arXiv:2101.07711v8 [cs.LO] 3 Oct 2022

Resource Bisimilarity in Petri Nets is Decidable

Irina A. Lomazova[*]

ialomazova@gmail.com

Vladimir A. Bashkin

Petr Jančar

Dept. of Computer Science, Faculty of Science

Palacký University, Olomouc, Czechia

petr.jancar@upol.cz

Abstract. Petri nets are a popular formalism for modeling and analyzing distributed systems. Tokens in Petri net models can represent the control flow state or resources produced/consumed by transition firings. We define a resource as a part (a submultiset) of Petri net markings and call two resources equivalent when replacing one of them with another in any marking does not change the observable Petri net behavior. We consider resource similarity and resource bisimilarity, two congruent restrictions of bisimulation equivalence on Petri net markings. Previously it was proved that resource similarity (the largest congruence included in bisimulation equivalence) is undecidable. Here we present an algorithm for checking resource bisimilarity, thereby proving that this relation (the largest congruence included in bisimulation equivalence that is a bisimulation) is decidable. We also give an example of two resources in a Petri net that are similar but not bisimilar.

Keywords: labeled Petri nets, bisimulation, congruence, decidability, resource bisimilarity

---

[*]Address for correspondence: ialomazova@gmail.com

1.   Introduction

The concept of process equivalence can be formalized in many different ways [1]. One of the most important is *bisimulation equivalence* [2, 3], which captures the main features of the observable behavior of a system. Generally, bisimulation equivalence, also called *bisimilarity*, is defined as a relation on sets of states of labeled transition systems (LTS). Two states are bisimilar if their behavior cannot be distinguished by an external observer.

Petri nets are a popular formalism for modeling and analyzing distributed systems, on which many notations used in practice are based. In Petri nets, states are represented as markings — multisets of tokens residing in Petri net places. The interleaving semantics associates a labeled Petri net (in which transitions are labeled with actions) and its initial marking with the corresponding LTS that describes the behavior of the net. For Petri nets, many important behavioral properties, e.g. reachability, are decidable, but behavioral equivalences like bisimilarity are undecidable [4].

As a mathematical formalism, Petri nets are equivalent to the subclass (**P**,**P**)-PRS of Process Rewrite Systems (PRS) introduced by R. Mayr [5]. PRS allow a (possibly infinite) LTS to be represented by a finite set of rewrite rules using algebraic operations of sequential and parallel composition. Several well-studied classes of infinite-state systems are included in the PRS hierarchy, which is based on operations used in rewrite rules to define the specific PRS: basic process algebras (BPA) are (**1**,**S**)-PRS, basic parallel processes (BPP) are (**1**,**P**)-PRS, pushdown automata (PDA) are (**S**,**S**)-PRS, Petri nets are (**P**,**P**)-PRS and process algebras (PA) are (**1**,**G**)-PRS.

The PRS hierarchy is strict (regarding expressivity) and sets a number of interesting decidability borders, in particular for bisimilarity. While bisimilarity is undecidable for (**P**,**P**)-PRS, corresponding to Petri nets, it is decidable for the class (**1**,**P**)-PRS [6], corresponding to labeled communication-free Petri nets, in which each transition has no more than one input place. Due to the restriction on the structure of communication-free Petri nets, bisimilarity for them is a congruence, and this property was crucial for the decidability proof in [6]. (This property is also implicitly used in the later proof [7].)

In Petri net models, places can be interpreted as resource repositories, and tokens represent resource availability or resources themselves. A transition firing, in turn, can be thought of as resource handling, relocation, deletion, and creation. Hence, Petri nets have a clear resource perspective and are widely used to model and analyze various types of resource-oriented systems such as production systems, resource allocation systems, etc.

Since bisimulation equivalence cannot be effectively checked for Petri nets, it is natural either to restrict the model (for example, to consider communication-free or one-counter Petri nets), or to strengthen the equivalence relation. In this paper we explore the second option. We consider resources as parts of Petri net markings (multisets of tokens residing in Petri net places) and study the possibility of replacing one resource with another in any marking without changing the observable behavior of the net.

Related work.

Numerous studies have been devoted to various aspects of resources in Petri nets. We name just a few of them.

In open nets special resource places are used for modeling a resource interface of the system [8, 9, 10, 11]. In workflow nets resource places represent resources that can be consumed and produced

during the business process execution. Obviously, resource places not only demonstrate a resource flow, but may substantially influence the system control flow [12, 13].

As a mathematical formalism, Petri nets are closely connected with Girard's linear logic [14], which also has a nice resource interpretation, and for different classes of Petri nets it is possible to express a Petri net as a linear logic formula [15, 16].

We have already mentioned bisimulation equivalence, which is defined as the largest bisimulation in a given LTS. In [17], an equivalence on places of a Petri net was defined which when lifted to the reachable markings preserves their token game and distribution over the places. This structural equivalence is called "strong bisimulation" in [17][1], because it is a non-trivial subrelation of bisimilarity.

The term *place bisimulation* comes from [18] where Olderog's concept was improved and used for a reduction of nets. In a Petri net, two bisimilar places can be fused without changing the observable net behavior.

In [19], a polynomial algorithm for computing the largest place bisimulation was presented. Equivalences on sets of places were further explored in [20, 21, 22]. In particular, it was proven (using the technique from [4]) that a weaker relation, called the *largest correct place fusion*, is undecidable. Places p and q can be correctly fused if for any marking M the two markings $M + p$ and $M + q$ are bisimilar.

Independently, in [23] a similar equivalence relation, called *structural bisimilarity*, was studied for labeled Place-Transition Nets. This relation is defined on the set of all vertices of a Petri net (both places and transitions) and also uses a kind of weak transfer property.

In [24, 25], a notion of *team bisimulation* was defined for communication-free Petri nets. Two distributed systems, each composed of a team of sequential non-cooperating agents (tokens in a communication-free net), are equivalent if it is possible to match each sequential component of one system with a team-bisimilar sequential component of another system (as in sports, where competing teams have the same number of equivalent player positions). For Petri nets, the team bisimulation coincides with the place bisimulation from [18].

