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Theory of Constructive Semigroups with Apartness – Foundations, Development and Practice

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"Semigroups aren't a barren, sterile flower on the tree of algebra, they are a natural algebraic approach to some of the most fundamental concepts of algebra (and mathematics in general), this is why they have been in existence for more then half a century, and this is why they are here to stay."

Boris M. Schein, [57]

Abstract. This paper has several purposes. We present through a critical review the results from already published papers on the constructive semigroup theory, and contribute to its further development by giving solutions to open problems. We also draw attention to its possible applications in other (constructive) mathematics disciplines, in computer science, social sciences, economics, etc. Another important goal of this paper is to provide a clear, understandable picture of constructive semigroups with apartness in Bishop's style both to (classical) algebraists and the ones who apply algebraic knowledge.

Keywords: Semigroup with apartness, set with apartness, co-quasiorder, co-equivalence, co-congruence.

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1. Introduction

A general answer to the question what constructive mathematics is could be formulated as follows: it is mathematics which can be implemented on a computer. There are two main ways of developing mathematics constructively. The first one uses classical traditional logic within a strict algorithmic framework. The second way is to replace classical logic with *intuitionistic logic*.

Throughout this paper *constructive mathematics* is understood as mathematics performed in the context of intuitionistic logic, that is, without the law of excluded middle (**LEM**). There are two main characteristics for a constructivist trend. The notion of *truth* is not taken as primitive, and *existence* means constructibility. From the classical mathematics (**CLASS**) point of view, mathematics consists of a preexisting mathematical truth. From a constructive viewpoint, the judgement φ *is true* means that *there is a proof of* φ. "What constitutes a proof is a social construct, an agreement among people as to what is a valid argument. The rules of logic codify a set of principles of reasoning that may be used in a valid proof. Constructive (intuitionistic) logic codifies the principles of mathematical reasoning as it is actually practiced," [26]. In constructive mathematics, the *status of existence statement* is much stronger than in **CLASS**. The classical interpretation is that an object exists if its non-existence is contradictory. In constructive mathematics when the existence of an object is proved, the proof also demonstrates how to find it. Thus, following further [26], the *constructive logic* can be described as logic of people matter, as distinct from the classical logic, which may be described as the logic of the mind of God. One of the main features of constructive mathematics is that the concepts that are equivalent in the presence of **LEM**, need not be equivalent any more. For example, we distinguish nonempty and inhabited sets, several types of inequalities, two complements of a given set, etc.

There is no doubt about deep connections between constructive mathematics and computer science. Moreover, "if programming is understood not as the writing of instructions for this or that computing machine but as the design of methods of computation that is the computer's duty to execute, then it no longer seems possible to distinguish the discipline of programming from constructive mathematics", [36].

Constructive mathematics is not a unique notion. Various forms of constructivism have been developed over time. The principal trends include the following varieties: **INT** - Brouwer's intuitionistic mathematics, **RUSS** - the constructive recursive mathematics of the Russian school of Markov, **BISH** - Bishop's constructive mathematics. Every form has intuitionistic logic at its core. Different schools have different additional principles or axioms given by the particular approach to constructivism. For example, the notion of an *algorithm* or a *finite routine* is taken as primitive in **INT** and **BISH**, while **RUSS** operates with a fixed programming language and an algorithm is a sequence of symbols in that language. We have to emphasize that Errett Bishop - style constructive mathematics, **BISH**, forms the framework for our work. **BISH** enables one to interpret the results both in classical mathematics and in other varieties of constructivism. **BISH** originated in 1967 with the publication of the book *Foundations of Constructive Mathematics*, [4], and with its second, much revised edition in 1985, [5]. There has been a steady stream of publications contributing to Bishop's programme since 1967. A ten-year long systematic research of computable topology, using apartness as the fundamental notion, resulted in the first book, [13], on topology within **BISH** framework. `Modern algebra`, as is noticed in [11], "contrary to Bishop's expectations, also proved amenable to natural, thoroughgoing, constructive treatment".

Working within the classical theory of semigroups several years ago we, [39], decided to change the classical background with the intuitionistic one. This meant, among other things, that the perfect safety of the classical theory with developed notions, notations and methodologies was left behind. Instead, we embarked on an adventure into exploring an algebraically new area (even without clearly stated notions and notations) of *constructive semigroups with apartness*. What we had "in hand" at that moment was the experience and knowledge coming from the classical semigroup theory, other constructive mathematics disciplines such as, for example, constructive analysis, and, especially, from constructive topology, as well as constructive theories of groups and rings with tight apartness and computer science. For classical algebraists who, like us, wonder "on the odd day" what constructive algebra is all about, and who want to find out what it feels like doing it, they will understand soon that *constructive algebra* is more complicated than classical algebra in various ways: algebraic structures as a rule do not carry a decidable equality relation (this difficulty is partly met by the introduction of a strong inequality relation, the so-called apartness relation); there is (sometime) the awkward abundance of all kinds of substructures, and hence of quotient structures, [61].

As highlighted by Romano DA *et al*, [41], *the theory of semigroup with apartness* is a new approach to semigroup theory and not a new class of semigroups. It presents a semigroup facet of some relatively well established direction of constructive mathematics which, to the best of our knowledge, has not yet been considered within the semigroup community. This paper has several purposes: to present through a critical review results from our already published papers, [17], [18], [41], on the constructive point of view on semigroup theory, to contribute to its further development giving solutions to the problems posted (in the open, or, somehow hidden way) within the scope of those papers, and to lay the foundation for further works.

The order theory provides one of the most basic tools of semigroup theory within **CLASS**. In particular, the structure of semigroups is usually most clearly revealed through the analysis of the behaviour of their appropriate orders. The most basic concept leads to the *quasiorders*, reflexive and transitive relations with the fundamental concepts being introduced whenever possible in their natural properties. Going through [17], [18], [41], we can conclude that one of the main objectives of those papers is to develop an appropriate constructive order theory for semigroups with apartness. We outline some of the basic concepts of semigroups with apartness, as special subsets on the one hand, and orders on the other. The strongly irreflexive and co-transitive relations are building blocks of the constructive order theory we develop. With a primitive notion of 'set with apartness' our main intention was to connect all relations defined on such a set. This is done by requiring them to be a part (subset) of an apartness. Such a relation is clearly strongly irreflexive. If, in addition, it is co-transitive, then it is called *co-quasiorder*.

In algebra within **CLASS**, the formulation of homomorphic images (together with substructures and direct products) is one of the principal tools used to manipulate algebraic structures. In the study of homomorphic images of an algebraic structure, a lot of help comes from the notion of a quotient structure, which captures all homomorphic images, at least up to isomorphism. On the other hand, the homomorphism is the concept which goes hand in hand with congruences. The relationship between quotients, homomorphisms and congruences is described by the celebrated *isomorphism theorems*, which are a general and important foundational part of abstract and universal algebras. The quotient structures are not part of **BISH**. The quotient structure does not, in general, have a natural apartness relation. So, *the Quotient Structure Problem* (**QSP**) is one of the very first problem which has to be considered for any structure with apartness. The solutions of **QSP** problem for sets and semigroups with apartness were given in [17]. Some examples of special cases can be found in [52] and [53]. *Co-equivalences*, symmetric co-quasiorders, and equivalences which can be associated to them play the main roles. As an example that a single concept of classical mathematics may split into two or more distinct concepts when working constructively we have logical $\neg Y$ and apartness $\sim Y$ complement of a given subset Y of a set or semigroup with apartness. The key for the solution of the **QSP** for a set and semigroup with apartness is given by the next theorem (Theorem 2.3, [17]).

Theorem 1.1. If κ is a co-equivalence on S, then the relation $\sim\kappa$ is an equivalence on S, and κ defines apartness on $S/\sim\kappa$.

Theorem 1.1 is the key ingredient to formulate and prove the apartness isomorphism theorem for a set with apartness (see [17], Theorem 2.5). Based on these results, the apartness isomorphism theorem for a semigroup with apartness ([17], Theorem 3.4) is formulated and proved as well. The just mentioned results are significantly improved in [41], where, among other things, it is proved that the two complements, logical and apartness, coincide for a co-quasiorder τ defined on a set or semigroup with apartness, i.e. we have $\sim\tau = \neg\tau$ (see Proposition 2.3, Theorem 2.4 from [41]).

Remark 1. In [40], an overview to the development of isomorphism theorems in certain algebraic structures - from classical to constructive - is given.

It is well known that within **CLASS** a number of subsets of a semigroup enjoy special properties relative to multiplication, for example, completely isolated subset, convex subset, subsemigroup, ideal. On the other hand, relations defined on a semigroup can be distinguished one from another according to the behaviour of their related elements to multiplication. From that point of view, *positive quasiorders* are of special interest. Going through literature with the theory of semigroups as the main topic, one can see that there is almost no method of studying semigroups without a certain type of positive quasiorders involved. It is often the case that results on connections between positive quasiorders and subsets defined above showed them fruitful as well. Partly inspired by classical results, in [18] we consider *complement positive co-quasiorders*, i.e. constructive counterparts of positive quasiorders, and their connections with special subsets of semigroups with apartness. Inspired by the existing notion from constructive analysis and topology, we use the complements (both of them) for the classification of subsets of a given set with apartness. *Strongly detachable subsets*, i.e. those subsets for which we can decide whether an element x from that set belongs to the subset in question or to its apartness complement, play a significant role within the scope of [18]. In Lemma 3.2, we prove that for any co-quasiorder τ, defined on a set with apartness, its left and right τ-classes of any element are strongly detachable subsets. The main result of this paper, Theorem 4.1, gives the description of a complement positive co-quasiorder, defined on a semigroup with apartness, via the behaviour of its left and right classes and their connections with special subsets.

Apart from the above issues, an addition problem is the so-called *constant domain axiom*: the folklore type of axiom in **CLASS** algebra

$$\varphi \vee \forall x\, \psi(x) \leftrightarrow \forall x\, (\varphi \vee \psi(x)),$$

its constructive version

$$\varphi \vee \forall_x\, \psi(x) \rightarrow \forall_x\, (\varphi \vee \psi(x))$$

can be a source of problems when doing algebra constructively. We choose intuitionistic logic of constant domains **CD** to be the background of [18]. Recall that intermediate logic, such as, for example **CD**, the logic that is stronger than intuitionistic logic but weaker than classical one, can be constructed by adding one or more axioms to intuitionistic logic. There is a continuum of such logics. For more details see [1], [23].

The presence of apartness implies the appearance of different types of substructures connected to it. We deal with strongly detachable subsets in [18]. In [41] we mentioned two more: detachable and quasi-detachable subsets. In Proposition 2.1 we show that a strongly detachable subset is detachable and quasi-detachable. Even more, from [41] the apartness and logical complements coincide for strongly detachable and quasi-detachable subsets.

Going through [18] and [41] it can be noticed that we can face several problems arising from their scope. For example,

- The relations between detachable, strongly detachable and quasi detachable subsets are only partially described in [41], Proposition 2.1. A complete description of their relationship remains an open problem.
- Are the results of [18] valid in intuitionistic logic if we work with quasi-detachable subsets instead of strongly detachable ones? Which of the presented results or their form(s) are valid for the intuitionistic background, if any?

To conclude, the theory of semigroup with apartness, its background and motivations, further development and its possible applications as well as the critical answers to a number of questions including those mentioned above will be the main topics throughout this paper.

