# Computing the Length of Sum of Squares and Pythagoras Element in a Global Field 

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#### Abstract

This paper presents algorithms for computing the length of a sum of squares and a Pythagoras element in a global field $K$ of characteristic different from 2. In the first part of the paper, we present algorithms for computing the length in a non-dyadic and dyadic (if $K$ is a number field) completion of $K$. These two algorithms serve as subsidiary steps for computing lengths in global fields. In the second part of the paper we present a procedure for constructing an element whose length equals the Pythagoras number of a global field, termed a Pythagoras element.


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## 1. Introduction

The problem of representing an element in a ring as a sum of squares is well known in mathematics, ranging from the works of Lagrange and Gauss, through the works of Waring and Hilbert, to contemporary papers (see e.g. [1] or [2]). For example, in papers [3] and [4] the Pythagoras number is considered, while [5] gives an answer to Hilbert's seventeenth problem, announced in 1900.

[^0]In this paper, we pursue the continuation of the recent work by P. Koprowski and A. Czogała in [6] on the computational aspects of the theory of quadratic forms over global fields. In [6] the authors focused on algorithms over number fields (i.e. finite extensions of $\mathbb{Q}$ ). The authors in their paper developed algorithms for checking the isotropy of forms, and computing some field invariants. The aim of this article is to present algorithms for computing the length of a sum of squares and a Pythagoras element (see Definition 1) in a global field of characteristic different from 2. In this paper we make a free use of the standard results from the theory of quadratic forms over global fields. The reader is referred to [7, 8, 9] for a proper exposition of the theory.

Throughout this paper, if $K$ is a number field whose multiplicative group of non-zero elements is $\dot{K}$, then $\mathcal{O}_{K}$ denotes the integral closure of $\mathbb{Z}$ in $K$, while if $K$ is a finite extension of $\mathbb{F}(X)$, where $\mathbb{F}$ is a finite field of characteristic not 2 , then $\mathcal{O}_{K}$ denotes the integral closure of $\mathbb{F}[X]$ in $K$. We denote by $\Omega(K)$ the set of all places of $K$. If $\mathfrak{p}$ is a place of $K$, then we call any valuation belonging to $\mathfrak{p}$ the $\mathfrak{p}$-adic valuation (if $\mathfrak{p}$ is finite, then we call it dyadic and non-dyadic if $\mathfrak{p}$ divides and does not divide 2 , respectively). The completion of $K$ under a $\mathfrak{p}$-adic valuation is denoted by $K_{\mathfrak{p}}$ and called the $\mathfrak{p}$-adic completion of $K$. We denote by $(\cdot, \cdot)_{\mathfrak{p}}$ the $\mathfrak{p}$-adic Hilbert symbol and by $h_{\mathfrak{p}}(q)$ the $\mathfrak{p}$-adic Hasse invariant of a quadratic form $q$ (for definitions and properties see [7]). If $q$ is a quadratic form over $K$ (over $K_{\mathfrak{p}}$, respectively), then we write $D(q)\left(D_{\mathfrak{p}}(q)\right.$, respectively) for the set of all elements of $K$ ( $K_{\mathfrak{p}}$, respectively) which are represented by $q$. The symbol $\left\langle a_{1}, \ldots, a_{n}\right\rangle$ denotes a diagonal quadratic form over $K$ (or $K_{\mathfrak{p}}$ ) of dimension $n$. Next, if $\mathfrak{p}$ is a finite place, then $v_{\mathfrak{p}}(a)$ denotes the $\mathfrak{p}$-adic valuation of an element $a$ in $K$. The square class group of the local field $K_{\mathfrak{p}}$ has the form $\dot{K}_{\mathfrak{p}} / \dot{K}_{\mathfrak{p}}^{2}=\left\{\dot{K}_{\mathfrak{p}}^{2}, u_{\mathfrak{p}} \dot{K}_{\mathfrak{p}}^{2}, \pi_{\mathfrak{p}} \dot{K}_{\mathfrak{p}}^{2}, u_{\mathfrak{p}} \pi_{\mathfrak{p}} \dot{K}_{\mathfrak{p}}^{2}\right\}$, where $v_{\mathfrak{p}}\left(u_{\mathfrak{p}}\right) \equiv 0(\bmod 2)$ is a $\mathfrak{p}$-adic unit, and $v_{\mathfrak{p}}\left(\pi_{\mathfrak{p}}\right) \equiv 1(\bmod 2)$ is a $\mathfrak{p}$-adic uniformizer (see e.g. [7], Theorem VI.2.2] for further details).