In [26], a new equivalence on Petri net resources, called *resource similarity* was proposed. As already mentioned, a resource in a Petri net is a part of its marking, and two resources are similar if replacing one of them by another in any marking does not change the observable net behavior; in our case it means that the behavior remains in the same class of bisimulation equivalence. It was shown that the correct place fusion equivalence from [22] is a special case of resource similarity, namely the place fusion is resource similarity for one-token resources. Hence the undecidability result for the place fusion extends to resource similarity. On the other hand, resource similarity is a congruence and thus can be generated by a finite basis. A special type of minimal basis, called the *ground basis*, was presented in [26].

In [26] also a stronger equivalence of resources in a Petri net, called *resource bisimilarity*[2] was defined; an equivalence of resources was called a resource bisimulation if its additive and transitive closure is a bisimulation. Properties of the mentioned resource equivalences were studied in [28, 29].

---

[1]Not related to the more common use of terms "strong bisimulation" and "weak bisimulation" for fundamental state bisimulation and state bisimulation in LTS with invisible (silent) transitions.

[2]Not related to the notion of "resource bisimulation" from [27].

In particular, it was shown that resource bisimulation is defined by a weak transfer property, similar to the weak transfer property of place bisimulation. It was proven that there exists the largest resource bisimulation, and it coincides with resource bisimilarity. The questions of whether the resource bisimilarity is decidable and whether it is strictly stronger than resource similarity remained open in [28, 29, 30].

Our contribution.

In this paper we give answers to these open questions. Based on the well-known "tableau method" (see, e.g., [31]), we construct an algorithm for checking resource bisimilarity in Petri nets. Thus, it is proved that resource bisimilarity is decidable in Petri nets and is strictly stronger than resource similarity. We also give an example of a labeled Petri net in which two similar resources are not resource bisimilar.

Interestingly, for resource similarity and resource bisimilarity we have that both are congruences, and thus both have finite bases, but one of them is decidable, and the other is not.

Organization of the paper. In Section 2 we recall basic notions and notations, including the result on finitely-based congruences. Section 3 clarifies how resource bisimilarity, resource similarity, and bisimilarity are related. Section 4 shows the announced algorithm deciding resource bisimilarity, and Section 5 provides conclusions and additional remarks.

2. Preliminaries

By $\mathbb{N}$ and $\mathbb{N}_+$ we denote the sets of non-negative integers and of positive integers, respectively.

Multisets. A *multiset* m over a set S is a mapping $m : S \to \mathbb{N}$; hence a multiset may contain several copies of the same element. By $\mathcal{M}(S)$ we denote the set of multisets over S (which could be also denoted by $\mathbb{N}^S$). The inclusion $\subseteq$ on $\mathcal{M}(S)$ coincides with the component-wise order $\leq$: we put $m \subseteq m'$, or $m \leq m'$, if $\forall s \in S : m(s) \leq m'(s)$. By $m = \emptyset$ we mean that $m(s) = 0$ for all $s \in S$.

Given $m, m' \in \mathcal{M}(S)$, the union $m \cup m'$ is generally different than the sum $m + m'$: for each $s \in S$ we have $(m \cup m')(s) = \max\{m(s), m'(s)\}$ and $(m + m')(s) = m(s) + m'(s)$. We also define the multiset subtraction $m - m'$: for each $s \in S$ we have $(m - m')(s) = \max\{m(s) - m'(s), 0\}$.

In this paper we only deal with multisets over finite sets S. In this case the cardinality $|m|$ of each $m \in \mathcal{M}(S)$ is finite: we have $|m| = \sum_{s \in S} m(s)$. We note that the partial order $\leq$ on $\mathcal{M}(S)$ can be naturally extended to a total order, the *cardinality-lexicographic* order $\sqsubseteq$:

$$\text{we put } m \sqsubseteq m' \text{ if } |m| < |m'|, \text{ or } |m| = |m'| \text{ and } m \leq_{\text{LEX}} m',$$

where $\leq_{\text{LEX}}$ is a lexicographic order. More precisely, the *lexicographic order* $\leq_{\text{LEX}}$ on $\mathcal{M}(S)$ assumes that the elements of S are ordered, in which case $m \in \mathcal{M}(S)$ can be also viewed as a vector $m = ((m)_1, (m)_2, \ldots, (m)_{|S|})$ in $\mathbb{N}^{|S|}$; we put $m \leq_{\text{LEX}} m'$ if $m = m'$ or $(m)_i < (m')_i$ for the least i for which $(m)_i$ and $(m')_i$ differ.

Labeled Petri nets. A *Petri net* is a tuple $N = (P, T, W)$ where P and T are finite disjoint sets of *places* and *transitions*, respectively, and $W : (P \times T) \cup (T \times P) \to \mathbb{N}$ is an *arc-weight function*. A *labeled Petri net* is a tuple $N = (P, T, W, l)$ where (P, T, W) is a Petri net and $l : T \to Act$ is a labeling function, mapping the transitions to *actions* (observed events) from a set *Act*. Figure 1 shows an example of a labeled Petri net, with three transitions depicted by boxes, all being labeled with the same action b; the places are depicted by circles, and the arc-weight function is presented by (multiple) directed arcs (there is no arc from $p \in P$ to $t \in T$ if $W(p, t) = 0$, and similarly no arc from t to p if $W(t, p) = 0$).

A *marking* in a Petri net $N = (P, T, W)$ is a function $M : P \to \mathbb{N}$, hence a multiset $M \in \mathcal{M}(P)$; it gives the number of *tokens* in each place. Tokens residing in a place are often interpreted as resources of some type, which are consumed or produced by transition firings (as defined below).

For a transition $t \in T$, the *preset* ${}^\bullet t$ and the *postset* $t^\bullet$ are defined as the multisets over P such that ${}^\bullet t(p) = W(p, t)$ and $t^\bullet(p) = W(t, p)$ for each $p \in P$. A *transition $t \in T$ is enabled in a marking M* if ${}^\bullet t \leq M$; such a *transition may fire in M*, yielding the marking $M' = (M - {}^\bullet t) + t^\bullet$ (hence $M'(p) = M(p) - W(p, t) + W(t, p)$ for each $p \in P$). We denote this firing by $M \xrightarrow{t} M'$.

Bisimulation equivalence (in labeled Petri nets). Informally speaking, two markings (states) in a labeled Petri net are considered equivalent if they generate the same observable behavior. Finding equivalent states may be very helpful for reducing the state space when analyzing behavioral properties of a given net. A classical behavioral equivalence is bisimulation equivalence, also called bisimilarity [3]. We recall its definition in the framework of labeled Petri nets; below we thus implicitly assume a fixed labeled Petri net $N = (P, T, W, l)$.