The paper is organized in the following way. In our work on constructive semigroups with apartness, as it is pointed out above, we have faced an algebraically completely new area. The background and motivation coming from the classical semigroup theory, other constructive mathematics disciplines and computer science are the content of **Section 2**. Some results on classical semigroups which can partly be seen as an inspiration for the constructive ones are also discussed here. In **Section 3**, the main one, we are going to give a critical review of some of the published results on sets and semigroups with apartness as well as the solutions to some of the open problems on sets and semigroups with apartness. One of the main results, Theorem 3.1, gives a complete description of the relationships between distinguished subsets of a set with apartness, which, in turn, justifies the constructive order theory we develop with those subsets with the main role in that framework. By Proposition 3.2, if any left/right-class of a co-quasiorder defined on a set with apartness is a (strongly) detachable subset then the limited principle of omniscience, **LPO**, holds. This shows that Theorem 4.1 (and Lemma 3.2

important for its proof) set in [18] cannot be proved in **BISH** without the logic of constant domains **CD**. Within intuitionistic logic, we can prove its weaker version, Theorem 3.7, which is another important result of this section. As for **QSP**, for sets and semigroups with apartness, we achieve a little progress in that direction. Theorem 3.2, the key theorem for the **QSP**'s solution, generalizes the similar ones from [17], [41]. In addition, as a generalization of the first apartness isomorphism theorem, the new theorem, Theorem 3.4, the second apartness isomorphism theorem for sets with apartness, is formulated and proved. Finally, in **Section 4**, examples of some already existing applications as well as future possible realizations of the ideas presented in the previous section are given.

More background on constructive mathematics can be found in [3], [4], [13], [61]. The standard reference for constructive algebra is [38]. For the classical case see [39], [43]. Examples of applications of these theoretical concepts can be found in [2], [12], [16], [25], [42].

2. Preliminaries: background, known results and motivation

Starting our work on constructive semigroups with apartness, as pointed out above, we faced an algebraically completely new area. What we had in "hand" at that moment were the experience and knowledge coming from the classical semigroup theory, other constructive mathematics disciplines, and computer science.

2.1. Algebra and semigroups within CLASS

"I was just going to say, when I was interrupted, that one of the many ways of classifying minds is under the heads of arithmetical and algebraical intellects. All economical and practical wisdom is an extension of the following arithmetical formula: 2 + 2 = 4. Every philosophical proposition has the more general character of the expression a + b = c. We are mere operatives, empirics, and egotists until we learn to think in letters instead of figures ."

Oliver Wendell Holmes: *The Autocrat of the Breakfast Table*

A very short account of abstract algebra and its development will be given here. Over the course of the 19th century, algebra made a transition from a subject concerned entirely with the solution of mostly polynomial equations to a discipline that deals with general structures within mathematics. The term abstract algebra as a name for this area appeared in the early 20th century. "In studying abstract algebra, a so called axiomatic approach is taken; that is, we take a collection of objects S and assume some rules about their structure. These rules are called axioms. Using the axioms for S, we wish to derive other information about S by using logical arguments. We require that our axioms be consistent; that is, they should not contradict one another. We also demand that there not be to many axioms. If a system of axioms is too restrictive, there will be few examples of the algebraic structure," [33].

An ***algebraic structure*** can be, informally, described as a set of some elements of objects with some (not necessarily, but often, binary) operations for combining them. A *set* is considered as a primitive notion which one does not define. We will take the intuitive approach that a set is some given collection of objects, called elements or members of the set. The cartesian product of a set S with itself, $S \times S$, is of special importance. A subset ρ of $S \times S$, or, equivalently, a property applicable to elements of $S \times S$, is called a *binary relation on* S. The ordered pair (S, ρ) is a particular *relational* *structure*. In general, there are many properties (for example: reflexivity, symmetry, transitivity) that binary relations may satisfy on a given set. As usual, for a relation ρ on S, $a\rho = \{x \in S : (a, x) \in \rho\}$, and $\rho a = \{x \in S : (x, a) \in \rho\}$ are the left and the right ρ-class of the element $a \in S$ respectively. The concept of an *equivalence*, i.e. reflexive, symmetric and transitive relation, is an extremely important one and plays a central role in mathematics. If ε is an equivalence on a set S, then $S/\varepsilon = \{x\varepsilon : x \in S\}$ is called the *quotient set of* S *by* ε. Classifying objects according to some property is a frequent procedure in many fields. Grouping elements in "a company" so that elements in each group are of the same prescribed property as performed by equivalence relations, and the classification gives the corresponding quotient sets. Thus, abstract algebra can show us how to identify objects with the same properties properly - we have to switch to a quotient structure (technique applicable, for example, to abstract data type theory).

Some fundamental concepts in abstract algebra are: set and operation(s) defined on that set; certain algebraic laws that all elements of a structure can respect, such as, for example, associativity, commutativity; some elements with special behaviour in connection with operation(s): idempotent elements, identity element, inverse elements, ... Combining the above concepts gives some of the most important structures in mathematics: groups, rings, semigroups, ... Centred around an algebraic structure are notions of: substructure, homomorphism, isomorphism, congruence, quotient structure. A mapping between two algebraic structures of the same type, that preserves the operation(s) or is compatible with the operation(s) of the structures is called *homomorphism*. Homomorphisms are essential to the study of any class of algebraic objects. An equivalence relation ρ on an algebraic structure S (such as a group, a ring, or a semigroup) that is compatible with the structure is called a *congruence*. Within **CLASS** the quotient set S/ρ becomes the structure of the same type in a natural way. The relationship between quotients, homomorphisms and congruences is described by the celebrated ***isomorphism theorems***. Isomorphism theorems are a general and important foundational part of abstract and universal algebra.

"Algebra is beautiful. It is so beautiful that many people forget that algebra can be very useful as well," [35]. Abstract algebra is the highest level of abstraction. Understanding it means, among other things, that one can think more clearly, more efficiently. With the development of computing in the last several decades, applications that involve algebraic structures have become increasingly important. To mention a few, lot of data structures form monoids (semigroups with the identity element); algebraic properties are important for parallel execution of programs - for example, combining a list of items with some binary operators can be easily parallelized if that operator is associative (commutativity is often required as well). Examples of applications given above lead to *semigroups*. In fact, following [34], (free) *semigroups* are the first mathematical objects every human being has to deal with - even before attending school.

2.1.1. More about the theory of semigroups

A ***semigroup*** is an algebraic structure consisting of a set with an associative binary operation defined on it. In the history of mathematics, *the algebraic theory of semigroups* is a relative newcomer, with the theory proper developing only in the second half of the twentieth century. Historically, it can be viewed as an algebraic abstraction of the properties of the composition of transformations on a set. But, there is no doubt about it, the main sources came from group and ring theories. However, semigroups are not a direct generalization of group theory as well as ring theory. Let us remember: congruences on groups are uniquely determined by their normal subgroups, and, on the other hand, there is a bijection between congruences and the ideals of rings. The study of congruences on semigroups is more complicated - no such device is available. One must study congruences as such. Thus, semigroups do not much resemble groups and rings. In fact, semigroups do not much resemble any other algebraic structure. Nowadays, semigroup theory is an enormously broad topic and has advanced on a very broad front. Following [37], "a huge variety of structures studied by mathematicians are sets endowed with associative binary operation." Even more, it appears that "semigroup theory provides a convenient general framework for unifying and clarifying a number of topics in fields that are seen, at first sight, unrelated", [34].

The capability and flexibility of semigroups from the point of view of modeling and problem-solving in extremely diverse situations have been already pointed out, and interesting new algebraic ideas arise with binary applications and connections to other areas of mathematics and sciences. Let us start our short journey through the applications of semigroups with the connections to the algebra of relations. The theory of semigroups is one of the main algebraic tools used in the theory of automata as well as the theory of formal languages. According to some authors, the role of the theory of semigroups for theoretical computer science is compared with the one which the philosophy has with the respect to science in general. Some investigations on transformation semigroups of synchronizing automata show up interesting implications for various applications for robotics, or more precisely, robotic manipulation. On the other hand, areas such as biology, biochemistry, sociology also make use of semigroups. For example, semigroups can be used in biology to describe certain aspects in the crossing of organisms, in genetics, and in consideration of metabolisms. Following [7], [35], the sociology includes the study of human interactive behaviour in group situations, in particular in underlying structures of societies. The study of such relations can be elegantly formulated in the language of semigroups. The book [8] is written for social scientists with the main aim to help readers to apply "interesting and powerful concepts" of semigroup theory to their own fields of expertise. However, the list of applications given above does not purport to mention all of the existing applications of semigroup theory. As it is pointed out in [37], it is often the case that "most applications make minimal use of the reach of the (classical) algebraic theory of semigroups." There is need for study of some more structures of semigroups which can find applications in different areas, [45]. This can bring very pretty mathematics to illustrate the interplay between certain scientific areas and semigroup-theoretic techniques. This type of research can be a topic on its own for certain types of papers.

In what follows some known results from the classical semigroup theory useful for our development will be presented.

A semigroup $(S, \cdot)$ is a set S together with an associative binary operation $\cdot$

(A) $\quad (\forall a, b, c \in S)\, [(a \cdot b) \cdot c \,=\, a \cdot (b \cdot c)].$

Where the nature of the multiplications is clear from the context, it is written S rather than $(S, \cdot)$. Frequently, xy is written rather than $x \cdot y$.

Various approaches have been developed over the years to construct frameworks for understanding the structure of semigroups. The fundamental concepts of semigroup theory elaborated by Suschkewitsch, Rees, Green, Clifford and other pioneers include as one of the main tools, *Green's quasiorders* (and equivalences generated by them), defined by the multiplication of semigroups and in terms of special subsemigroups. The notion of an order plays an important role throughout mathematics as well as in some adjacent disciplines such as logic and computer science. Order theory provides one of the most basic tools of semigroup theory as well. In particular, the structure of semigroups is usually most clearly revealed through the analysis of the behaviour of their appropriate orders. A pure order theory is concerned with a single undefined binary relation ρ. This relation is assumed to have certain properties (such as, for example, reflexivity, transitivity, symmetry, antisymmetry), the most basic of which leads to the concept of *quasiorder*. A quasiorder plays a central role throughout this short exposition with the fundamental concepts being introduced whenever possible in their natural properties.

Distinguishing subsets

A number of subsets of a semigroup enjoy special properties relative to the multiplication. A subset T of a semigroup S is:

- *completely isolated* if $ab \in T$ implies $a \in T$ or $b \in T$ for any $a, b \in S$,
- *convex* if $ab \in T$ implies both $a, b \in T$ for any $a, b \in S$,
- *subsemigroup* if for any $a, b \in T$ we have $ab \in T$,
- *ideal* if for any $a \in T$ and $s \in S$ we have $as, sa \in T$.

A subsemigroup T of S which is convex (resp. completely isolated) as a subset is called a *convex* (resp. *completely isolated*) *subsemigroup*. In an analogous way, we define a complex (completely isolated) ideal of S. Some of their existing properties are listed in the lemma below.

Lemma 2.1. Let S be a semigroup. Then:

(i) An ideal I of S is completely isolated if and only if $\neg I = S \setminus I$ is either a subsemigroup of S or is empty.

(ii) An nonempty subset F of S is convex if and only if $\neg F = S \setminus F$ is either a completely isolated ideal or is empty.

Within **CLASS** semigroups can historically be viewed as an algebraic abstraction of the transformations on a set. Of great importance is the role of the subsemigroups given above in describing the structure of transformation semigroups. We refer the reader to [22], [58] for more details about definitions, properties and applications of such subsemigroups.

Describing a semigroup and its structure is a formidable task. There are many different techniques developed for that purpose. *Semilattice decomposition of semigroups* is one of the methods with general applications. For more information on semilattice decomposition of semigroups see [39], [43]. It is shown in [43] that this method leads to the study of completely isolated ideals and convex subsemigroups.

Quasiorders

By definition, a binary relation ρ of set S is a subset of $S \times S$. To describe the relation defined on a semigroup S, we have to say which order pairs belong to ρ. In other words, for any $a \in S$, we have to know the following subsets of S:

$a\rho = \{x \in S \mid (a, x) \in \rho\}$,
$\rho a = \{x \in S \mid (x, a) \in \rho\}$,

called the left and right ρ-class of an element a. That is how we connect a study of binary relations defined on a given set with its subsets.