Recall that the level $s(K)$ of $K$ is defined as the smallest positive integer $n$ (if it exists) such that -1 is a sum of $n$ squares of elements of $K$.

Let $a \in \dot{K}$. If $a$ is not a sum of squares of elements of $K$, then we say that $a$ has length $\infty$ and write $\ell(a)=\infty$. Otherwise, we define it's length $\ell(a)$ to be the minimal number of summands needed to express $a$ as a sum of squares of elements of $K$. If $\mathfrak{p} \in \Omega(K)$ is a place of $K$, then similarly we define the length of $a$ in the field $K_{\mathfrak{p}}$ and denote it by $\ell_{\mathfrak{p}}(a)$

The Pythagoras number of $K$ (see e.g. [7, XI.5.5]), denoted $P(K)$, is the smallest positive integer $n$ such that every sum of squares in $K$ is a sum of $n$ squares. If no such integer $n$ exists, then $P(K):=\infty$. A Pythagoras element is defined as follows.

Definition 1. A Pythagoras element of a global field $K$, denoted $a_{K}$, is defined to be an element whose length is equal to the Pythagoras number of $K$. Thus $\ell\left(a_{K}\right)=P(K)$.

For example, the Pythagoras number of the rationals is $P(\mathbb{Q})=4$ and $7 \in \mathbb{Q}$ is a Pythagoras element. The Pythagoras element is not unique, e.g. 15 is another Pythagoras element of $\mathbb{Q}$.

The paper is organized as follows: in Section 2 we present algorithms (see Algorithms 3 and 4) for computing the length of a sum of squares in a number and global function field, respectively. These algorithms use subsidiary procedures (Algorithms 1 and 2) for deciding the lengths in a nondyadic and dyadic completion of $K$, respectively. Next, in Section 3, Algorithms 5 and 6 construct a Pythagoras element in a given number field and global function field, respectively.

## 2. Length of a sum of squares

Let $K$ be a global field.
Observation 2. If $a \in \dot{K}$, then $a \in D(\langle 1,1, \ldots, 1\rangle)$ for $\langle 1,1, \ldots, 1\rangle$ of dimension $n$ if and only if the quadratic form $\langle a,-1,-1, \ldots,-1\rangle$ of dimension $n+1$ is isotropic.

The above observation implies that if $\ell(a)<\infty$, then

$$
\begin{gather*}
\ell(a)=\min \{n \in \mathbb{N} \mid\langle a,-1,-1, \ldots,-1\rangle \text { of dimension }  \tag{1}\\
n+1 \text { is isotropic }\}
\end{gather*}
$$

Obviously, it is true for $\ell_{\mathfrak{p}}(a)$ and every $\mathfrak{p} \in \Omega(K)$.
Assume $\mathfrak{p} \in \Omega(K)$ is a place of $K$. If $\mathfrak{p}$ is finite, then the $u$-invariant of $K_{\mathfrak{p}}$ is 4 (see e.g. [7] Theorem VI.2.12]), so the form $\langle 1,1,1,1\rangle$ is universal over $K_{\mathfrak{p}}$. Therefore $a \in D_{\mathfrak{p}}(\langle 1,1,1,1\rangle)$ and $\ell_{\mathfrak{p}}(a) \leq 4$. If $K$ is a number field and $\mathfrak{p}$ is infinite, then

$$
\ell_{\mathfrak{p}}(a)= \begin{cases}1 & \text { if }(a,-1)_{\mathfrak{p}}=1 \\ \infty & \text { if }(a,-1)_{\mathfrak{p}}=-1\end{cases}
$$