We say that a *relation $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ satisfies* the *transfer property w.r.t. a relation $R' \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$* if for each pair $(M_1, M_2) \in R$ and for each firing $M_1 \xrightarrow{t} M_1'$ there exists an ("imitating") firing $M_2 \xrightarrow{u} M_2'$ such that $l(u) = l(t)$ and $(M_1', M_2') \in R'$. We can illustrate this property by the following diagram.

$$
\begin{array}{ccc}
M_1 & R & M_2 \\
\downarrow t & & \downarrow (\exists) u : l(u) = l(t) \\
M_1' & R' & M_2'
\end{array}
$$

By saying that R *has the transfer property* we mean that R satisfies the transfer property w.r.t. itself (hence $R' = R$ in the diagram).

A relation $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a *bisimulation* if R has the transfer property and also R^{-1} has the transfer property. Markings M_1 and M_2 are *bisimilar* (or *bisimulation equivalent*), written $M_1 \sim M_2$, if there exists a bisimulation R such that $(M_1, M_2) \in R$; the relation $\sim$, called *bisimilarity* (or *bisimulation equivalence*), is thus the union of all bisimulations. It is easy to check that bisimilarity is an equivalence (a reflexive, symmetric, and transitive relation), and that it is itself a bisimulation. Hence the relation $\sim$ is the largest bisimulation (related to our fixed labeled Petri net $N = (P, T, W, l)$).

It is also standard to define *stratified bisimilarity relations* [32] $\sim_i \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$, for $i \in \mathbb{N}$:

- $\sim_0 = \mathcal{M}(P) \times \mathcal{M}(P)$ (hence $M_1 \sim_0 M_2$ for all $M_1, M_2 \in \mathcal{M}(P)$);

- $\sim_{i+1}$ is the largest symmetric relation that satisfies the transfer property w.r.t. $\sim_i$ (hence $M_1 \sim_{i+1} M_2$ iff for each $M_1 \xrightarrow{t} M_1'$ there is $M_2 \xrightarrow{u} M_2'$ where $l(u) = l(t)$ and $M_1' \sim_i M_2'$ and for each $M_2 \xrightarrow{t} M_2'$ there is $M_1 \xrightarrow{u} M_1'$ where $l(u) = l(t)$ and $M_1' \sim_i M_2'$).

It is easy to note that $\sim_i$ are equivalences (for all $i \in \mathbb{N}$), $\sim_0 \supseteq \sim_1 \supseteq \sim_2 \supseteq \cdots$, and $\sim = \bigcap_{i \in \mathbb{N}} \sim_i$; hence $M_1 \sim M_2$ iff $M_1 \sim_i M_2$ for all $i \in \mathbb{N}$ (since labeled Petri nets generate image-finite labeled transition systems [32]).

While bisimilarity is undecidable for labeled Petri nets [4], we note that it is decidable for so called *communication-free labeled Petri nets* [6] (also know as (**1**,**P**)-systems [5], or Basic Parallel Processes [33]). The communication-free Petri nets (P, T, W) satisfy $|{}^\bullet t| \leq 1$ for all $t \in T$, and it is thus easy to observe that in this case bisimulation equivalence is a congruence w.r.t. the marking addition. For our aims it is useful to look at congruences on $\mathcal{M}(P)$ for finite sets P in general; we do this in the next paragraph, using rather symbols r, s instead of M for elements of $\mathcal{M}(P)$, to stress that it is not important here whether or not P is the set of places of a Petri net.

Congruences on $\mathcal{M}(P)$. Given a finite set P (which can be the set of places of a Petri net), an equivalence relation ρ on $\mathcal{M}(P)$ is a *congruence* if

$$r_1 \, \rho \, r_2 \text{ implies } (r_1 + s) \, \rho \, (r_2 + s) \text{ for all } s \in \mathcal{M}(P).$$

(Hence $r_1 \, \rho \, r_2$ and $r_1' \, \rho \, r_2'$ implies $(r_1 + r_1') \, \rho \, (r_2 + r_1')$ and $(r_1' + r_2) \, \rho \, (r_2' + r_2)$, and thus also $(r_1 + r_1') \, \rho \, (r_2 + r_2')$.)

An important known fact is captured by the next proposition, which says that each congruence on $\mathcal{M}(P)$ has a finite basis (by which the congruence is generated); this was an important ingredient in the decidability proof in [6]. We can refer, e.g., to [34, 35], but we provide a self-contained proof for convenience.

Proposition 2.1. Each congruence ρ on $\mathcal{M}(P)$, where P is a finite set, is generated by a finite subset of ρ, in particular by the set ρ_B (called a *basis of* ρ) consisting of the minimal elements of the set $\{(r, s) \in \rho \mid r \neq s\}$ w.r.t. the partial order $\leq$, where we put $(r, s) \leq (r', s')$ if $r \leq r'$ and $s \leq s'$. (Finiteness of ρ_B follows by Dickson's lemma.)

Proof:
For the sake of contradiction, we assume that the least congruence ρ' containing ρ_B is a proper subset of ρ. We choose a pair $(r_0, s_0) \in \rho \smallsetminus \rho'$ such that $r_0 + s_0$ is the least in the set $\{r + s \mid (r, s) \in \rho \smallsetminus \rho'\}$ w.r.t. the cardinality-lexicographic order $\sqsubseteq$ on $\mathcal{M}(P)$; we note that the pair (s_0, r_0) also has this property. Since $(r_0, s_0) \in \rho \smallsetminus \rho'$, we have $r_0 \neq s_0$ and $(r, s) \leq (r_0, s_0)$ for some $(r, s) \in \rho_\mathrm{B} \subseteq \rho'$; w.l.o.g. we assume $r \sqsubseteq s$ (since otherwise we would consider the pairs $(s, r) \leq (s_0, r_0)$, where $(s, r) \in \rho'$ and $(s_0, r_0) \in \rho \smallsetminus \rho'$).