The relations defined on a semigroup S are distinguished one from another according to the behaviour of their related elements to the multiplication. A relation ρ defined on a semigroup S is

- *positive* if $(a, ab), (a, ba) \in \rho$, for any $a, b \in S$,
- *with common multiply property*, or, also called for short, with *cm-property* if $(a, c), (b, c) \in \rho$ implies $(ab, c) \in \rho$, for any $a, b, c \in S$.
- *with compatibility property* if $(x, y), (u, v) \in \rho$ implies $(xu, yv) \in \rho$ for any $x, y, u, v \in S$.

In the sequel, the positive quasiorders will also be considered. Recall that the *division relation* $|$ on a semigroup S, defined by

$$a \,|\, b \overset{\text{def}}{\Leftrightarrow} (\exists x, y \in S^1)\, b = xay,$$

for $a, b \in S$, is the smallest positive quasi-order defined on S. The positive quasi-orders were introduced in [56]. In [60] their link to semilattice decompositions of semigroups was established. In [44] their possible applications in psychology were announced. Finally, close connections between positive quasiorders and subsemigroups defined above were given in [6]. Here we mention some of these results:

Theorem 2.1. Let ρ be a quasiorder on S. The following conditions on a semigroup S are equivalent:

(i) ρ is a positive quasiorder;

(ii) $(\forall a, b \in S)\, (ab)\rho \subseteq a\rho \cap b\rho$;

(iii) $(\forall a, b \in S)\, \rho a \cup \rho b \subseteq \rho(ab)$;

(iv) $a\rho$ is an ideal for any $a \in S$;

(v) ρa is a convex subset of S for any $a \in S$.

Theorem 2.2. Let ρ be a quasiorder on S. The following conditions on a semigroup S are equivalent:

(i) ρ is a positive quasiorder with cm-property;

(ii) ρa is a convex subsemigroup of S for any $a \in S$;

(iii) $(\forall a, b \in S)\, (ab)\rho = a\rho \cap b\rho$.

Finally, we can say that, from the point of view of the classical semigroup theory, the interrelations between the following notions are of interest:

- semilattice decomposition of semigroups,
- completely isolated and convex subsemigroups and/or ideals,
- positive quasiorders.

Isomorphism theorems for semigroups

Let us remember that congruences on groups are uniquely determined by their normal subgroups, and, on the other hand, there is a bijection between congruences and the ideals of rings. The study of congruences on semigroups is more complicated - no such device is available. One must study congruences as such. A *congruence* ρ on a semigroup S is an equivalence, i.e. symmetric quasiorder, with the *compatibility property*. Classically, the quotient set S/ρ is then provided with a semigroup structure.

Theorem 2.3. Let S be a semigroup and ρ a congruence on it.Then S/ρ is a semigroup with respect to the operation defined by $(x\rho)(y\rho) = (xy)\rho$, and the mapping $\pi : S \to S/\rho$, $\pi(x) = x\rho$, $x \in S$, is an *onto* homomorphism.

Provided that the (*first*) *isomorphism theorem for semigroups* follows.

Theorem 2.4. Let $f : S \to T$ be a homomorphism between semigroups S and T. Then

(i) $ker\, f = f \circ f^{-1} = \{(x, y) \in S \times S : f(x) = f(y)\}$ is a congruence on S;

(ii) the mapping $\theta : S/ker\, f \to T$ defined by $\theta(x(ker\, f)) = f(x)$ is an embedding such that $f = \theta \circ \pi$;

(iii) if f maps S onto T, then θ is an isomorphism.

The theorem which follows is concerned with a more general situation.

Theorem 2.5. Let ρ be a congruence on a semigroup S, and let $f : S \to T$ be a homomorphism between semigroups S and T such that $\rho \subseteq ker\, f$. Then there exists a homomorphism of semigroups $\theta : S/\rho \to T$, such that $f = \theta \circ \pi$. If, in addition, f is *onto*, then θ is an isomorphism.

2.2. Constructive algebra

Constructive algebra is a relatively old discipline developed among others by L. Kronecker, van der Waerden, A. Heyting. For more information on the history see [38], [61]. One of the main topics in constructive algebra is constructive algebraic structures with the relation of (tight) apartness $\#$, the second most important relation in constructive mathematics. The principal novelty in treating basic algebraic structures constructively is that (tight) apartness becomes a fundamental notion. (Consider the reals: we cannot assert that x^{-1} exists unless we know that x is apart from zero, i.e. $|x| > 0$ - constructively that is not the same thing as $x \neq 0$. Furthermore, in fields x^{-1} exists only if x is apart from 0, [3]) The study of algebraic structures in the presence of *tight* apartness was started by Heyting, [27]. Heyting gave the theory a firm base in [29]. Roughly, the descriptive definition of a structure with apartness includes two main parts:

- the notion of a certain classical algebraic structure is straightforwardly adopted;
- a structure is equipped with an apartness with standard operations respecting that apartness.

Quotient structures are not part of **BISH**. A quotient structure does not, in general, have a natural apartness relation. So, *the Quotient Structure Problem* - **QSP** is one of the very first problems which has to be considered for any structure with apartness. Talking about the **QSP** for sets and semigroups with apartness and its history - solution of the **QSP** for sets with apartness is for the first time given in [17]. The **QSP**'s solutions for groups with *tight* apartness and commutative rings with *tight* apartness presented in [3], [19], [46], [48], [49], [54], [61] inspired us to give solutions of **QSP** for *sets with apartness* in 2013, [17], which, in turn, imply the solution of QSP for semigroups with apartness as its consequence.

A lot of ideas, notions and notations come from, for example, the constructive analysis, and, especially, from the constructive topology, as well as from constructive theories of groups and rings with *tight* apartness. Although the area of constuctive semigroups with apartness is still in its infancy, we can already conclude that, similarly to the clasical case, the semigroups with apartness do not much resemble groups and rings. In fact, they do not much resemble any other constructive algebraic structures with apartness.

2.3. Computer science

It is well known that formalization is a general method in science. Although it was created as a technique in logic and mathematics, it has entered into engineering as well. Formal engineering methods can be understood as mathematically-based techniques for the functional specification, development and verification in the engineering of software and hardware systems. Despite some initial suspicion, it was proved that formal methods are powerful enough to deal with real life systems. For example, it is shown that "software of the size and complexity as we find in modern cars today can be formally specified and verified by applying computer based tools for modeling and interactive theorem proving," [15].

Proof assistants are computer systems which give a user the possibility to do mathematics on a computer: from (numerical and symbolical) computing aspects to the aspects of defining and proving. The latter ones, doing proofs, are the main focus. It is believed that, besides their great future within the area of mathematics formalization, their applications within computer-aided modelling and verification of the systems are and will be more important. One of the most popular, with the intuitionistic background, is the proof assistant computer system *Coq*.

Coq is used for formal proves of well known mathematical theorems, such as, for example, the Fundamental Theorem of Algebra, FTA, [24]. For that purpose, the *constructive algebraic hierarchy* for Coq was developed, [25], consisting of constructive basic algebraic structures (semigroups, monoids, groups, rings, fields) with *tight* apartness. In addition, all these structures are limited to the commutative case. As it is noticed in [25] "that algebraic hierarchy has been designed to prove FTA. This means that it is not rich as one would like. For instance, we do not have noncommutative structure because they did not occur in our work." ... So, a question which arises from this is:

> *What can be done in connection with noncommutative semigroups with apartness where apartness is only "ordinary" and not the tight one?*

We put noncommutative constructive semigroups with "ordinary" apartness in the core of our study, proving first, of course, that such semigroups do exist, [17]. As in [5], we made "every effort to follow classical development along the lines suggested by familiar classical theories or in all together new directions."

The results of our several years long investigations, [17], [18], [41], present a semigroup facet of some relatively well established directions of constructive mathematics which, to the best of our knowledge, have not yet been considered within the semigroup community. The initial step towards grounding the theory done through our papers will be developed through the scope of this paper. We are going to give a critical review of some of those results as well as the solutions to some of the open problems arising from those papers.

3. Main results: sets and semigoups with apartness

Before starting our constructive examination of sets and semigroups with apartness, we should clarify its setting. By constructive mathematics we mean Bishop-style mathematics, **BISH**. We adopt Fred Richman's viewpoint, [47], where constructive mathematics is simply mathematics carried out with intuitionistic logic. The Bishop-style of constructive mathematics enables one to interpret the results both in classical mathematics, **CLASS**, and other varieties of constructivism. We regard classical mathematics as Bishop-style mathematics plus the law of excluded middle, **LEM**. This logical principle can be regarded as the main source of nonconstructivity. It was Brouwer, [14], who first observed that **LEM** was extended without justification to statements about infinite sets. Several consequences of **LEM** are not accepted in Bishop's constructivism. We will mention three such nonconstructive principles - the ones which will be used latter.

- **The limited principle of omniscience**, **LPO**: for each binary sequence $(a_n)_{n\geq 1}$, either $a_n = 0$ for all $n \in \mathbb{N}$, or else there exists n with $a_n = 1$.
- **The lesser limited principle of omniscience**, **LLPO**: if $(a_n)_{n\in\mathbb{N}}$ is a binary sequence containing at most one term equal to 1, then either $a_{2n} = 0$ for all $n \in \mathbb{N}$, or else $a_{2n+1} = 0$ for all $n \in \mathbb{N}$.
- **Markov's principle**, **MP**: For each binary sequence $(a_n)_{n\geq 1}$, if it is impossible that $a_n = 0$ for all $n \in \mathbb{N}$, then there exists n with $a_n = 1$.

Remark 2. **LPO** is equivalent to the decidability of equality on the real number line $\mathbb{R}$.

$$\forall_{x\in\mathbb{R}}\,(x = 0 \vee x \neq 0).$$

A detailed constructive study of $\mathbb{R}$ can be found in [10].

Within constructive mathematics, a statement P, as in classical mathematics, can be disproved by giving a counterexample. However, it is also possible to give a *Brouwerian counterexample* to show that the statement is nonconstructive. A Brouwerian counterexample to a statement P is a constructive proof that P implies some nonconstructive principle, such as, for example, **LEM**, and its weaker versions **LPO**, **LLPO**, **MP**. It is not a counterexample in the true sense of the word - it is just an indication that P does not admit a constructive proof. More details about nonconstructive principles and various classical theorems that are not constructively valid can be found in [31].

3.1. Set with apartness

The cornerstones for **BISH** include the notion of positive integers, sets and functions. The set $\mathbb{N}$ of positive numbers is regarded as a basic set, and it is assumed that the positive numbers have the usual algebraic and order properties, including mathematical induction.

Contrary to the classical case, a set exists only when it is defined. To define a set S, we have to give a property that enables us to construct members of S, and to describe the equality $=$ between elements of S – which is a matter of convention, except that it must be an equivalence. A set $(S, =)$ is an *inhabited* set if we can construct an element of S. The distinction between the notions of a nonempty set and an inhabited set is a key in constructive set theories. The notion of equality of different sets is not defined. The only way in which elements of two different sets can be regarded as equal is by requiring them to be subsets of a third set. For this reason, the operations of union and intersection are defined only for sets which are given as subsets of a given set. There is another problem to face when we consider families of sets that are closed under a suitable operation of complementation. Following [5] "we do not wish to define complementation in the terms of negation; but on the other hand, this seems to be the only method available. The way out of this awkward position is to have a very flexible notion based on the concept of *a set with apartness*."