in the case when $K_{\mathfrak{p}} \cong \mathbb{R}$ and $\ell_{\mathfrak{p}}(a)=1$ in the case when $K_{\mathfrak{p}} \cong \mathbb{C}$.
Proposition 3. Let $K$ be a global field and $a \in \dot{K}$ with $\ell(a)<\infty$, then

$$
\ell(a)=\max _{\mathfrak{p} \in \Omega(K)} \ell_{\mathfrak{p}}(a)
$$

## Proof:

From (1), it follows that $\ell(a)$ is the minimal natural number such that the form $\langle a,-1,-1, \ldots,-1\rangle$ of dimension $\ell(a)+1$ is isotropic. By the Local-global principle [7] Principle VI.3.1], $\langle a,-1,-1, \ldots,-1\rangle$ is isotropic over $K$ if and only if it is isotropic over $K_{\mathfrak{p}}$ for all $\mathfrak{p} \in \Omega(K)$. Therefore $\langle a,-1,-1, \ldots,-1\rangle$ of dimension $\ell(a)+1$ is isotropic over $K_{\mathfrak{p}}$ for all $\mathfrak{p} \in \Omega(K)$. Again from (1) and the Local-global principle, it follows that there is at least one $\mathfrak{q} \in \Omega(K)$ such that the form $\langle a,-1,-1, \ldots,-1\rangle$ of dimension $\ell(a)$ is not isotropic over $K_{\mathfrak{q}}$, hence $\ell_{\mathfrak{q}}(a)=\ell(a)$ and $\ell(a)$ is the maximal length among all $\ell_{\mathfrak{p}}(a), \mathfrak{p} \in \Omega(K)$.

Now we present the first algorithm for computing the length of $a$ in a non-dyadic completion of the field $K$.

## Proof of correctness of Algorithm 1:

Assume that $v_{\mathfrak{p}}(a)$ is even. Then either $a=1$ or $a=u_{\mathfrak{p}}$ (modulo squares). If $a$ is a square, then $\ell_{\mathfrak{p}}(a)=1$. Otherwise, $(-1, a)_{\mathfrak{p}}=\left(-1, u_{\mathfrak{p}}\right)_{\mathfrak{p}}=1$, i.e. $1 \in D_{\mathfrak{p}}(\langle-1, a\rangle)$, which is equivalent to the fact that $a \in D_{\mathfrak{p}}(\langle 1,1\rangle)$. Therefore $\ell_{\mathfrak{p}}(a)=2$.

```
Algorithm 1: Length in a non-dyadic completion
    Input: A nonzero element \(a\) of a global field \(K\) and a finite non-dyadic place \(\mathfrak{p}\) of \(K\).
    Output: Length of \(a\) in the completion \(K_{\mathfrak{p}}\)
    if \(v_{\mathfrak{p}}(a) \equiv 1(\bmod 2)\) then
        if -1 is a local square in \(K_{\mathfrak{p}}\) then
            return 2;
        else
            return 3;
    else
        if \(a\) is a local square in \(K_{\mathfrak{p}}\) then
            return 1;
        else
            return 2;
```

Now assume that $v_{\mathfrak{p}}(a)$ is odd. Then either $a=\pi_{\mathfrak{p}}$ or $u_{\mathfrak{p}} \pi_{\mathfrak{p}}$ (modulo squares) and hence

$$
(-1, a)_{\mathfrak{p}}=\left(-1, u_{\mathfrak{p}} \pi_{\mathfrak{p}}\right)_{\mathfrak{p}}=\left(-1, \pi_{\mathfrak{p}}\right)_{\mathfrak{p}}= \begin{cases}1 & \text { if }-1 \in \dot{K}_{\mathfrak{p}}^{2} \\ -1 & \text { if }-1 \notin \dot{K}_{\mathfrak{p}}^{2}\end{cases}
$$

If $-1 \in \dot{K}_{\mathfrak{p}}^{2}$, then similarly as in the previous paragraph, $\ell_{\mathfrak{p}}(a)=2$. If $-1 \notin \dot{K}_{\mathfrak{p}}^{2}$, then the level of $K_{\mathfrak{p}}$ is equal to 2 , so the form $\langle 1,1,1\rangle$ is isotropic over $K_{\mathfrak{p}}$. Hence $a \in D_{\mathfrak{p}}(\langle 1,1,1\rangle)$ and $\ell_{\mathfrak{p}}(a)=3$.