We derive $(r + (s_0 - s), s + (s_0 - s)) \in \rho'$, i.e., $(r + (s_0 - s), s_0) \in \rho' \subseteq \rho$; since $(r_0, s_0) \in \rho$, we deduce $(r_0, r + (s_0 - s)) \in \rho$. Since $r \sqsubseteq s$ and $r \neq s$, we have either $|r| < |s|$, or $|r| = |s|$ and $r <_\mathrm{LEX} s$; this entails that either $|r_0 + (r + (s_0 - s))| < |r_0 + s_0|$, or $|r_0 + (r + (s_0 - s))| = |r_0 + s_0|$ and $r_0 + (r + (s_0 - s)) <_\mathrm{LEX} r_0 + s_0$. Due to our choice of (r_0, s_0) we deduce that $(r_0, r + (s_0 - s)) \in \rho'$; together with the above fact $(r + (s_0 - s), s_0)) \in \rho'$ this entails $(r_0, s_0) \in \rho'$ – a contradiction. $\qquad\square$

Remark.
We might also note that the above defined basis ρ_{B} could be made smaller by including just one of symmetric pairs (r, s) and (s, r).

We note that bisimilarity, i.e. the equivalence $\sim$, in labeled Petri nets is not a congruence in general (unlike the case of communication-free labeled Petri nets); we can use the simple example in Figure 2, where $\{X\} \sim \{Y\}$, but $(\{X\} + \{X\}) \not\sim (\{Y\} + \{X\})$ (since the firing $\{X, X\} \xrightarrow{t} \emptyset$, where t is labeled with b, has no imitating firing from $\{X, Y\}$, where no b-labeled transition is enabled).

Instead of looking at subclasses where $\sim$ is a congruence, we will look at natural equivalences refining $\sim$ that are congruences for the whole class of labeled Petri nets. The finite-basis result (Proposition 2.1) might give some hope regarding the decidability of such congruences.

3. Resource similarity and resource bisimilarity

In Section 3.1 we look at the largest congruence that refines bisimilarity (the equivalence $\sim$) in labeled Petri nets. This relation is known as resource similarity (denoted here by $\approx$), and it is also known to be undecidable (which entails that its finite basis is not effectively computable). In Section 3.2 we then recall another congruence, known as resource bisimilarity (denoted here by $\simeq$). In fact, it is the largest congruence refining $\sim$ that is a bisimulation. In Section 4 we show the decidability of resource bisimilarity (thus answering a question that was left open in the literature); nevertheless, the effective computability of its finite basis is not established by this result. (We add further comments in Section 5.)

3.1. Resource similarity

The relation of *resource similarity*, introduced in [26] for labeled Petri nets, is a congruent strengthening of bisimulation equivalence. Intuitively, a resource in a Petri net can be viewed as a part of its marking; formally it is also a multiset of places, hence its definition does not differ from the definition of a marking. The notions "markings" and "resources" are viewed as different only due to their different interpretation in particular contexts. Resources are parts of markings that may or may not provide
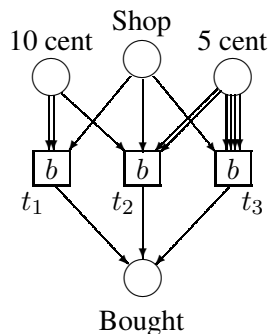

Figure 1. Buying goods for 20 cents

this or that kind of net behavior; e. g., in the Petri net example in Fig. 1 from [30], two ten-cent coins form a resource — enough to buy an item of goods.

Two resources are viewed as similar for a given labeled Petri net if replacing one of them by another in any marking (i.e., in any context) does not change the observable behavior of the net. In this sense, resource similarity is a congruence that preserves visible behaviors up to bisimilarity. Now we capture this description formally. We use symbols r, s, w for elements of $\mathcal{M}(P)$ (rather than M) to stress our "resource motivation" here.

Definition 3.1. Let $N = (P, T, W, l)$ be a labeled Petri net. A *resource* r in N is a multiset over the set P of places; hence $r \in \mathcal{M}(P)$. Two resources r and s in N are called *similar*, which is denoted by $r \approx s$, if for all $w \in \mathcal{M}(P)$ we have $(r + w) \sim (s + w)$, where $\sim$ is bisimulation equivalence. The relation $\approx$ is called *resource similarity*.

Proposition 3.2. For any labeled Petri net $N = (P, T, W, l)$, the relation $\approx$ (resource similarity) is the largest congruence included in $\sim$ (bisimilarity).

Proof:
It is straightforward to check that $\approx$ is an equivalence relation (reflexive, symmetric, and transitive), since $\sim$ is an equivalence. (In particular, if $r_1 \approx r_2$ and $r_2 \approx r_3$, then for each $w \in \mathcal{M}(P)$ we have $(r_1 + w) \sim (r_2 + w)$ and $(r_2 + w) \sim (r_3 + w)$, and thus $(r_1 + w) \sim (r_3 + w)$; hence $r_1 \approx r_3$.) Moreover, $\approx$ is a congruence w.r.t. multiset addition, i.e., $r \approx s$ implies $(r + w) \approx (s + w)$; indeed, for any w' we have $((r + w) + w') \sim ((s + w) + w')$, since $r \approx s$ entails $(r + (w + w')) \sim (s + (w + w'))$. We trivially have $\approx \, \subseteq \, \sim$. Moreover, each congruence $\rho \subseteq \sim$ satisfies that $r \, \rho \, s$ implies $(r + w) \, \rho \, (s + w)$, and thus $(r + w) \sim (s + w)$, for all $w \in \mathcal{M}(P)$; this entails $\rho \subseteq \approx$.   □

Since resource similarity is a congruence on $\mathcal{M}(P)$, it has a finite basis by Proposition 2.1. However, the place fusion (studied in [21, 22]) is a special case of resource similarity for resources of capacity one. Place fusion has been proven to be undecidable, and this implies undecidability of resource similarity; it also entails that the finite basis of $\approx$ is not effectively computable.

Remark.
The problem of deciding non-bisimilarity, i.e. the relation $\not\sim$, is semidecidable (for labeled Petri nets); this follows from the fact that the equivalences $\sim_i$ are decidable for all $i \in \mathbb{N}$, and $\sim \, = \bigcap_{i \in \mathbb{N}} \sim_i$. On the other hand, the halting problem for Minsky machines can be reduced to deciding $\not\sim$, which entails that $\sim$ is not semidecidable. Since $r \not\approx s$ means that $(r + w) \not\sim (s + w)$ for some w, we easily derive the semidecidability of $\not\approx$. Moreover, the mentioned reduction of the halting problem to deciding $\not\sim$ can be adjusted so that it reduces to deciding $\not\approx$. This entails that $\approx$ is not semidecidable, and thus the finite basis of $\approx$ cannot be effectively computable.