A property P, which is applicable to the elements of a set S, determines a subset of S denoted by $\{x \in S : P(x)\}$. Furthermore, we will be interested only in properties $P(x)$ which are *extensional* in the sense that for all $x_1, x_2 \in S$ with $x_1 = x_2$, $P(x_1)$ and $P(x_2)$ are equivalent. Informally, it means that "it does not depend on the particular description by which x is given to us", [13].

An inhabited subset of $S \times S$, or, equivalently, a property applicable to elements of $S \times S$, is called a *binary relation* on S. In general, there are many properties that binary relations may satisfy on a given set. For instance, reflexivity, symmetry, transitivity, irreflexivity, strong irreflexivity, co-transitivity play a role under constructive rules.

In **CLASS**, equivalence is the natural generalization of equality. A theory with equivalence involves equivalence and functions, and relations respecting this equivalence. In constructive mathematics the same works without difficulty, [55].

Many sets come with a binary relation called inequality satisfying certain properties, and denoted by $\neq$, $\#$ or $\not\simeq$. In general, more computational information is required to distinguish elements of a set S, than to show that elements are equal. Comparing with **CLASS**, the situation for inequality is more complicated. There are different types of inequalities (denial inequality, diversity, apartness, tight apartness - to mention a few), some of them completely independent, which only in **CLASS** are equal to one standard inequality. So, in **CLASS** the study of the equivalence relation suffices, but in constructive mathematics, an inequality becomes a "basic notion in intuitionistic axiomatics". Apartness, as a positive version of inequality, "is yet another fundamental notion developed in intuitionism which shows up in computer science," [32].

Let $(S, =)$ be an *inhabited* set. By an ***apartness*** on S we mean a binary relation $\#$ on S which satisfies the axioms of irreflexivity, symmetry and cotransitivity:

(Ap1) $\neg(x\#x)$

(Ap2) $x\#y \Rightarrow y\#x$,

(Ap3) $x\#z \Rightarrow \forall_y (x\#y \vee y\#z)$.

If $x\#y$, then x and y are different, or distinct. Roughly speaking, $x = y$ means that we have a proof that x equals y while $x\#y$ means that we have a proof that x and y are different. Therefore, the negation of $x = y$ does not necessarily imply that $x\#y$ and vice versa: given x and y, we may have neither a proof that $x = y$ nor a proof that $x\#y$.

The negation of apartness is an equivalence $(\approx) \stackrel{def}{=} (\neg\,\#)$ called *weak equality* on S.

Remark 3. The statement that every equivalence relation is the negation of some apartness relation is equivalent to the excluded middle. The statement that the negation of an equivalence relation is always an apartness relation is equivalent to the nonconstructive de Morgan law.

The apartness on a set S is *tight* if

(Ap4) $\neg(x\#y) \Rightarrow x = y$.

Apartness is tight just when $\approx$ and $=$ are the same, that is $\neg(x\#y) \Leftrightarrow x = y$.

In some books and papers, such as [61], the term "preapartness" is used for an apartness relation, while "apartness" means tight apartness. The tight apartness on the real numbers was introduced by L. E. J. Brouwer in the early 1920s. Brouwer introduced the notion of apartness as a positive intuitionistic basic concept. A formal treatment of apartness relations began with A. Heyting's formalization of elementary intuitionistic geometry in [28]. The intuitionistic axiomatization of apartness is given in [30].

By extensionality, we have

(Ap5) $x\#y \wedge y = z \Rightarrow x\#z$,

the equivalent form of which is

(Ap5') $x\#y \wedge x = x' \wedge y = y' \Rightarrow x'\#y'$.

A *set with apartness* $(S, =, \#)$ is the starting point for our considerations, and will be simply denoted by S. The existence of an apartness relation on a structure often gives rise to an apartness relation on another structure. For example, given two sets with apartness $(S, =_S, \#_S)$ and $(T, =_T, \#_T)$, it is permissible to construct the set of mappings between them. Following [13], *a mapping*

$f : S \to T$ is an algorithm which produces an element $f(x)$ of T when applied to an element x of S, which is extensional, that is

$$\forall_{x,y \in S} (x =_S y \;\Rightarrow\; f(x) =_T f(y)).$$

A mapping $f : S \to T$ is:
- *onto* S or *surjection*: $\forall_{y \in T} \, \exists_{x \in S} \, (y =_T f(x))$;
- *one-one* or *injection*: $\forall_{x,y \in S} \, (f(x) =_T f(y) \;\Rightarrow\; x =_S y)$;
- *bijection* between S and T: it is a one-one and onto.

An important property applicable to mapping f is that of strong extensionality. Namely, a mapping $f : S \to T$ is a *strongly extensional* mapping, or, for short, an *se-mapping*, if

$$\forall_{x,y \in S} (f(x) \#_T f(y) \;\Rightarrow\; x \#_S y).$$

(The strong extensionality of all mappings from $\mathbb{R}$ to $\mathbb{R}$ implies Markov principle, **MP**, see [13].)

Furthermore, f is
- *apartness injective*, shortly *a-injective*: $\forall_{x,y \in S} \, (x \#_S y \;\Rightarrow\; f(x) \#_T f(y))$;
- *apartness bijective*: a-injective, se-bijective.

Given two sets with apartness S and T it is permissible to construct the set of ordered pairs $(S \times T, =, \#)$ of these sets defining apartness by

$$(s,t) \# (u,v) \;\stackrel{\text{def}}{\Leftrightarrow}\; s \#_S u \vee t \#_T v.$$

3.1.1. Distinguishing subsets

The presence of apartness implies the appearance of different types of substructures connected to it. Inspired by the constructive topology with apartness [13], we define the relation $\bowtie$ between an element $x \in S$ and a subset Y of S by

$$x \bowtie Y \;\stackrel{\text{def}}{\Leftrightarrow}\; \forall_{y \in Y} (x \# y).$$

A subset Y of S has two natural complementary subsets: *the logical complement* of Y

$$\neg Y \stackrel{\text{def}}{=} \{x \in S : x \notin Y\},$$

and *the apartness complement* or, shortly, the *a-complement* of Y

$$\sim Y \;\stackrel{\text{def}}{=}\; \{x \in S \,:\, x \bowtie Y\}.$$

Denote by $\widetilde{x}$ the a-complement of the singleton $\{x\}$. Then it can be easily shown that $x \in \sim Y$ if and only if $Y \subseteq \widetilde{x}$.

If the apartness is not tight we can find subsets Y with $\sim Y \subset \neg Y$ as in the following example.

Example 1. Let $S = \{a, b, c\}$ be a set with apartness defined by $\{(a,c), (c,a), (b,c), (c,b)\}$ and let $Y = \{a\}$. Then the a-complement $\sim Y = \{c\}$ is a proper subset of its logical complement $\neg Y = \{b, c\}$.

For a tight apartness, the two complements are constructive counterparts of the classical complement. In general, we have $\sim Y \subseteq \neg Y$. However, even for a tight apartness, the converse inclusion entails the Markov principle, **MP**. This result illustrates a main feature of constructive mathematics: classically equivalent notions could be no longer equivalent constructively. For which type of subset of a set with apartness do we have equality between its two complements? It turns out that the answer initiated a development of *order theory* for sets and semigroups with apartness

The complements are used for the classification of subsets of a given set. A subset Y of S is

- *a detachable* subset in S or, in short, a *d-subset* in S if

$$\forall_{x\in S}\,(x \in Y \vee x \in \neg Y);$$

- *a strongly detachable* subset of S, shortly *an sd-subset of* S, if

$$\forall_{x\in S}\,(x \in Y \vee x \in \sim\!Y),$$

- *a quasi-detachable* subset of S, shortly *a qd-subset of* S, if

$$\forall_{x\in S}\,\forall_{y\in Y}\,(x \in Y \vee x\#y).$$

The relations between detachable, strongly detachable and quasi-detachable subsets are partially described in [41], Proposition 2.1. A description of the relationships between those subsets of set with apartness, which, in turn, justifies the constructive order theory for sets and semigroups with apartness we develop, is given in the next theorem which is one of the main results of this paper.

Theorem 3.1. Let Y be a subset of S. Then:

(i) Any sd-subset is a qd-subset of S. The converse implication entails **LPO** .

(ii) Any qd-subset Y of S satisfies $\sim Y = \neg Y$.

(iii) If any qd-subset is a d-subset, then **LPO** holds.

(iv) If any d-subset is a qd-subset, then **MP** holds.

(v) Any sd-subset is a d-subset of S. The converse implication entails **MP**.

(vi) If any subset of a set with apartness S is a qd-subset, then **LPO** holds.

Proof:
(i). Let Y be an sd-subset of S. Then, applying the definition and logical axiom we have

$$\begin{aligned}\forall_{x\in S}\,(x \in Y \vee x \in \sim\!Y) \quad &\Leftrightarrow \quad \forall_{x\in S}\,(x \in Y \;\vee\; \forall_{y\in Y}(x\#y))\\ &\Rightarrow \quad \forall_{x\in S}\,\forall_{y\in Y}\,(x \in Y \vee x\#y).\end{aligned}$$

In order to prove the second part of this statement, we consider the real number set $\mathbb{R}$ with the usual (tight) apartness and the subset $Y = \widetilde{0}$. Then, for each real number x and for each $y \in Y$ it follows, from the co-transitivity of $\#$, either $y \# x$ or $x \# 0$, that is, either $x \in Y$ or $x \# y$. Consequently, Y is a qd-subset of $\mathbb{R}$. On the other hand, if Y is an sd-subset of $\mathbb{R}$, then for each $x \in \mathbb{R}$, either $x \in Y$ or $x \in \sim Y$. In the former case, $x \# 0$ and in the latter $x = 0$, hence **LPO** holds.

(ii). Let Y be a qd-subset, and let $a \in \neg Y$. By assumption we have

$$\forall_{x \in S} \forall_{y \in Y} \, (x \in Y \vee x \# y),$$

so substituting a for x, we get $\forall_{y \in Y} \, (a \in Y \vee a \# y)$, and since, by assumption, $\neg(a \in Y)$, it follows that $a \# y$ for all $y \in Y$. Hence $a \in \sim Y$. See also [41].

(iii). Let S be the real number set $\mathbb{R}$ with the usual apartness $\#$. As in the proof of (i), consider the qd-subset $\widetilde{0}$ of $\mathbb{R}$. If $\widetilde{0}$ is a d-subset of $\mathbb{R}$, then $x \in \widetilde{0}$ or $\neg(x \in \widetilde{0})$, for all real numbers x. In the latter case $\neg(x \# 0)$, which is equivalent to $x = 0$. Thus we obtain the property $\forall_{x \in \mathbb{R}} \, (x \# 0 \vee x = 0)$ which, in turn, is equivalent to **LPO**.

(iv). Consider a real number a with $\neg(a = 0)$ and let S be the set $\{0, a\}$ endowed with the usual apartness of $\mathbb{R}$. For $Y = \{0\}$, since $0 \in Y$ and $a \in \neg Y$, it follows that Y is a d-subset of S. On the other hand, if Y is a qd-subset of S, then $a \# 0$. It follows that for any real number with $\neg(a = 0)$, $a \# 0$ which entails the Markov Principle, **MP**.

(v). The first part follows immediately from (i), (ii) and the definition of d-subsets. The converse follows from (i) and (iv).

(vi). Consider again $\mathbb{R}$ with the usual apartness and define $Y = \{0\}$. If Y is a qd-subset of $\mathbb{R}$, then for all $x \in \mathbb{R}$ we have $x = 0$ or $x \# 0$, hence **LPO** holds. □

If the apartness is not tight, we can find subsets which are not qd-subsets, let alone sd-subsets. To show this, let us consider the set $S = \{a, b, c\}$ with the apartness defined in Example 1 and define $Y = \{a\}$. Then Y is not a qd-subset of S. If we work with a tight apartness, although vacuously true in classical mathematics, the properties of detachability are not automatically satisfied in **BISH**. The Brouwerian examples from Theorem 3.1 motivate the use of qd-subsets. Constructive mathematics brings to the light some notions which are invisible to the classical eye (here, the three notions of detachability).