Remark 4. A procedure for testing whether an element $a$ is a square in a completion $K_{\mathfrak{p}}$ is equivalent to testing whether $x^{2}-a$ is irreducible in $K_{\mathfrak{p}}[x]$. Algorithms for testing the irreducibility of polynomials are already in existence and can be found for example in [10], [11] or [12].

Next, we present an algorithm for computing the length of $a$ in a dyadic completion of $K$ (if $K$ is a number field).

```
Algorithm 2: Length in a dyadic completion
    Input: A nonzero element \(a\) of a number field \(K\) and a dyadic place \(\mathfrak{d}\) of \(K\).
    Output: Length of \(a\) in the completion \(K_{\mathfrak{D}}\)
    if \(a\) is a local square in \(K_{\mathfrak{d}}\) then
        return 1;
    Compute the Hilbert symbol \((-1, a)_{\mathfrak{o}}\);
    if \((-1, a)_{\mathfrak{o}}=1\) then
        return 2;
    Compute the Hilbert symbol \((-1,-1)_{\mathfrak{D}}\);
    if \((-1,-1)_{\mathfrak{d}}=1\) or \(-a\) is not a square in \(K_{\mathfrak{d}}\) then
        return 3;
    return 4;
```

The proof of correctness is preceded by the following lemma.

Lemma 5. Let $a$ be a nonzero element of a number field $K$, and $\mathfrak{d}$ a dyadic place of $K$. The form $\langle a,-1,-1,-1\rangle$ is isotropic over $K_{\mathfrak{d}}$ if and only if either $(-1,-1)_{\mathfrak{d}}=1$ or $-a \notin \dot{K}_{\mathfrak{d}}^{2}$.

## Proof:

Assume $-a \in \dot{K}_{\mathfrak{d}}^{2}$. Then $\langle a,-1,-1,-1\rangle \cong\langle-1,-1,-1,-1\rangle$ over $K_{\mathfrak{D}}$ and $h_{\mathfrak{d}}(\langle-1,-1,-1,-1\rangle)=$ $(-1,-1)_{\mathfrak{d}}^{6}=1$. From the assumption it follows that $\langle-1,-1,-1,-1\rangle$ is isotropic, so by the means of [7, Proposition V.3.23] we have $(-1,-1)_{\mathfrak{d}}=1$.

Conversely, suppose that $(-1,-1)_{\mathfrak{d}}=1$. Then $1 \in D_{\mathfrak{d}}(\langle-1,-1\rangle)$, so the the form $\langle a,-1,-1,-1\rangle$ is isotropic over $K_{\mathfrak{d}}$. Now assume $-a \notin \dot{K}_{\mathfrak{d}}^{2}$ and consider the quadratic extension $L_{\mathfrak{D}}:=K_{\mathfrak{d}}(\sqrt{-a})$ of $K_{\mathfrak{d}}$. From [7, Example XI.2.4(7)], it follows that $(-1,-1)_{\mathfrak{D}}=1$ since $\left[L_{\mathfrak{Q}}: \mathbb{Q}_{2}\right]=\left[K_{\mathfrak{d}}(\sqrt{-a}): \mathbb{Q}_{2}\right]$ is even. Moreover, $\langle a,-1,-1,-1\rangle \cong\langle-1,-1,-1,-1\rangle$ over $L_{\mathfrak{D}}$, and $h_{\mathfrak{D}}(\langle-1,-1,-1,-1\rangle)=1$. Finally, we have $h_{\mathfrak{D}}(\langle a,-1,-1,-1\rangle)=(-1,-1)_{\mathfrak{D}}$, hence by [7, Remark V.3.24] $\langle a,-1,-1,-1\rangle$ is isotropic over $K_{\mathfrak{d}}$.