We note that resource similarity is not a bisimulation in general; the relation $\approx$ may not have the transfer property. This is exemplified by the labeled Petri net in Figure 3: we shall later show that $\{X_1\} \approx \{Y_1\}$, but the firing $\{X_1\} \xrightarrow{t_4} \{X_3\}$ has no "imitating" firing from $\{Y_1\}$; indeed, both candidates $\{Y_1\} \xrightarrow{u_1} \emptyset$ and $\{Y_1\} \xrightarrow{u_2} \{Y_2\}$ fail, since $\{X_3\} \not\approx \emptyset$ (because $(\{X_3\} + \{Z\}) \not\sim_1 (\emptyset + \{Z\})$) and $\{X_3\} \not\approx \{Y_2\}$ (because $(\{X_3\} + \emptyset) \not\sim_1 (\{Y_2\} + \emptyset)$).

3.2.   Resource bisimilarity

In view of the above facts, that the largest congruence $\approx$ refining $\sim$ is undecidable and is not a bisimulation, it is natural to look at the largest congruence $\simeq$ refining $\sim$ that *is* a bisimulation (whence we have $\simeq \,\subseteq\, \approx \,\subseteq\, \sim$). In fact, the relation $\simeq$, named *resource bisimilarity*, was already defined in [26], though technically slightly differently than we do this below. We also use a new term, namely *the resource transfer property*; this property is stronger than *the transfer property* defined for relations on $\mathcal{M}(P)$ in Section 2. An analogous notion was called *the weak transfer property* in [26], due to a different viewpoint (related to the variant of the transfer property for place bisimulation defined in [19]).

Definition 3.3. Let $N = (P, T, W, l)$ be a labeled Petri net. We say that a *relation* $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ *satisfies* the *resource transfer property w.r.t. a relation* $R' \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ if for each pair $(r, s) \in R$ and for each firing $(r + (^\bullet t - r)) \xrightarrow{t} r'$ there exists an ("imitating") firing $(s + (^\bullet t - r)) \xrightarrow{u} s'$ such that $l(u) = l(t)$ and $(r', s') \in R'$. We can illustrate this property by the following diagram.

$$
\begin{array}{ccc}
r & R & s \\[4pt]
r + (^\bullet t - r) & & s + (^\bullet t - r) \\[4pt]
\downarrow t & & \downarrow (\exists) u : \; l(u) = l(t) \\[4pt]
r' & R' & s'
\end{array}
$$

(By our multiset definitions, $r + (^\bullet t - r) = r \cup {}^\bullet t$. We also note that $^\bullet t \leq r$ entails that $^\bullet t - r = \emptyset$, hence $r + (^\bullet t - r) = r$ and $s + (^\bullet t - r) = s$; therefore the resource transfer property implies the transfer property.)

By saying that R *has the resource transfer property* we mean that R satisfies the resource transfer property w.r.t. itself (hence $R' = R$ in the diagram).

A relation $R \subseteq \mathcal{M}(P) \times \mathcal{M}(P)$ is a *resource bisimulation* if R has the resource transfer property and also R^{-1} has the resource transfer property. Resources (or markings) $r, s \in \mathcal{M}(P)$ are *resource bisimilar*, written $r \simeq s$, if there exists a resource bisimulation R such that $(r, s) \in R$; the relation $\simeq$, called *resource bisimilarity*, is thus the union of all resource bisimulations.

Example 3.4. In the net in Figure 1 we can present any multiset r over the set of places by the vector $(r(\texttt{10cent}), r(\texttt{Shop}), r(\texttt{5cent}), r(\texttt{Bought}))$.

We can show that $r_0 \simeq s_0$ where $r_0 = (1, 0, 0, 0)$ and $s_0 = (0, 0, 2, 0)$, by verifying that the relation $\{\big((1, 0, 0, k), (0, 0, 2, k)\big) \mid k \geq 0\} \cup \{\big((0, 0, 0, k), (0, 0, 0, k)\big) \mid k \geq 1\}$ is a resource bisimulation.

E.g., $^\bullet t_1 - r_0 = (1, 1, 0, 0)$, and we have $r_0 + (^\bullet t_1 - r_0) = (2, 1, 0, 0) \xrightarrow{t_1} (0, 0, 0, 1)$. This firing can be imitated from $s_0 + (^\bullet t_1 - r_0) = (1, 1, 2, 0)$ by the firing $(1, 1, 2, 0) \xrightarrow{t_2} (0, 0, 0, 1)$ (since both t_1 and t_2 have the same label b). It is a routine to finish this verification.

Proposition 3.5. For any labeled Petri net $N = (P, T, W, l)$, the relation $\simeq$ (resource bisimilarity) is the largest congruence included in $\sim$ that is a bisimulation; hence we have $\simeq \,\subseteq\, \approx \,\subseteq\, \sim$.

Proof:

Since the relation $\simeq$ is the union of all resource bisimulations, i.e.

$$\simeq = \bigcup \{\, R \subseteq \mathcal{M}(P) \times \mathcal{M}(P) \mid R \text{ and } R^{-1} \text{ have the resource transfer property} \,\},$$

it is obvious that also $\simeq$ and $\simeq^{-1}$ have the resource transfer property; hence $\simeq$ is the largest resource bisimulation (related to the considered labeled Petri net $N = (P, T, W, l)$). Since the resource transfer property is stronger than the transfer property (if R has the resource transfer property, then R also has the transfer property), each resource bisimulation is a (standard) bisimulation; hence $\simeq$ is a bisimulation, and we have $\simeq \subseteq \sim$.

The reflexivity and symmetry of $\simeq$ is straightforward; the next two points show that $\simeq$ is a congruence:

1. We prove that $r \simeq s$ implies $(r + w) \simeq (s + w)$ by showing that the set

$$R = \{(r + w, s + w) \mid r, s, w \in \mathcal{M}(P), r \simeq s\}$$

   is a resource bisimulation (hence $R = \simeq$). To this aim we fix some $(r + w, s + w) \in R$, where $r \simeq s$, and consider a firing $(r + w + ({}^\bullet t - (r + w))) \xrightarrow{t} r'$; we look for an imitating firing from $s + w + ({}^\bullet t - (r + w))$. We note that $w + ({}^\bullet t - (r + w)) = ({}^\bullet t - r) + w'$ for a (maybe nonempty) multiset $w' \in \mathcal{M}(P)$; we thus have $(r + ({}^\bullet t - r) + w') \xrightarrow{t} r'$, and look for an imitating firing from $s + ({}^\bullet t - r) + w'$. We have $(r + ({}^\bullet t - r)) \xrightarrow{t} r''$, and by $r \simeq s$ we deduce that there is an imitating firing $(s + ({}^\bullet t - r)) \xrightarrow{t'} s''$, where $r'' \simeq s''$. Since $(r + ({}^\bullet t - r) + w') \xrightarrow{t} r'' + w'$ (for the above r' we have $r' = r'' + w'$), the firing $(s + ({}^\bullet t - r) + w') \xrightarrow{t'} s'' + w'$ satisfies our need: $(r'' + w', s'' + w') \in R$.