3.1.2. Co-quasiorders

Let $(S \times S, =, \#)$ be a set with apartness. An inhabited subset of $S \times S$, or, equivalently, a property applicable to the elements of $S \times S$, is called a *binary relation* on S. Let α be a relation on S. Then

$$(a, b) \bowtie \alpha \Leftrightarrow \forall_{(x,y) \in \alpha} \, ((a, b) \# (x, y)),$$

for any $(a, b) \in S \times S$. The apartness complement of α is the relation

$$\sim \alpha = \{(x, y) \in S \times S : (x, y) \bowtie \alpha\}.$$

In general, we have $\sim \alpha \subseteq \neg \alpha$, which is shown by the following example.

Example 2. Let $S = \{a, b, c\}$ be a set with apartness defined by $\{(a, c), (c, a), (b, c), (c, b)\}$. Let $\alpha = \{(a, c), (c, a)\}$ be a relation on S. Its a-complement

$$\sim \alpha = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$$

is a proper subset of its logical complement $\neg\alpha$.

The relation α defined on a set with apartness S is

- irreflexive if $\forall_{x \in S}\, \neg((x, x) \in \alpha)$;
- strongly irreflexive if $(x, y) \in \alpha \;\Rightarrow\; x\#y$;
- co-transitive if $(x, y) \in \alpha \;\Rightarrow\; \forall_{z \in S}\, ((x, z) \in \alpha \,\vee\, (z, y) \in \alpha)$.

It is easy to check that a strongly irreflexive relation is also irreflexive. For a tight apartness, the two notions of irreflexivity are classically equivalent but not so constructively. More precisely, if each irreflexive relation were strongly irreflexive then **MP** would hold.

In the constructive order theory, the notion of co-transitivity, that is the property that for every pair of related elements, any other element is related to one of the original elements in the same order as the original pair is a constructive counterpart to classical transitivity, [17].

Lemma 3.1. Let α be a relation on S. Then:

(i) α is strongly irreflexive if and only if $\sim \alpha$ is reflexive;

(ii) if α is reflexive then $\sim \alpha$ is strongly irreflexive;

(iii) if α is symmetric then $\sim \alpha$ is symmetric;

(iv) if α is co-transitive then $\sim \alpha$ is transitive.

Proof:
(i). Let α be a strongly irreflexive relation on S. For each $a \in S$, it can be easily proved that $(a, a)\#(x, y)$ for all $(x, y) \in \alpha$.

Let $\sim \alpha$ be reflexive, that is $(x, x) \in\sim \alpha$, for any $x \in S$. On the other hand, the definition of the a-complement implies $(x, y)\#(x, x)$ for any $(x, y) \in \alpha$. So, $x\#x$ or $x\#y$. Thus, $x\#y$, that is, α is strongly irreflexive.

(ii). Let α be reflexive. Let (x, y) be an element of $\sim \alpha$. Since α is reflexive, $(y, y) \in \alpha$ hence $(x, y)\#(y, y)$ which implies $x\#y$. Consequently, $\sim \alpha$ is strongly irreflexive.

(iii). If α is symmetric, then

$$\begin{aligned}
(x, y) \in\sim \alpha \quad &\Leftrightarrow \quad \forall_{(a,b) \in \alpha}\, ((x, y)\#(a, b)) \\
&\Rightarrow \quad \forall_{(b,a) \in \alpha}\, ((x, y)\#(b, a)) \\
&\Rightarrow \quad \forall_{(b,a) \in \alpha}\, (x\#b \,\vee\, y\#a) \\
&\Rightarrow \quad \forall_{(a,b) \in \alpha}\, ((y, x)\#(a, b)) \\
&\Leftrightarrow \quad (y, x) \in\sim \alpha.
\end{aligned}$$

(iv). If $(x, y) \in\sim \alpha$ and $(y, z) \in\sim \alpha$, then, by the definition of $\sim \alpha$, we have that $(x, y) \bowtie \alpha$ and $(y, z) \bowtie \alpha$. For an element $(a, b) \in \alpha$, by co-transitivity of α, we have $(a, x) \in \alpha$ or $(x, y) \in \alpha$ or $(y, z) \in \alpha$ or $(z, b) \in \alpha$. Thus $(a, x) \in \alpha$ or $(z, b) \in \alpha$, which implies that $a\#x$ or $b\#z$, that is $(x, z)\#(a, b)$. So, $(x, z) \bowtie \alpha$ and $(x, z) \in \sim \alpha$. Therefore, $\sim \alpha$ is transitive. □

Remark 4. As it is shown in Lemma 3.1, it can be proved that the logical complement of each co-transitive relation is transitive. However, if the logical complement of any transitive relation were co-transitive, then **LLPO** would hold.

To prove this, let us consider the strict order $<$ on the real number line and assume that its logical complement $\geq$ is cotransitive. Then, taking into account that $x \geq x$, it follows that

$$\forall x \in \mathbf{R}\ (x \geq 0 \vee 0 \geq x)$$

which, in turn, is equivalent to **LLPO**. This Brouwerian example can be found in books on constructive mathematics, see, for example, [13].

The apartness complement $\sim \alpha$ of a relation α of S can be transitive without assuming co-transitivity of α. So, the converse statement from Lemma 3.1(iv), in general, is not true.

Example 3. Let $(S, =, \#)$ be a set with apartness defined in Example 2.

(1.) A strongly irreflexive (symmetric) relation $\alpha = \{(a, c), (c, a)\}$, which is not co-transitive has the a-complement

$$\sim \alpha = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$$

which is transitive.

(2.) A strongly irreflexive (nonsymmetric) relation $\alpha = \{(a, c), (c, a), (b, c)\}$, which is not co-transitive, has the a-complement

$$\sim \alpha = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$$

which is transitive.

Remark 5. The Brouwerian example from Remark 4 also shows that, even for a tight apartness, a relation whose co-transitivity cannot be proved constructively might have a transitive a-complement. To prove this, we need only observe that the a-complement of the relation $\geq$ is $<$.

A relation τ defined on a set with apartness S is a

- *weak co-quasiorder* if it is irreflexive and co-transitive,
- *co-quasiorder* if it is strongly irreflexive and co-transitive.

Remark 6. "One might expect that the splitting of notions leads to an enormous proliferation of results in the various parts of constructive mathematics when compared with their classical counterparts. In particular, usually only very few constructive versions of a classical notion are worth developing since other variants do not lead to a mathematically satisfactory theory," [61].

Even if the two classically (but not constructively) equivalent variants of a co-quasiorder are constructive counterparts of a quasiorder in the case of (a tight) apartness, the stronger variant, co-quasiorder, is, of course, the most appropriate for a constructive development of the theory of semigroups with apartness we develop, which will be evident in the continuation of this paper. The weaker variant, that is, weak co-quasiorder, could be relevant in analysis.

As in Example 2 the a-complement of a relation can be a proper subset of its logical complement. If the relation in question is a co-quasiorder, then we have the following important properties.

Proposition 3.1. Let τ be a co-quasiorder on S. Then:

(i) τ is a qd-subset of $S \times S$;

(ii) $\sim \tau = \neg \tau$.

Proof:
(i). Let $(x, y) \in S \times S$. Then, for all $(a, b) \in \tau$,

$$\begin{aligned} a\tau x \vee x\tau b &\Rightarrow a\tau x \vee x\tau y \vee y\tau b \\ &\Rightarrow a\#x \vee x\tau y \vee y\#b \\ &\Rightarrow (a,b)\#(x,y) \vee x\tau y, \end{aligned}$$

that is, τ is a qd-subset.

(ii). It follows from (i) and Theorem 3.1(ii). □

See also [41].

The co-quasiorder is one of the main building blocks for the order theory of semigroups with apartness we develop.

In general, to describe the relation we have to determine which ordered pairs belong to τ, that is, we have to determine $a\tau$ and τa, the left and the right τ-class of each element a from S. That is the way to connect (in **CLASS** and in **BISH** as well) a relation defined on a given set with certain subsets of the set. Starting from an sd-subset T of S, we are able to construct co-quasiorders as follows.

Lemma 3.2. Let T be an sd-subset of a set with apartness S. Then, the relation τ on S, defined by

$$(a, b) \in \tau \overset{\text{def}}{\Leftrightarrow} a \in\sim T \wedge b \in T,$$

is a co-quasiorder on S.

Proof:
Let $(a, b) \in \tau$, that is $a \in\sim T$ and $b \in T$, and let $x \in S$. By the assumption, T is an sd-subset, so we have $x \in T$ or $x \in\sim T$. If $x \in T$, then, by the definition of τ, we have $(a, x) \in \tau$. Similarly, if $x \in\sim T$, then $(x, b) \in \tau$. Thus, co-transitivity of τ is proved. By the definition of τ, the strong irreflexivity follows immediately. Thus, τ is a co-quasiorder on S. □

Example 4. Let $S = \{a, b, c, d, e\}$ be a set with the diagonal

$$\triangle_S = \{(a, a), (b, b), (c, c), (d, d), (e, e)\}$$

as the equality relation. If we denote by K the set $\triangle_S \cup \{(a, b), (b, a)\}$, then we can define an apartness $\#$ on S to be $(S \times S) \setminus K$. Thus, $(S, =, \#)$ is a set with apartness. The relation $\tau \subseteq S \times S$, defined by

$$\tau = \{(c, a), (c, b), (d, a), (d, b), (d, c), (e, a), (e, b), (e, c), (e, d)\},$$

is a co-quasiorder on S. (Left) τ-classes of S are: $a\tau = b\tau = \emptyset$, $c\tau = \{a, b\}$, $d\tau = \{a, b, c\}$, $e\tau = \{a, b, c, d\}$. It can be easily checked that all those τ-classes are sd-subsets of S.

Generally speaking, for a co-quasiorder defined on a set with apartness we can not prove that its left and/or right classes are d-subsets or sd-subsets. More precisely, we can prove the following result.

Proposition 3.2. Let τ be a co-quasiorder. Then:

(i) if $a\tau$ is a d-subset of S for any $a \in S$, then **LPO** holds;

(ii) if $a\tau$ is an sd-subset of S for any $a \in S$, then **LPO** holds.

Proof:
(i). Similar to the proof of Theorem 3.1(iii). It suffices to let τ be the usual apartness on the real number set and $a = 0$.

(ii). We can use the same example as above and apply Theorem 3.1(i). □

Having in mind what is just proved, we cannot expect to prove Lemma 3.2 from [18], as stated in [18], with d-subsets or sd-subsets without the Constant Domain Axiom, **CDA**.

3.1.3. Intuitionistic logic of constant domains CD as a background

Following [1], the intuitionistic logic of constant domains **CD** arises from a very natural Kripke-style semantics, which was proposed in [23] as a philosophically plausible interpretation of intuitionistic logic. **CD** can be formalized as intuitionistic logic extended with from the classical algebra point of view pretty strong principle, the Constant Domain Axiom, CDA,

$$\Vdash \forall_x (P \vee R(x)) \rightarrow (P \vee \forall_x R(x)),$$

where x is not a free variable of P. The intermediate logic obtained in this way, as it is pointed out in [1], further proves intuitionistically as well as classically valid theorems, yet they often possess a strong constructive flavour.

From a given co-quasiorder τ, with **CD** as a logical background, we are able to prove the connection of its classes with sd-subsets of S.

Lemma 3.3. Let τ be a co-quasiorder on a set S. Then $a\tau$ (respectively τa) is an sd-subset of S, such that $a \bowtie a\tau$ (respectively $a \bowtie \tau a$), for any $a \in S$. Moreover, if $(a, b) \in \tau$, then $a\tau \cup \tau b = S$ is true for all $a, b \in S$.