## Proof of correctness of Algorithm 2;

Let $\mathfrak{d}$ be a dyadic place. If $a \in \dot{K}_{\mathfrak{d}}^{2}$, then of course $\ell_{\mathfrak{d}}(a)=1$. Assume $a \notin \dot{K}_{\mathfrak{d}}^{2}$. We consider the $\mathfrak{d}$-adic Hilbert symbol $(-1, a)_{\mathfrak{d}}$. If $(-1, a)_{\mathfrak{d}}=1$, then $a \in D_{\mathfrak{d}}(\langle 1,1\rangle)$ and $\ell_{\mathfrak{d}}(a)=2$. Suppose $(-1, a)_{\mathfrak{d}}=-1$. Then $a \notin D_{\mathfrak{d}}(\langle 1,1\rangle)$. By Lemma 5] if either $(-1,-1)_{\mathfrak{d}}=1$ or $-a \notin \dot{K}_{\mathfrak{d}}^{2}$, then $\ell_{\mathfrak{0}}(a)=3$. Otherwise, $\ell_{\mathfrak{d}}(a)=4$.

Remark 6. An algorithm for computing the Hilbert symbol in a completion of a number field can be found in [13, Algorithm 6.6].

Remark 7. In the next algorithms, Algorithms 3, 4, 5 and 6, we perform two kinds of factorization in like manner as in [6]. The first one is to find all dyadic primes of a given field, i.e. to factor $2 \mathcal{O}_{K}$. The second one is to find all primes dividing an element in a given field. Algorithms for factorization of ideals are well known. One may refer for example to [14, §6.2.5], [15] or in [16, §2.2].

Now we present an algorithm for computing the length of a sum of squares in a number field.

## Proof of correctness of Algorithm 3:

Let $\rho_{1}, \ldots, \rho_{r}$ be all real embeddings of $K$ for $r \geq 0$. If $\rho_{i}(a)<0$ for some $i \leq r$, then $a$ is not a sum of squares in the corresponding completion, hence it cannot be a sum of squares in $K$ either.

Assume either $r=0$ or $\rho_{i}(a)>0$ for all $i \in\{1, \ldots, r\}$. If $a \in \dot{K}^{2}$, then $\ell(a)=1$. Therefore suppose $a \notin \dot{K}^{2}$. Let $\mathfrak{D}$ and $\mathfrak{Q}$ be the set of prime factors of 2 and the set of prime factors of $a$ in $\mathcal{O}_{K}$ that do not divide 2 , respectively. Moreover, put $\mathfrak{P}:=\mathfrak{D} \cup \mathfrak{Q}$ and fix a finite place $\mathfrak{p} \in \Omega(K)$. If $\mathfrak{p} \notin \mathfrak{P}$, then $v_{\mathfrak{p}}(a)=0$. It implies either $a=1$ or $a=u_{\mathfrak{p}}$ (modulo squares). If $a$ is a square, then $\ell_{\mathfrak{p}}(a)=1$. Otherwise, similarly as in the proof of correctness of Algorithm $11 \ell_{\mathfrak{p}}(a)=2$. Suppose $\mathfrak{p} \in \mathfrak{P}$. If $\mathfrak{p}$ is a dyadic place, then we use Algorithm 2, Otherwise, we use Algorithm 1. Finally, by Proposition 3 , $\ell(a)=\max _{\mathfrak{p} \in \Omega(K)} \ell_{\mathfrak{p}}(a)$.