2. Transitivity of $\simeq$ (i.e., $r_1 \simeq r_2$ and $r_2 \simeq r_3$ entails $r_1 \simeq r_3$) is demonstrated by showing that the relation

$$R = \{(r_1, r_3) \mid r_1 \simeq r_2 \text{ and } r_2 \simeq r_3 \text{ for some } r_2 \in \mathcal{M}(P)\}$$

   is a resource bisimulation. Let us consider a pair $(r_1, r_3) \in R$, where $r_1 \simeq r_2$ and $r_2 \simeq r_3$, and a firing $(r_1 + ({}^\bullet t_1 - r_1)) \xrightarrow{t_1} r'_1$. Since $r_1 \simeq r_2$, there is an imitating firing $(r_2 + ({}^\bullet t_1 - r_1)) \xrightarrow{t_2} r'_2$, where $r'_1 \simeq r'_2$. By Point 1, $r_2 \simeq r_3$ entails $(r_2 + ({}^\bullet t_1 - r_1)) \simeq (r_3 + ({}^\bullet t_1 - r_1))$, hence $(r_2 + ({}^\bullet t_1 - r_1)) \xrightarrow{t_2} r'_2$ (where ${}^\bullet t_2 - (r_2 + ({}^\bullet t_1 - r_1)) = \emptyset$) has an imitating firing $(r_3 + ({}^\bullet t_1 - r_1)) \xrightarrow{t_3} r'_3$, where $r'_2 \simeq r'_3$. Since $(r'_1, r'_3) \in R$, we are done.

Finally we note that each congruence ρ on $\mathcal{M}(P)$ that refines $\sim$ (i.e., $\rho \subseteq \sim$) and is a (standard) bisimulation is also a resource bisimulation (since $r \, \rho \, s$ implies $(r + ({}^\bullet t - r)) \, \rho \, (s + ({}^\bullet t - r)))$; hence we have $\rho \subseteq \simeq$ for all such congruences. $\qquad\qquad\square$

Similarly as for (standard) bisimilarity, we define the stratified versions for resource bisimilarity; we also observe their properties that underpin our algorithm in Section 4.

Definition 3.6. (of equivalences $\simeq_i$ and equivalence-levels $\mathrm{EQLEV}(r,s)$)
Assuming a labeled Petri net $N = (P, T, W, l)$, we define the relations $\simeq_i$, $i \in \mathbb{N}$, as follows:

- $\simeq_0 = \mathcal{M}(P) \times \mathcal{M}(P)$;

- $\simeq_{i+1}$ is the largest symmetric relation that satisfies the resource transfer property w.r.t. $\simeq_i$.

For $r, s \in \mathcal{M}(P)$, by $\mathrm{EQLEV}(r,s)$ we denote the largest i such that $r \simeq_i s$, stipulating $\mathrm{EQLEV}(r,s) = \omega$ (where $i < \omega$ for all $i \in \mathbb{N}$) if $r \simeq_i s$ for all $i \in \mathbb{N}$.

Proposition 3.7.

1. For all $i \in \mathbb{N}$, $\simeq_i$ are congruences on $\mathcal{M}(P)$.

2. $\simeq_0 \supseteq \simeq_1 \supseteq \simeq_2 \supseteq \cdots$, and $\simeq \, = \bigcap_{i \in \mathbb{N}} \simeq_i$.

3. If $\mathrm{EQLEV}(s, s') > \mathrm{EQLEV}(r, s)$, then $\mathrm{EQLEV}(r, s') = \mathrm{EQLEV}(r, s)$.

Proof:
  1) We can proceed by induction on i, and analogously as in the proof of Proposition 3.5.

  2) The fact $\simeq_i \, \supseteq \, \simeq_{i+1}$ and the inclusion $\simeq \, \subseteq (\bigcap_{i \in \mathbb{N}} \simeq_i)$ are obvious. Now we observe that the relation $\bigcap_{i \in \mathbb{N}} \simeq_i$ is a resource bisimulation (which entails $(\bigcap_{i \in \mathbb{N}} \simeq_i) \subseteq \, \simeq$); here we use image-finiteness, i.e., the fact that each firing has only finitely many candidates for imitating firings (with the same action-label). Hence if $(r, s) \in \bigcap_{i \in \mathbb{N}} \simeq_i$, and $(r + (^\bullet t - r)) \xrightarrow{t} r'$, then there is an imitating firing $(s + (^\bullet t - r)) \xrightarrow{t'} s'$ (with $l(t') = l(t)$) such that $r' \simeq_i s'$ for infinitely many $i \in \mathbb{N}$; but this entails that $(r', s') \in \bigcap_{i \in \mathbb{N}} \simeq_i$.

  3) Let $\mathrm{EQLEV}(s, s') > \mathrm{EQLEV}(r, s) = i$; hence $r \simeq_i s$, $r \not\simeq_{i+1} s$, and $s \simeq_{i+1} s'$ (and $s \simeq_i s'$). Since $\simeq_i$ and $\simeq_{i+1}$ are equivalences, we have $r \simeq_i s'$ and $r \not\simeq_{i+1} s'$ (since $r \simeq_{i+1} s'$ would entail $r \simeq_{i+1} s$). $\qquad\square$

  The next theorem summarizes the previously derived facts on the equivalences $\simeq$ (resource bisimilarity), $\approx$ (resource similarity), $\sim$ (bisimilarity), and adds that these equivalences really differ in general.

Theorem 3.8. (Mutual relations of resource bisimilarity, resource similarity, and bisimilarity)
  1. For each labeled Petri net we have $\simeq \, \subseteq \, \approx \, \subseteq \, \sim$; moreover, $\approx$ is the largest congruence included in $\sim$, and $\simeq$ is the largest congruence included in $\sim$ that is a bisimulation.