Proof:
See [18]. □

Lemma 3.3, due to Proposition 3.2, cannot be proved outside **CD** as logical background.

Remark 7. In intuitionistic logic of constant domain **CD**, the notions of sd-subset and qd-subset coincide.

3.1.4. QSP for sets with apartness

The Quotient Structure Problem, **QSP**, is one of the very first problems which has to be considered for any structure with apartness. The solutions of the **QSP** problem for sets and semigroups with apartness was given in [17]. Those results are improved in [41]. In what follows, we achieve a little progress in that direction. Theorem 3.2, the key theorem for the **QSP**'s solution generalizes the similar ones from [17], [41]. In addition, as a generalization of the Theorem 3.3, the first apartness isomorphism theorem, the new Theorem 3.4, that we call the second apartness isomorphism theorem for sets with apartness is formulated and proved.

The quotient structures are not part of **BISH**. A quotient structure does not have, in general, a natural apartness relation. For most purposes, we overcome this problem using a ***co-equivalence***–symmetric co-quasiorder–instead of an equivalence. Existing properties of a co-equivalence guarantee its a-complement is an equivalence as well as the quotient set of that equivalence will inherit an apartness. The following notion will be necessary. For any two relations α and β on S we can say that α *defines an apartness on* S/β if we have

(Ap6) $x\beta \# y\beta \overset{\text{def}}{\Leftrightarrow} (x, y) \in \alpha.$

If in addition α is a co-quasiorder and β is an equivalence, then (Ap6) implies

(Ap6') $((x, a) \in \beta \wedge (y, b) \in \beta) \Rightarrow ((x, y) \in \alpha \Leftrightarrow (a, b) \in \alpha).$

Indeed, let α be a co-quasiorder and β an equivalence on S such that α *defines an apartness on* S/β. Let $(x, a), (y, b) \in \beta$, i.e. $a \in x\beta$ and $b \in y\beta$, which, by the assumption, gives $a\beta = x\beta$ and $b\beta = y\beta$. If $(x, y) \in \alpha$, then, by (Ap6), $x\beta \# y\beta$, which, by (Ap5'), gives $a\beta \# b\beta$. By (Ap6) we have $(a, b) \in \alpha$. In a similar manner, starting from $(a, b) \in \alpha$ we can conclude $(x, y) \in \alpha$.

The next theorem is the key for the solution of **QSP** for sets with apartness. It generalizes the results from [17], [41].

Theorem 3.2. Let S be a set with apartness. Then:

(i) Let ε be an equivalence, and κ a co-equivalence on S. Then, κ defines an apartness on the factor set S/ε if and only if $\varepsilon \cap \kappa = \emptyset$.

(ii) The quotient mapping $\pi : S \to S/\varepsilon$, defined by $\pi(x) = x\varepsilon$, is an onto se-mapping.

Proof:
(i). Let $x, y \in S$ and assume that $(x, y) \in \varepsilon \cap \kappa$. Then $(x, y) \in \varepsilon$ and $(y, y) \in \varepsilon$, which, by (Ap6') and and $(x, y) \in \kappa$, gives $(y, y) \in \kappa$, which is impossible. Thus, $\varepsilon \cap \kappa = \emptyset$.

Let $(x, a), (y, b) \in \varepsilon$ and $(x, y) \in \kappa$. Then, by co-transitivity of κ and by assumption, we have

$$\begin{aligned} (x, y) \in \kappa \quad &\Rightarrow \quad (x, a) \in \kappa \vee (a, y) \in \kappa \\ &\Rightarrow \quad (x, a) \in \kappa \vee (a, b) \in \kappa \vee (b, y) \in \kappa \\ &\Rightarrow \quad (a, b) \in \kappa. \end{aligned}$$

(ii). Let $\pi(x)\#\pi(y)$, that is $x\varepsilon\#y\varepsilon$, which, by (i), means that $(x, y) \in \kappa$. Then, by the strong irreflexivity of κ, we have $x\#y$. So π is an se-mapping.

Let $a\varepsilon \in S/\varepsilon$ and $x \in a\varepsilon$. Then $(a, x) \in \varepsilon$, i.e. $a\varepsilon = x\varepsilon$, which implies that $a\varepsilon = x\varepsilon = \pi(x)$. Thus π is an onto mapping. □

Corollary 3.1. If κ is a co-equivalence on S, then the relation $\sim\kappa (= \neg\kappa)$ is an equivalence on S, and κ defines an apartness on $S/\sim\kappa$.

Proof:
By Lemma 3.1, $\sim\kappa$ is an equivalence, by Proposition 3.1, $(\sim\kappa) = (\neg\kappa)$, and, by Theorem 3.2, κ defines an apartness on $S/\sim\kappa$. □

Let $f : S \to T$ be an se-mapping between sets with apartness. Then the relation

$$\text{coker } f \overset{\text{def}}{=} \{(x, y) \in S \times S : f(x)\#f(y)\}$$

defined on S is called the *co-kernel* of f. Now, *the first apartness isomorphism theorem* for sets with apartness follows.

Theorem 3.3. Let $f : S \to T$ be an se-mapping between sets with apartness. Then

(i) the co-kernel of f is a co-equivalence on S which defines an apartness on $S/\ker f$;

(ii) the mapping $\theta : S/\ker f \to T$, defined by $\theta(x(\ker f)) = f(x)$, is a one-one, a-injective se-mapping such that $f = \theta \circ \pi$;

(iii) if f maps S onto T, then θ is an apartness bijection.

Proof:
See [41]. □

Now, *the second apartness isomorphism theorem*, a generalised version of Theorem 3.3, for sets with apartness follows.

Theorem 3.4. Let $f : S \to T$ be a mapping between sets with apartness, and let κ be a co-equivalence on S such that $\kappa \cap \ker f = \emptyset$. Then:

(i) κ defines apartness on factor set $S/\ker f$;

(ii) the projection $\pi : S \to S/\ker f$ defined by $\pi(x) = x(\ker f)$ is an onto se-mapping;

(iii) the mapping f induces a one-one mapping $\theta : S/\ker f \to T$ given by $\theta(x(\ker f)) = f(x)$, and $f = \theta \circ \pi$;

(iv) θ is an se-mapping if and only if $\text{coker}\, f \subseteq \kappa$;

(v) θ is a-injective if and only if $\kappa \subseteq \text{coker}\, f$.

Proof:
(i). It follows from Theorem 3.2(i).
(ii). It follows from Theorem 3.2(ii).
(iii). This was shown in Theorem 3.3.
(iv). Let θ be an se-mapping. Let $(x, y) \in \text{coker}\, f$ for some $x, y \in S$. Then, by definition of $\text{coker}\, f$ and θ, the assumption and (i), we have

$$\begin{aligned} f(x)\#f(y) &\Leftrightarrow \theta(x(\ker f))\#\theta(y(\ker f)) \\ &\Rightarrow x(\ker f)\#y(\ker f) \\ &\Leftrightarrow (x, y) \in \kappa. \end{aligned}$$

Conversely, let $\text{coker}\, f \subseteq \kappa$. By assumption, (i), (iii) and the definitions of θ and $\text{coker}\, f$, we have

$$\begin{aligned} \theta(x(\ker f))\#\theta(y(\ker f)) &\Leftrightarrow f(x)\#f(y) \\ &\Leftrightarrow (x, y) \in \text{coker}\, f \\ &\Rightarrow (x, y) \in \kappa \\ &\Leftrightarrow x(\ker f)\#y(\ker f). \end{aligned}$$

(v). Let θ be a-injective, and let $(x, y) \in \kappa$. Then, by (iii), we have

$$\begin{aligned} x(\ker f)\#y(\ker f) &\Rightarrow \theta(x(\ker f))\#\theta(y(\ker f)) \\ &\Leftrightarrow f(x)\#f(y) \\ &\Leftrightarrow (x, y) \in \text{coker}\, f. \end{aligned}$$

Conversely, let $\kappa \subseteq \text{coker}\, f$. Then

$$\begin{aligned} x(\ker f)\#y(\ker f) &\Leftrightarrow (x, y) \in \kappa \\ &\Rightarrow (x, y) \in \text{coker}\, f \\ &\Leftrightarrow f(x)\#f(y) \\ &\Leftrightarrow \theta(x(\ker f))\#\theta(y(\ker f)). \end{aligned}$$

□

Corollary 3.2. Let $f : S \to T$ be a mapping between sets with apartness, and let κ be a co-equivalence on S such that $\kappa \cap \ker f = \emptyset$. Then:

(i) f is an se-mapping if and only if $\text{coker}\, f$ is strongly irreflexive;

(ii) if $\theta : S/\ker f \to T$, defined by $\theta(x(\ker f)) = f(x)$, is an se-mapping, then f is an se-mapping too.

Proof:

(i). Let f be an se-mapping. Then, by Theorem 3.3, $\operatorname{coker} f$ is strongly irreflexive. The converse is almost obvious.

(ii). If θ is an se-mapping then, by Theorem 3.4(iv), we have that $\operatorname{coker} f \subseteq \kappa$. So, the strong irreflexivity of κ implies the strong irreflexivity of $\operatorname{coker} f$, which, by (i), implies f is an se-mapping. □

Remark 8. The notion of co-quasiorder first appeared in [50]. However, let us mention that the results reported from [50]: Theorem 0.4, Lemma 0.4.1, Lemma 0.4.2, Theorem 0.5 and Corollary 0.5.1 (pages 10-11 in [51]) are not correct. Indeed, the mentioned filled product is not associative in general. The notion of co-equivalence, i.e. a symmetric co-quasiorder, first appeared in [9].

3.2. Semigroups with apartness

Given a set with apartness $(S, =, \#)$, the tuple $(S, =, \#, \cdot)$ is a *semigroup with apartness* if the binary operation $\cdot$ is associative

(A) $\quad \forall_{a,b,c \in S}\,[(a \cdot b) \cdot c = a \cdot (b \cdot c)]$,

and strongly extensional

(S) $\quad \forall_{a,b,x,y \in S}\,(a \cdot x \# b \cdot y \Rightarrow (a \# b \vee x \# y))$.

As usual, we are going to write ab instead of $a \cdot b$. For example, for a given set with apartness A we can construct a semigroup with apartness $S = A^A$ in the following way.

Theorem 3.5. Let S be the set of all se-functions from A to A with the standard equality $=$

$$f = g \iff \forall_{x \in A}\,(f(x) = g(x))$$

and apartness

$$f \# g \iff \exists_{x \in A}\,(f(x) \# g(x)).$$

Then $(S, =, \#, \circ)$ is a semigroup with respect to the binary operation $\circ$ of composition of functions.

Proof:

See [18]. □

Until the end of of this paper, we adopt the convention that *semigroup* means *semigroup with apartness*. Apartness from Theorem 3.5 does not have to be tight, [17].

Let S and T be semigroups with apartness. A mapping $f : S \to T$ is a homomorphism if

$$\forall_{x,y \in S}\,(f(xy) = f(x)f(y)).$$

A homomorphism f is

- an *se-embedding* if it is one-one and strongly extensional;

- an *apartness embedding* if it is a-injective se-embedding;
- an *apartness isomorphism* if it is apartness bijection and se-homomorphism.

Within **CLASS**, the semigroups can be viewed, historically, as an algebraic abstraction of the properties of the composition of transformations on a set. Cayley's theorem for semigroups (which can be seen as an extension of the celebrated Cayley's theorem on groups) stated that every semigroup can be embedded in a semigroup of all self-maps on a set. As a consequence of the Theorem 3.5, we can formulate *the constructive Cayley's theorem for semigroups with apartness* as follows.

Theorem 3.6. Every semigroup with apartness se-embeds into the semigroup of all strongly extensional self-maps on a set.