Next, we present an algorithm for computing the length of a sum of squares in a global function field.

```
Algorithm 3: Length of a sum of squares in a number field
    Input: A nonzero element \(a\) of a number field \(K\)
    Output: Length of \(a\) in \(K\)
    if \(K\) is formally real then
        Let \(\mathfrak{R}=\left\{\rho_{1}, \ldots, \rho_{r}\right\}\) be the list of all real embeddings of \(K, r \in \mathbb{N}\);
        for \(\rho \in \mathfrak{R}\) do
            if \(\rho(a)<0\) then
                return \(\infty\)
    if \(a\) is a square in \(K\) then
        return 1
    Let \(\mathfrak{D}=\left\{\mathfrak{d}_{1}, \ldots, \mathfrak{d}_{m}\right\}\) be the list of prime factors of 2 in \(\mathcal{O}_{K}\);
    \(\ell \leftarrow 2\);
    for \(\mathfrak{d} \in \mathfrak{D}\) do
        Compute \(\ell_{\mathfrak{J}}(a)\) in \(K_{\mathfrak{D}}\) using Algorithm 2;
        if \(\ell_{\mathfrak{d}}(a)=4\) then
            L return 4
            \(\ell \leftarrow \max \left\{\ell, \ell_{\mathfrak{0}}(a)\right\} ;\)
    if \(\ell=3\) then
        return 3
    Let \(\mathfrak{Q}=\left\{\mathfrak{q}_{1}, \ldots, \mathfrak{q}_{n}\right\}\) be the list of prime factors of \(a\) in \(\mathcal{O}_{K}\) that do not divide 2;
    for \(\mathfrak{q} \in \mathfrak{Q}\) do
        Compute \(\ell_{\mathfrak{q}}(a)\) in \(K_{\mathfrak{q}}\) using Algorithm 1;
        if \(\ell_{\mathfrak{d}}(a)=3\) then
            return 3
    return 2 ;
```

```
Algorithm 4: Length of a sum of squares in a global function field
    Input: A nonzero element \(a\) of a global function field \(K\)
    Output: Length of \(a\) in \(K\)
    if \(a\) is a square in \(K\) then
        return 1
    Let \(\mathfrak{P}=\left\{\mathfrak{q}_{1}, \ldots, \mathfrak{q}_{n}\right\}\) be the list of places dividing \(a\) in \(K\);
    for \(\mathfrak{q} \in \mathfrak{P}\) do
        Compute \(\ell_{\mathfrak{q}}(a)\) in \(K_{\mathfrak{q}}\) using Algorithm 1;
        if \(\ell_{\mathfrak{q}}(a)=3\) then
            return 3
    return 2;
```


## Proof of correctness of Algorithm 4:

Assume $\mathfrak{P}$ is the set of places dividing $a$ in $K$. The proof of correctness is similar to the proof of correctness of Algorithm 3, the second paragraph.

## 3. Pythagoras element

In this section, we devise algorithms that construct Pythagoras elements in global fields. We start with the following theorem.

Theorem 8. Let $K$ be a number field, then
(i) $P(K)=2$ iff $s(K)=1$.
(ii) $P(K)=3$ iff $s(K) \neq 1$ and every dyadic place of $K$ has even degree.
(iii) $P(K)=4$ iff there is a dyadic place of $K$ of odd degree.

## Proof:

(i) Similarly as in the proofs of correctness of Algorithms 1 and 2, $P(K)=2$ iff $-1 \in \dot{K}_{\mathfrak{p}}^{2}$ for every finite non-dyadic place $\mathfrak{p}$ of $K$, and $(-1, a)_{\mathfrak{d}}=1$ for every dyadic place $\mathfrak{d}$ of $K$ and any $a \in \dot{K}$. Hence $P(K)=2$ iff $-1 \in \dot{K}_{\mathfrak{q}}^{2}$ for every finite place $\mathfrak{q} \in \Omega(K)$ which is equivalent to $s(K)=1$.
(iii) By Lemma 5, $P(K)=4$ iff $(-1,-1)_{\mathfrak{d}}=-1$ for some dyadic place $\mathfrak{d}$ of $K$, so $P(K)=4$ iff there is a dyadic place of $K$ of odd degree by means of [7, Example XI.2.4(7)].
(ii) Follows from (i) and (iii).