  2. There exists a labeled Petri net for which $\approx \, \neq \, \sim$.

  3. There exists a labeled Petri net for which $\simeq \, \neq \, \approx$.

  4. For each labeled communication-free Petri net we have $\simeq \, = \, \approx \, = \, \sim$.

Proof:
  Point 1 summarizes Propositions 3.2 and 3.5.
  Point 4 follows by the fact that $\sim$ is a congruence for labeled communication-free Petri nets.
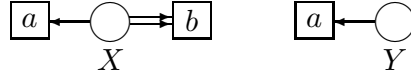
Figure 2.   Resource similarity does not coincide with bisimilarity: $\{X\} \sim \{Y\}$, but $\{X\} \not\approx \{Y\}$.
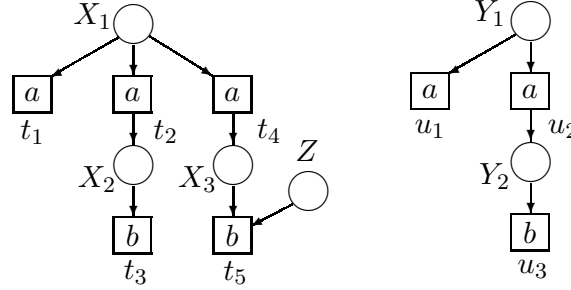

Figure 3.   Resource similarity does not coincide with resource bisimilarity: $\{X_1\} \approx \{Y_1\}$, but $\{X_1\} \not\simeq \{Y_1\}$.

Point 2 is shown by the example depicted in Fig. 2. We have $\{X\} \sim \{Y\}$, since the relation $\{(\{X\}, \{Y\}), (\emptyset, \emptyset)\}$ is a bisimulation; but $\{X\} \not\approx \{Y\}$, since $(\{X\} + \{X\}) \not\sim (\{Y\} + \{X\})$ (we even have $\{X, X\} \not\sim_1 \{X, Y\}$ due to the action b that is enabled just in one of these multisets).

Point 3 is shown by the example depicted in Fig. 3. At the end of Section 3.1 we have, in fact, already shown that $\{X_1\} \not\simeq \{Y_1\}$: since the pair $(\{X_1\}, \{Y_1\})$ (viewed as a relation with a single element) does not satisfy the transfer property w.r.t. $\approx$, it surely does not satisfy the resource transfer property w.r.t. $\simeq$ (since $\simeq \subseteq \approx$, and the resource transfer property is stronger than the transfer property). It thus remains to show that $\{X_1\} \approx \{Y_1\}$, i.e. that

$$(\{X_1\} + w) \sim (\{Y_1\} + w) \text{ for each multiset } w \text{ over the set } P = \{X_1, X_2, X_3, Y_1, Y_2, Z\}.$$

We show this by verifying that the following relation R on $\mathcal{M}(P)$ is a bisimulation; R contains the pairs (r, s) for which one of the following conditions is satisfied (where the condition 1 guarantees that $(\{X_1\} + w, \{Y_1\} + w) \in R$):

1. $r(X_1) - s(X_1) = 1$, $s(Y_1) - r(Y_1) = 1$, and r, s coincide on the set $\{X_2, X_3, Y_2, Z\}$;

2. $r(X_1) = s(X_1)$, $r(Y_1) = s(Y_1)$, and
   $r(X_2) + \min\{r(X_3), r(Z)\} + r(Y_2) = s(X_2) + \min\{s(X_3), s(Z)\} + s(Y_2)$.

It is a routine to check that both R and R^{-1} have the transfer property. The only interesting cases are the firings from one side of $(r, s) \in R$ that cannot be matched ("imitated") by firing the same transition from the other side. Such cases are described in what follows:

- (r, s) satisfies 2:
  Here any b-transition (i.e., t_3, t_5, u_3) fired from r (or from s) can be matched by firing any b-transition from s (or from r, respectively); the resulting pair (r', s') obviously satisfies 2 as well.

For the firing $r \xrightarrow{t_4} r'$ (t_4 is an a-transition) we have

- either $\min\{r'(X_3), r'(Z)\} = \min\{r(X_3), r(Z)\} + 1$ (when $r(X_3) < r(Z)$), in which case the firing $r \xrightarrow{t_4} r'$ is matched by $s \xrightarrow{t_2} s'$,
- or $\min\{r'(X_3), r'(Z)\} = \min\{r(X_3), r(Z))\}$ (when $r(X_3) \geq r(Z)$), in which case the firing $r \xrightarrow{t_4} r'$ is matched by $s \xrightarrow{t_1} s'$.

In both cases (r', s') satisfies 2. The firing $s \xrightarrow{t_4} s'$ is matched analogously.

- (r, s) satisfies 1 and $r(Y_1) = 0$ or $s(X_1) = 0$:

  If $r(Y_1) = 0$ (hence $s(Y_1) = 1$), then the firing $s \xrightarrow{u_1} s'$ (or $s \xrightarrow{u_2} s'$) is matched by $r \xrightarrow{t_1} r'$ (or $r \xrightarrow{t_2} r'$, respectively); in both cases (r', s') satisfies 2.

  If $r(X_1) = 1$ (hence $s(X_1) = 0$), then the firing $r \xrightarrow{t_1} r'$ (or $r \xrightarrow{t_2} r'$) is matched by $s \xrightarrow{u_1} s'$ (or $s \xrightarrow{u_2} s'$, respectively).

  The firing $r \xrightarrow{t_4} r'$ is matched similarly as discussed above:

  - if $\min\{r'(X_3), r'(Z)\} = \min\{r(X_3), r(Z)\} + 1$, then $r \xrightarrow{t_4} r'$ is matched by $s \xrightarrow{u_2} s'$;
  - if $\min\{r'(X_3), r'(Z)\} = \min\{r(X_3), r(Z)\}$, then $r \xrightarrow{t_4} r'$ is matched by $s \xrightarrow{u_1} s'$.