Proof:
See [18]. □

Remark 9. Following [47], the term "constructive theorem" refers to a theorem with constructive proof. A classical theorem that is proven in a constructive manner is a constructive theorem.

It is a pretty common point of view that classical theorem becomes more enlightening when it is seen from the constructive viewpoint. On the other hand, it can not be said that the theory of constructive semigroups with apartness aims at revising the whole classical framework in nature.

3.2.1. Co-quasiorders defined on a semigroup

We are going to encounter sd-subsets or sd-subsemigroups which have some of the properties mentioned in Section 2.1.1. A strongly detachable convex (respectively completely isolated) subsemigroup of S is called, in short, an *sd-convex* (respectively sd-completely isolated) subsemigroup of S. Similarly, there are sd-convex and sd-completely isolated ideals of S.

Lemma 3.4. Let S be a semigroup with apartness. The following conditions are true:

(i) Let T be an sd-convex subset of a semigroup with apartness S. If $\sim T$ is inhabited, then it is an ideal of S.

(ii) If I is an sd-completely isolated ideal of a semigroup with apartness S, then $\sim I$ is a convex subsemigroup of S.

Proof:
(i). Let $x, y, \in \sim T$. Let $a \in \sim T$ and $x \in S$. By the assumption we have that $ax \in T$ or $ax \in\sim T$. If $ax \in T$, then, as T is convex, we have $a \in T$, which is impossible. Similarly, one can prove that $xa \in \sim T$. So, $\sim T$ is an ideal of S.

(ii). In a similar manner as in (i) we can prove that $\sim I$ is a subsemigroup of S.

Let $xy \in \sim I$. By the assumption, we have $x \in I$ or $x \in\sim I$. If $x \in I$, then, as I is an ideal, we have $xy \in I$, which is impossible. Thus $x \in\sim I$. Similarly, we can prove that $y \in \sim I$. So, $\sim I$ is convex. □

Let us start with an example of a co-quasiorder defined on a semigroup with apartness S.

Example 5. Let S be a semigroup given by

·	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

Let the equality on S be the diagonal $\triangle_S = \{(a,a),(b,b),(c,c),(d,d),(e,e)\}$. If we denote by $K = \triangle_S \cup \{(a,b),(b,a)\}$, then we can define an apartness $\#$ on S by $(S \times S) \setminus K$. The relation $\tau \subseteq S \times S$, defined by

$$\tau = \{(c,a),(c,b),(d,a),(d,b),(d,c),(e,a),(e,b),(e,c),(e,d)\},$$

is a co-quasiorder on S.

Let τ be a co-quasiorder defined on a semigroup S with apartness. Following the classical results as much as possible, we can start with the following definition.

A co-quasiorder τ on a semigroup S is

- *complement positive* if $(a,ab),(a,ba) \in\sim \tau$ for any $a,b \in S$,
- with *constructive common multiple property*, or, in short, with *constructive cm-property* if $(ab,c) \in \tau \Rightarrow (a,c) \in \tau \vee (b,c) \in \tau$ for all $a,b,c \in S$,
- with *complement common multiple property*, or, in short, with *complement cm-property* if $(a,c),(b,c) \in\sim \tau \Rightarrow (ab,c) \in\sim \tau$ for all $a,b,c \in S$.

Recall, by the Proposition 3.1, $(\sim \tau) = (\neg\tau)$.

Example 6. The co-quasiorder α defined on the semigroup S considered in Example 5 is not complement positive because we have $(e,ea) = (e,d) \in \alpha$.

Example 7. Let S be the three element semilattice given by

·	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

Let the equality on S be the diagonal $\triangle_S = \{(a,a),(b,b),(c,c)\}$. We can define an apartness $\#$ on S to be $(S \times S) \setminus \triangle_S$. Thus, $(S, =, \#, \cdot)$ is a semigroup with apartness. The relation $\tau \subseteq S \times S$, defined by

$$\tau = \{(a,b),(c,a),(c,b)\},$$

is a complement positive co-quasiorder on S.

On the other hand, from $(ab,a) = (c,a) \in \tau$ neither (a,a) nor (b,a) are in τ, so τ does not have the constructive cm-property. From $(a,a) \bowtie \tau$ and $(b,a) \bowtie \tau$, we have $(ab,a) = (c,a) \in \tau$, and τ does not have the complement cm-property as well.

The following lemma shows how some sd-subsets lead us to positive co-quasiorders.

Lemma 3.5. Let S be a semigroup with apartness S.

(i) If K is an sd-convex subset of S, then the relation τ defined by

$$(a,b) \in \tau \overset{\text{def}}{\Leftrightarrow} a \in\sim K \wedge b \in K$$

is a complement positive co-quasiorder on S.

(ii) If J is an sd-ideal of S such that $J \subset S$, then the relation τ defined by

$$(a,b) \in \tau \overset{\text{def}}{\Leftrightarrow} a \in J \wedge b \in\sim J$$

is a complement positive co-quasiorder on S.

Proof:
(i). By Lemma 3.2, τ is a co-quasiorder on S. Let $(x,y) \in \tau$. By the co-transitivity of τ, we have $(x,a) \in \tau \vee (a,ab) \in \tau \vee (ab,y) \in \tau$, for any $a, b \in S$. If $(a,ab) \in \tau$, then, by the definition of τ, we have $a \in\sim K$ and $ab \in K$, and, as K is a convex subset, we have $a \in K$ and $b \in K$, which is impossible. So, we have $(x,a) \in \tau \vee (ab,y) \in \tau$. By the strong irreflexivity of τ we have $x\#a \vee ab\#y$, i.e. $(x,y)\#(a,ab)$. Thus, we have proved that $(a,ab) \bowtie \tau$ for any $a,b \in S$. The proof of $(a,ba) \bowtie \tau$ is similar. Therefore, τ is a complement positive co-quasiorder on S.

(ii). By Lemma 3.2, τ is a co-quasiorder on S. Let $(x,y) \in \tau$. By the co-transitivity of τ, we have $(x,a) \in \tau \vee (a,ab) \in \tau \vee (ab,y) \in \tau$, for any $a,b \in S$. If $(a,ab) \in \tau$, then, by the definition of τ, we have $a \in J$ and $ab \in\sim J$, which, as J is an ideal, further implies $ab \in J$, which is a contradiction. The rest of the proof is similar to the arguments in the proof of (i). □

By Proposition 3.2, if any left/right-class of a co-quasiorder defined on a set with apartness is a (strongly) detachable subset, then **LPO** holds. This shows that Theorem 4.1 on a complement positive co-quasiorder (and Lemma 3.2 important for its proof) from [18] cannot be proved outside intuitionistic logic of constant domains **CD**. Nevertheless, we can prove within intuitionistic logic the next theorem, which is its weaker version, and another important result of this section. The description of a complement positive co-quasiorder via its classes follows.

Theorem 3.7. Let τ be a co-quasiorder τ on a semigroup S.

(i) If τ is complement positive, then

$$\forall_{a,b\in S}\ \ (\tau(ab) \subseteq \tau a \cap \tau b).$$

(ii) If τa is an sd-ideal of S and $a \bowtie \tau a$ for every $a \in S$, then τ is complement positive and

$$\forall_{a,b\in S}\ \ (a\tau \cup b\tau \subseteq (ab)\tau).$$

(iii) If $a\tau$ is an sd-convex subset of S, and $a \bowtie a\tau$ for every $a \in S$, then τ is a complement positive co-quasiorder.

Proof:
(i). Let τ be a complement positive co-quasiorder. For all $a, b, x \in S$ such that $x \in \tau(ab)$, that is $(x, ab) \in \tau$, by the co-transitivity of τ, we have

$$((x, a) \in \tau \vee (a, ab) \in \tau) \wedge ((x, b) \in \tau \vee (b, ab) \in \tau).$$

But, τ is complement positive, so that we have $(x, a) \in \tau \wedge (x, b) \in \tau$, i.e. $x \in \tau a \cap \tau b$.

(ii). Let $(x, y) \in \tau$ and $a, b \in S$. Then, by the co-transitivity of τ,

$$(x, a) \in \tau \vee (a, ab) \in \tau \vee (ab, y) \in \tau.$$

If $a \in \tau(ab)$, then, as $\tau(ab)$ is an ideal, we have $ab \in \tau(ab)$, which is, by the assumption, impossible. Now, by the strong irreflexivity of τ, we have $x\#a$ or $ab\#y$, that is $(x, y)\#(a, ab)$. Thus, $(a, ab) \bowtie \tau$ for any $a, b \in S$. $(a, ba) \bowtie \tau$ can be proved similarly. Thus, τ is a complement positive co-quasiorder.

Let $x \in a\tau \cup b$, $x \in S$. By the co-transitivity and complement positivity of τ, we have

$$\begin{aligned} x \in a\tau \cup b\tau &\Leftrightarrow x \in a\tau \vee x \in b\tau \\ &\Leftrightarrow (a, x) \in \tau \vee (b, x) \in \tau \\ &\Rightarrow ((a, ab) \in \tau \vee (ab, x) \in \tau) \vee ((b, ab) \in \tau \vee (ab, x) \in \tau) \\ &\Rightarrow (ab, x) \in \tau \\ &\Leftrightarrow x \in (ab)\tau. \end{aligned}$$

(iii). Let $(x, y) \in \tau$. Then, by the co-transitivity of τ,

$$(x, a) \in \tau \vee (a, ab) \in \tau \vee (ab, y) \in \tau,$$

for any $a, b \in S$. Let $(a, ab) \in \tau$, that is $ab \in a\tau$. Then, by assumption, $a \in a\tau$ (and $b \in a\tau$), which is impossible. Now, by the strong irreflexivity of τ, we have $x\#a$ or $ab\#y$, that is, $(x, y)\#(a, ab)$. Thus $(a, ab) \bowtie \tau$ for any $a, b \in S$. $(a, ba) \bowtie \tau$ can be proved similarly. Thus, τ is a complement positive co-quasiorder. □

Theorem 3.8. A complement positive co-quasiorder with the constructive cm-property has the complement cm-property.

Proof:
Let τ be a complement positive co-quasiorder with the constructive cm-property on a semigroup S and let $a, b, c, x, y \in S$ be such that $(a, c), (b, c) \bowtie \tau$ and $(x, y) \in \tau$. Then we have

$$\begin{aligned}
(x, y) \in \tau &\Rightarrow (x, ab) \in \tau \vee (ab, c) \in \tau \vee (c, y) \in \tau && \text{by co-transitivity} \\
&\Rightarrow x\#ab \vee (a, c) \in \tau \vee (b, c) \in \tau \vee c\#y && \text{by strong reflexivity} \\
& && \text{and by constructive cm-property} \\
&\Rightarrow (ab, c)\#(x, y) && \text{since } (a, c) \bowtie \tau \text{ and } (b, c) \bowtie \tau.
\end{aligned}$$

Hence $(ab, c) \bowtie \tau$, i.e. $(ab, c) \in\sim \tau$. □

3.2.2. Intuitionistic logic of constant domains CD as a background

If we have a complement positive co-quasiorder τ on a semigroup with apartness S, we can construct special subsets and semigroups mentioned above. Some other criteria for a co-quasiorder to be complement positive will be given too.

Theorem 3.9. The following conditions for a co-quasiorder τ on a semigroup S are equivalent:

(i) τ is complement positive;

(ii) $\forall_{a,b \in S}\ (a\tau \cup b\tau \subseteq (ab)\tau)$;

(iii) $\forall_{a,b \in S}\ (\tau(ab) \subseteq \tau a \cap \tau b)$;

(iv) $a\tau$ is an sd-convex subset of S and $a \bowtie a\tau$ for every $a \in S$;

(v) τa is an sd-ideal of S and $a \bowtie \tau a$ for every $a \in S$.