Remark 9. Observe that if $K$ is a number field and $P(K)=2$, then $K$ is a nonreal field.

```
Algorithm 5: Pythagoras element in a number field
    Input: A number field \(K\)
    Output: A Pythagoras element in \(K\)
    if \(s(K)=1\) then
        return any \(a \in \dot{K} \backslash \dot{K}^{2} ;\)
    Let \(\mathfrak{D}=\left\{\mathfrak{d}_{1}, \ldots, \mathfrak{d}_{m}\right\}\) be the list of prime factors of 2 in \(\mathcal{O}_{K}\);
    for \(\mathfrak{d} \in \mathfrak{D}\) do
        if \((-1,-1)_{\mathfrak{o}}=-1\) then
            return 7
    Set flag \(\leftarrow\) FALSE;
    Set \(p \leftarrow 2\);
    while flag \(=\) FALSE do
        Let \(p \leftarrow\) smallest prime number \(\geq p+1\);
        if \(p \equiv 3(\bmod 4)\) then
            Let \(\mathfrak{P}=\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{n}\right\}\) be the list of prime factors of \(p\) in \(\mathcal{O}_{K}\);
            for \(\mathfrak{p} \in \mathfrak{P}\) do
                if \(v_{\mathfrak{p}}(p) \equiv 1(\bmod 2)\) then
                    Set flag \(\leftarrow\) TRUE;
    return \(p\);
```


## Proof of correctness of Algorithm 5:

Assume $P(K)=2$ and let $a \in \dot{K} \backslash \dot{K}^{2}$. Then $s(K)=1$ and $-1 \in \dot{K}^{2}$, so $-1 \in \dot{K}_{\mathfrak{q}}^{2}$ for every finite place $\mathfrak{q} \in \Omega(K)$. If $\mathfrak{q}$ is a finite place such that $a \in \dot{K}_{\mathfrak{q}}^{2}$, then $\ell_{\mathfrak{q}}(a)=1$. Otherwise, similarly as in the proofs of correctness of Algorithms 1 and $2, \ell_{\mathfrak{q}}(a)=2$ so $\ell(a)=2$ and $a$ is a Pythagoras element.

Next, assume $P(K)=3$, then $s(K) \neq 1$ and $(-1,-1)_{\mathfrak{d}}=1$ for every dyadic place $\mathfrak{d}$ of $K$. Since $-1 \notin \dot{K}^{2}$, there exists a finite non-dyadic place $\mathfrak{p}$ of $K$ such that $-1 \notin \dot{K}_{\mathfrak{p}}^{2}$. If $a$ is an element of $K$ such that $v_{\mathfrak{p}}(a)$ is odd, then $a=\pi_{\mathfrak{p}}$ or $u_{\mathfrak{p}} \pi_{\mathfrak{p}}$ (modulo squares) and $\ell_{\mathfrak{p}}(a)=3$. If $\mathfrak{d}$ is a dyadic place of $K$, then $(-1,-1)_{\mathfrak{d}}=1$ so by Lemma 5, $\ell_{\mathfrak{J}}(a) \leq 3$. Moreover, if $a$ is a totally positive element (in the case when $K$ is formally real), then $\ell(a)=3$ and $a$ is a Pythagoras element of $K$. Now let $p \equiv 3(\bmod 4)$ be any prime number factoring into prime ideals of the form $p \mathcal{O}_{K}=\mathfrak{p}_{1}^{e_{1}} \cdots \mathfrak{p}_{n}^{e_{n}}$, and let $\mathfrak{p}$ be any of those factors such that $e(\mathfrak{p} \mid p) \equiv 1(\bmod 2)$. Then by [7, Corollary VI.2.6], since $p \equiv 3(\bmod 4)$, it implies that $-1 \notin \dot{K}_{\mathfrak{p}}^{2}$, and $p$ is the sought element.

Finally, if $P(K)=4$, then choose a dyadic place $\mathfrak{d}$ of $K$ such that $(-1,-1)_{\mathfrak{d}}=-1$. We prove that 7 is a Pythagoras element of $K$. Indeed, $-7 \in \dot{\mathbb{Q}}_{2}^{2} \subset \dot{K}_{\mathfrak{d}}^{2}$. By Lemma 5 , it implies that $\langle 7,-1,-1,-1\rangle$ is anisotropic over $K_{\mathfrak{d}}$ and $\ell_{\mathfrak{J}}(7)=4$. Therefore $\ell(7)=4$ and 7 is a Pythagoras element.

