \text{run}_{\mathcal{B}}(\text{Rep}(U))$. As proofs are effective, we can build a defining MSO sentence describing it.

If $\text{Tail}(U)$ is not empty, the construction the same, except that $s \triangleright U^+$ is then the arrangement $(X, \leq_{in}, \text{lab})$ prefixed by τ_U labelled by $(\text{run}_{\mathcal{B}}(\text{Rep}(U)), 1)$. Hence it is regular and is MSO-definable. \square

Hence Construction 4.15 and Lemmas 4.16 and 4.17 prove the following.

Proposition 4.18: From a finite automaton \mathcal{B} accepting a term $t \in T^{\infty}(F)$ that defines an SOA-tree, one can construct a regular description scheme for $\text{val}(t)$.

Theorem 4.19 : The following properties of an O-tree J are equivalent:

- (1) J is regular,
- (2) J is described by a regular description scheme,
- (3) J is MSO_{fin} -definable.

Proof:

(1) \implies (2). If J is regular, a regular term defines a structuring of it (Definition 4.4(d)). By Proposition 4.18 this structuring has a regular description scheme, that describes J according to Definition 3.13.

(2) \implies (3) If J is described by a regular description scheme, then, it is MSO_{fin} -definable by Theorem 3.18.

(3) \implies (1). By Definition 4.4, the mapping α that transforms the relational structure $[t]$ for t in $T^\infty(F)$ into the O-forest $val(t) = (N, \leq)$ (where $N = \text{Occ}(t, *)$) is an MSO-transduction, because an MSO formula can identify the nodes of $val(t)$ among the positions of t and other formulas can define \leq .

Let $J = (N, \leq)$ be an MSO_{fin} -definable O-tree. It is, up to isomorphism, the unique model of an MSO_{fin} sentence ψ . It follows by a standard argument¹⁷ that the set of terms t in $T^\infty(F)$ such that $\alpha([t]) \models \psi$ is MSO_{fin} definable. Since the structures $[t]$ are linearly ordered by \leq_{in} that is MSO-definable, this set is also MSO-definable (cf. Definition 2.8), and thus, contains a regular term, by a result due to Rabin [18]. This term defines J . Hence J is regular. \square

As for Corollaries 4.22 and 4.31 in [11] we have :

Corollary 4.20 : The isomorphism problem for regular O-trees is decidable.

Proof:

A regular O-tree can be defined by a regular term or by an MSO_{fin} sentence. The proof of Theorem 4.19 is effective: algorithms can convert any of these specifications into another one. Hence, two regular O-trees can be given, one by an MSO_{fin} sentence ψ , the other by a regular term t . They are isomorphic if and only if $\alpha([t]) \models \psi$ (cf. the proof of (3) \implies (1) of Theorem 4.19) if and only if $[t] \models \psi'$ where ψ' obtained by applying Backwards Translation [13] to the sentence ψ and the transduction α . This is decidable [18]. \square

Corollary 4.21 : An O-forest is regular if and only if it is MSO_{fin} -definable. The isomorphism of two regular O-forests is decidable.

Proof:

If J is an O-forest, then $J \bullet *$ is an O-tree. It is easy to prove that J is MSO_{fin} -definable if and only if $J \bullet *$ is. Furthermore, J is regular if and only if $J \bullet *$ is. The results follow then from Theorem 4.19 and Corollary 4.20. \square

Corollary 4.22 : (1) The MSO_{fin} -theory of a regular O-forest is decidable.

(2) It is decidable whether some O-forest satisfies a given MSO_{fin} -sentence. If so, this sentence is satisfied by some regular O-forest.

Proof:

We use routine constructions as for (3) \implies (1) of Theorem 4.19. \square

¹⁷If α is an MSO-transduction and ψ is an MSO_{fin} sentence, then the set of structures S such that $\alpha(S) \models \psi$ is MSO_{fin} -definable, by Backwards Translation [13].

5. Conclusion

We have shown how to describe O-trees and O-forests, *u.t.i.*, by infinite terms over three operations. We have generalized to regular O-forests the results of [11] that concern regular join-trees.

More complex terms than the regular ones have finitary descriptions (see [4]) and thus, can yield finitary descriptions of other types of O-forests. As they have decidable MSO theories, the MSO-theories of the defined O-forests are decidable.

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