Proof:
(i) ⇒ (iii), (v) ⇒ (i), (v) ⇒ (ii), (iv) ⇒ (ii). Those implications are proved in the Theorem 3.7.

(iii) ⇒ (iv). By Lemma 3.3, $a\tau$ is an sd-subset of S such that $a \bowtie a\tau$ for any $a \in S$. We have

$$\begin{aligned}
xy \in a\tau &\Leftrightarrow (a, xy) \in \tau \\
&\Leftrightarrow a \in \tau(xy) \subseteq \tau x \cap \tau y \\
&\Rightarrow a \in \tau x \wedge a \in \tau y \\
&\Leftrightarrow x \in a\tau \wedge y \in a\tau.
\end{aligned}$$

So, $a\tau$ is an sd-convex subset for any $a \in S$.

(i) ⇒ (v). By Lemma 3.3, τa is an sd-subset of S such that $a \bowtie \tau a$ for any $a \in S$. Let $a, x \in S$ be such that $x \in \tau a$, i.e. $(x, a) \in \tau$. By the co-transitivity of τ, we have

$$(x, xs) \in \tau \vee (xs, a) \in \tau,$$

for any $s \in S$. But, as τ is positive, we have only $(xs, a) \in \tau$, i.e $xs \in \tau a$. In the same way one can prove that $sx \in \tau a$. Thus, τa is an ideal of S for any $a \in S$.

(ii) $\Rightarrow$ (v). By Lemma 3.3, τa is an sd-subset of S, and $a \bowtie \tau a$ for any $a \in S$. Now, let $x \in \tau a$ and $s \in S$. Then, by the co-transitivity of τ, we have $(x, xs) \in \tau$ or $(xs, a) \in \tau$. If $(x, xs) \in \tau$, then $xs \in x\tau \subseteq x\tau \cup s\tau \subseteq (xs)\tau$, which is, by Lemma 3.3, impossible. Thus $(xs, a) \in \tau$. As $(sx, a) \in \tau$ can be proved similarly, we have proved that τa is an sd-ideal of S. □

Theorem 3.10. Let τ be a complement positive co-quasiorder on a semigroup S. The following conditions are equivalent:

(i) τ has the constructive cm-property;

(ii) $\forall_{a,b \in S}\ ((ab)\tau = a\tau \cup b\tau)$;

(iii) τa is an sd-completely isolated ideal of S such that $a \bowtie \tau a$ for any $a \in S$.

Proof:
(i) $\Rightarrow$ (ii). By Theorem 3.9, $a\tau \cup b\tau \subseteq (ab)\tau$ for all $a, b \in S$. To prove the converse inclusion, take $x \in (ab)\tau$. Then we have

$$\begin{aligned} x \in (ab)\tau &\Leftrightarrow (ab, x) \in \tau \\ &\Rightarrow (a, x) \in \tau \vee (b, x) \in \tau && \text{by the cm-property} \\ &\Leftrightarrow x \in a\tau \vee x \in b\tau \\ &\Leftrightarrow x \in a\tau \cup b\tau. \end{aligned}$$

(ii) $\Rightarrow$ (iii). By Theorem 3.9, τa is an sd-ideal of S such that $a \bowtie \tau a$, for any $a \in S$. Let $x, y \in S$ be such that $xy \in \tau a$. Then $a \in (xy)\tau = x\tau \cup y\tau$ by the assumption. Thus, $a \in x\tau$ or $a \in y\tau$. So, $x \in \tau a$ or $y \in \tau a$, and τa is an sd-completely isolated ideal of S for any $a \in S$.

(iii) $\Rightarrow$ (i). Let $a, b, c \in S$ be such that $(ab, c) \in \tau$. Then, $ab \in \tau c$ and, since τc is completely isolated, $a \in \tau c$ or $b \in \tau c$, which means that $(a, c) \in \tau$ or $(b, c) \in \tau$. □

Following Bishop, every classical theorem presents the challenge: find a constructive version with a constructive proof. This constructive version can be obtained by strengthening the conditions or weakening the conclusion of the theorem. There are, often, several constructively different versions of the same classical theorem.

Comparing the obtained results for complement positive co-quasiorders with the parallel ones for positive quasiorders in the classical background, we can conclude that the classical Theorem 2.1 breaks into two new ones in the constructive setting:

- Theorem 3.7 obtained by weakening the conclusions,
- Theorem 3.9 obtained by strengthening the conditions - here strengthening the logical background. Recall that intermediate logic proves intuitionistically as well as classically valid theorems, yet they often possess a strong constructive flavour.

In addition, there are two definitions: those of the constructive cm-property, and the complement cm-property. Nevertheless, the last definition, Theorem 3.8, is stronger.

Remark 10. For some classical theorems it is shown that they are not provable constructively. Some classical theorems are neither provable nor disprovable, that is, they are independent of **BISH**.

3.2.3. QSP for semigroups with apartness

Let us remember that in **CLASS** the compatibility property is an important condition for providing the semigroup structure on quotient sets. Now we are looking for the tools for introducing an apartness relation on a factor semigroup. Our starting point is the results from Subsection 3.1.4, as well as the next definition.

A co-equivalence $\boldsymbol{\kappa}$ is a *co-congruence* if it is *co-compatible*

$$\forall_{a,b,x,y\in S}\,((ax,by)\in\kappa\;\Rightarrow\;(a,b)\in\kappa\vee(x,y)\in\kappa)$$

Theorem 3.11. Let S be a semigroup with apartness. Then

(i) Let μ be a congruence, and κ a co-congruence on S. Then, κ defines an apartness on the factor set S/μ if and only if $\mu\cap\kappa=\emptyset$.

(ii) The quotient mapping $\pi: S\to S/\mu$, defined by $\pi(x)=x\mu$, is an onto se-homomorphism.

Proof:
(i). If κ defines an apartness on S/μ, then, by Theorem 3.2(i), $\mu\cap\kappa=\emptyset$.

Let μ be a congruence and κ a co-congruence on a semigroup with apartness S such that $\mu\cap\kappa=\emptyset$. Then, by Theorem 3.2(i), κ defines apartness onS/μ.

Let $a\mu\, x\mu\#b\mu\, y\mu$, then $(ax)\mu\#(bx)\mu$ which further, by the definition of apartness on S/μ, ensures that $(ax,by)\in\kappa$. But κ is a co-congruence, so either $(a,b)\in\kappa$ or $(x,y)\in\kappa$. Thus, by the definition of apartness in S/μ again, either $a\mu\#b\mu$ or $x\mu\#y\mu$. So $(S/\mu,=,\#,\cdot)$ is a semigroup with apartness.

(ii). By Theorem 3.2(ii), π is an onto se-mapping. By (i) and assumption, we have

$$\pi(xy)=(xy)(\sim\kappa)=x(\sim\kappa)\,y(\sim\kappa)=\pi(x)\pi(y).$$

Hence π is a homomorphism. □

As a consequence of Theorem 3.11 and Corollary 3.1 we have the next corollary.

Corollary 3.3. If κ is a co-congruence on S, then the relation $\sim\kappa(=\neg\kappa)$ is a congruence on S, and κ defines an apartness on $S/\sim\kappa$.

The apartness isomorphism theorem for semigroups with apartness follows.

Theorem 3.12. Let $f: S\to T$ be an se-homomorphism between semigroups with apartness. Then:

(i) coker f is a co-congruence on S, which defines an apartness on $S/\ker\, f$,

(ii) the mapping $\theta: S/\ker\, f\to T$, defined by $\theta(x(\ker\, f))=f(x)$, is an apartness embedding such that $f=\theta\circ\pi$; and

(iii) if f maps S onto T, then θ is an apartness isomorphism.

Proof:
See [41]. □

Recall, following [47], **BISH** (and constructive mathematics in general) is not the study of constructive things, it is a constructive study of things. In constructive proofs of classical theorems, only constructive methods are used.

Although constructive theorems might look like the corresponding classical versions, they often have more complicated hypotheses and proofs. Comparing Theorem 2.4 (respectively Theorem 2.3) for classical semigroups and Theorem 3.12 (respectively Theorem 3.11) for semigroup with apartness, we have evidence for that.

4. Concluding remarks

During the implementation of the FTA Project [25], the notion of commutative constructive semigroups with tight apartness appeared. We put noncommutative constructive semigroups with "ordinary" apartness in the centre of our study, proving first, of course, that such semigroups do exist. Once again we want to emphasize that semigroups with apartness are a **new approach**, and not a new class of semigroups.

Let us give some examples of applications of ideas presented in the previous section. We will start with constructive analysis. The proof of one of the directions of the constructive version of the Spectral Mapping Theorem is based on some elementary constructive semigroups with inequality techniques, [12]. It is also worth mentioning the applications of commutative basic algebraic structures with tight apartness within the automated reasoning area, [16]. For possible applications within computational linguistic see [42]. Some topics from mathematical economics can be approached constructively too (using some order theory for sets with apartness), [2]. Contrary to the classical case, the applications of constructive semigroups with apartness, due to their novelty, constitute an unexplored area. In what follows some possible connections between semigroups with apartness and computer science are sketched, [41].

semigroups with apartness	semigroups with apartness
$\updownarrow$	$\updownarrow$
bisimulation	automated theorem proving
$\updownarrow$	$\updownarrow$
formal reasoning about processes	knowledge representation and automated reasoning
$\updownarrow$	$\updownarrow$
process algebra	artificial intelligence
$\updownarrow$	
transactions and concurrency	
$\updownarrow$	
databases	

One of the directions of future work is to be able to say more about those links. The study of basic constructive algebraic structures with apartness as well as constructive algebra as a whole can impact the development of other areas of constructive mathematics. On the other hand, it can make both proof engineering and programming more flexibile.

Although the classical theory of semigroups has been considerably developed in the last decades, constructive mathematics has not paid much attention to semigroup theory. One of our main scientific activities will be to further develop of the constructive theory of semigroups with apartness. Semigroups will be examined constructively, that is with intuitionistic logic. To develop this constructive theory of semigroups with apartness, we need first to clarify the notion of a set with apartness. The initial step towards grounding the theory is done by our contributing papers [17], [18], [40], [41], [20] - a critical review of some of those results as well as the solutions to some of the open problems arising from those papers are presented in Section 3.

Why should a mathematician choose to work in this manner? As it is written in one of the reviews of Errett Bishop's monograph *Foundations of functional analysis*, [59], "to replace the classical system by the constructive one does not in any way mutilate the great classical theories of mathematics. Not at all. If anything, it strengthens them, and shows them, in a truer light, to be far grander than we had known." At heart, Bishop's constructive mathematics is simply mathematics done with intuitionistic logic, and may be regarded as "constructive mathematics for the working mathematician", [61]. The main activity in the field consists in proving theorems rather than demonstrating the unprovability of theorems (or making other metamathematical observations), [3]. "Theorems are tools that make new and productive applications of mathematics possible," [33].

The theory of semigroups with apartness is, of course, in its infancy, but, as we have already pointed out, it promises a prospective of applications in other (constructive) mathematics disciplines, certain areas of computer science, social sciences, economics.

To conclude, although one of the main motivators for initiating and developing the theory of semigroups with apartness comes from the computer science area, in order to have profound applications, a certain amount of the theory, which can be applied, is necessary first. Among priorities, besides the growing the general theory, are further developments of: constructive relational structures - (co)quotient structures in the first place, constructive order theory, theory of ordered semigroups with apartness, etc.

The Summary of the European Commission's *Mathematics for Europe*, June 2016, [21], states that "mathematics should not only focus on nowadays' applications but should leave room for development, even theoretical, that may be vital tomorrow." With a strong belief in the tomorrow's vitalness of the theory of semigroups with apartness, the focus should be on its further development. On the other hand, it is useful to "leave room" for "nowadays' applications" as well. All those will represent the core of our forthcoming papers.

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