Theorem 10. Let $K$ be a global function field with full field of constants $\mathbb{F}_{q}$ of order $q$, then
(i) $P(K)=2$ iff $q \equiv 1(\bmod 4)$
(ii) $P(K)=3$ iff $q \equiv 3(\bmod 4)$

## Proof:

(i) Similarly as in the proof of correctness of Algorithm [1, $P(K)=2$ if and only if $-1 \in \dot{K}_{\mathfrak{p}}^{2}$ for every place $\mathfrak{p} \in \Omega(K)$. Therefore

$$
P(K)=2 \Longleftrightarrow-1 \in \dot{\mathbb{F}}_{q}{ }^{2} \subset \dot{K}^{2} \Longleftrightarrow q \equiv 1(\bmod 4)
$$

(ii) Follows from (i).

## Proof of correctness of Algorithm 6:

The full field $\mathbb{F}_{q}$ of constants is algebraically closed in $K$. Hence $a \notin \dot{\mathbb{F}}_{q}{ }^{2}$ implies that $a \notin \dot{K}^{2}$. Therefore if $P(K)=2$, we have $\ell(a)=2=P(K)$.

Conversely, if $q \equiv 3(\bmod 4)$, then $-1 \notin \dot{K}^{2}$ by Theorem 10, so there is a place $\mathfrak{p} \in \Omega(K)$ such that $-1 \notin \dot{K}_{\mathfrak{p}}^{2}$. If $a \in K$ and $v_{\mathfrak{p}}(a)$ is odd, then similarly as in the proof of correctness of Algorithm 5, $\ell_{\mathfrak{p}}(a)=3$ which is the maximal length in $K$ and $a$ is a Pythagoras element. Further, for any positive integer $m$, it is well known that the polynomial $x^{q^{m}}-x$ in $\mathbb{F}_{q}[x]$ factors into a product $\prod_{d \mid m} P(d, q)$ of monic irreducible polynomials $P$ of degree $d$ (see e.g. [17, Chap. 7, Theorem 2]). If $P$ is any monic irreducible polynomial factoring into a product of powers of prime ideals $\mathfrak{p}_{1}^{e_{1}} \cdots \mathfrak{p}_{k}^{e_{k}}$ in $\mathcal{O}_{K}$ such that $e_{i}\left(\mathfrak{p}_{i} \mid P\right) \equiv 1(\bmod 2)$ for any $i \leq k$, then $(-1, P)_{\mathfrak{p}_{i}}=-1$ which implies $-1 \notin \dot{K}_{\mathfrak{p}_{i}}^{2}$, and $P$ is the sought element.

```
Algorithm 6: Pythagoras element in a global function field
    Input: A global function field \(K\) with full field of constants \(\mathbb{F}_{q}\) of order \(q\).
    Output: A Pythagoras element in \(K\)
    if \(q \equiv 1(\bmod 4)\) then
        return any \(a \in \dot{\mathbb{F}}_{q} \backslash \dot{\mathbb{F}}_{q}^{2} ;\)
    Set flag \(\leftarrow\) FALSE;
    Set \(m \leftarrow 0\);
    while flag \(=\) FALSE do
        \(m \leftarrow m+1\);
        Factor \(\left(x^{q^{m}}-x\right)\) into monic irreducible polynomials in the form of a list: \(\mathcal{P}=\left\{p_{1}, \ldots, p_{k}\right\} ;\)
        for \(p \in \mathcal{P}\) do
            Let \(\mathfrak{P}=\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{n}\right\}\) be the list of places dividing \(p\) in \(K\);
            for \(\mathfrak{p} \in \mathfrak{P}\) do
                if \(v_{\mathfrak{p}}(p) \equiv 1(\bmod 2)\) then
                Set flag \(\leftarrow\) TRUE;
    return \(p\);
```

The presented algorithms can be implemented in existing computer algebra systems. In fact, they have currently been implemented in CQF - a free, open-source Magma package for doing computations in quadratic forms theory (see [18]). CQF determines the length of an element and a Pythagoras element in a global field using the functions LengthOfSumOfSquares (or LengthOfSOS for short) and PythagorasElement, respectively.

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