  In both cases (r', s') satisfies 2. □

As we already mentioned, bisimilarity (the relation $\sim$) and resource similarity (the relation $\approx$) are undecidable. In the next section (Section 4) we show that resource bisimilarity (the relation $\simeq$) is decidable. Here we still show that the decidability does not immediately follow from the fact that $\simeq = \bigcap_{i \in \mathbb{N}} \simeq_i$ (where $\simeq_0 \supseteq \simeq_1 \supseteq \simeq_2 \supseteq \cdots$), even though all $\simeq_i$, as well as $\simeq$, are congruences and thus have finite bases (by Proposition 2.1). We could even easily derive that finite bases for $\simeq_i$, $i \in \mathbb{N}$, are effectively computable; nevertheless, it is not immediately clear how they "converge" to a finite basis for $\sim$. The following simple example aims to illustrate this, even for the case of a labeled communication-free Petri net (where $\simeq$ coincides with $\sim$).
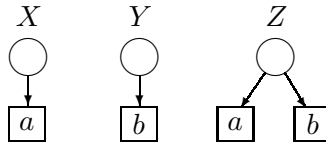

Figure 4. A labeled communication-free Petri net

Example 3.9. Let us consider the labeled Petri net in Fig. 4; it is a communication-free net, since the preset of each transition is a (multi)set containing a single place. Here bisimilarity (the relation $\sim$) is a congruence (thus coinciding with $\simeq$), and it is not difficult to derive that $\sim$ is the identity relation (we can thus take the empty set as its finite basis).

We now look at finite bases of congruences $\simeq_i$. To this aim we present multisets r over $\{X, Y, Z\}$ as the vectors $(r(X), r(Y), r(Z))$. For $\simeq_0$, a basis is the set

$$\{\big((0,0,0),(0,0,1)\big), \big((0,0,0),(0,1,0)\big), \big((0,0,0),(1,0,0)\big)\}$$

(where we do not include the symmetric pairs).

For $\simeq_1$, a basis contains elements like

$$\big((0,0,1),(0,0,2)\big), \big((0,0,1),(0,1,1)\big), \big((0,0,1),(1,0,1)\big), \big((0,0,1),(1,1,0)\big), \dots$$

For $\simeq_2$, a basis can not contain, e.g., $\big((0,0,1),(0,0,2)\big)$, but "instead" it contains elements like $\big((0,0,2),(0,0,3)\big), \big((1,1,1),(1,1,2)\big), \dots$.

It seems to be a subtle problem in general, to derive a basis for $\simeq$ by constructing a row of bases for $\simeq_1, \simeq_2, \dots$. Nevertheless, for labeled *communication-free* Petri nets, a finite basis for $\sim$ (coinciding with $\simeq$) can be effectively computed; further remarks about this are in Section 5.

4.   Algorithm for checking resource bisimilarity

The problem:
Given a labeled Petri net $N = (P, T, W, l)$ and two resources $r_0, s_0 \in \mathcal{M}(P)$, we need to check whether they are resource bisimilar, i.e. whether $r_0 \simeq s_0$. We assume that $P \neq \emptyset$ and $T \neq \emptyset$ (otherwise the problem is trivial).

The algorithm ALG we present here to solve this problem uses the well-known tableau technique (see e. g., [31]), adapted to the resource bisimilarity relation and its resource transfer property.

Below we give a pseudocode of ALG (with some comments inside $(* \ *)$). In its computation, ALG will be also comparing the cardinalities $|r|$, $|s|$ of multisets $r, s \in \mathcal{M}(P)$, and in the case $|r| = |s|$ it will use the *lexicographic order* $r \leq_{\text{LEX}} s$. In other words, it will use the cardinality-lexicographic order $\sqsubseteq$ that was defined in Section 2 for multisets. Besides that, ALG will also use the standard component-wise order $r \leq s$, extended to the pairs of multisets as in Proposition 2.1: we put $(r, s) \leq (r', s')$ if $r \leq r'$ and $s \leq s'$.

> **Nondeterministic algorithm ALG deciding resource bisimilarity**
>
> *Input:* A labeled Petri net $N = (P, T, W, l)$ and two resources $r_0, s_0 \in \mathcal{M}(P)$.
>
> *Output:* If $r_0 \simeq s_0$, then at least one computation returns YES; if $r_0 \not\simeq s_0$, then all computations return NO.
>
> **Procedure:**
>
> $(*$ It stepwise constructs a tree (a *proof tree*, also called a *tableau*), which is stored in a "program variable" CT (Current Tree); its nodes are labeled with pairs of resources. A node of CT is called an *identity-node* if its label is of the form (r, r); each such node will be a *successful leaf* of the constructed tree. The outcome YES will be returned iff a *successful tree*, i.e. a tree whose all leaves are successful, will be constructed. $*)$

1. Create the root labeled with the input pair (r_0, s_0); this node constitutes the initial current tree, hence the initial value of CT.

2. **while** there is a non-identity leaf in CT **do**
   begin
   In CT choose a leaf N labeled with (r, s) where $r \neq s$, and process it as follows:

   - **if** the rule REDUCE ($*$ described below $*$) is applicable to N
     then apply it; by doing this, N gets exactly one child-node $\bar{N}$
     ($*$ $\bar{N}$ becomes a new leaf in (the extended) CT, and is labeled with some $(\bar{r}, \bar{s})$
     where $\bar{r} + \bar{s}$ is strictly smaller than $r + s$ in the cardinality-lexicographic order
     $\sqsubseteq$ $*$);
   - **otherwise** ($*$ when REDUCE is not applicable to N $*$), apply EXPAND to N
     ($*$ which fails if $r \not\simeq_1 s$, in which case the computation returns NO, and other-
     wise creates (at most) $2 \cdot |T|$ children of N $*$).

   end
   RETURN YES ($*$ here CT is successful, since all leaves are identity-nodes $*$).

To formulate the rule EXPAND, we introduce the following definitions, referring to the underlying labeled Petri net $N = (P, T, W, l)$. For $t \in T$, and $r, s, r', s' \in \mathcal{M}(P)$, the pair (r', s') is a *t-child of* (r, s) if there is $u \in T$ such that $l(u) = l(t)$, $(r + (\bullet t - r)) \overset{t}{\to} r'$, and $(s + (\bullet t - r)) \overset{u}{\to} s'$. (Recall the diagram in Definition 3.3.) By $next_t(r, s)$ we denote the set of all t-children of (r, s).

We observe a trivial fact that shows the soundness of the following description of EXPAND.

Claim 4.1. We have $r \not\simeq_1 s$ iff $next_t(r, s)$ or $next_t(s, r)$ is empty for some $t \in T$.

Application of EXPAND to a node N labeled with (r, s) (where $r \neq s$):

if $r \not\simeq_1 s$ **then** RETURN NO

($*$ hence the algorithm ALG invoking EXPAND returns NO in this case $*$),

otherwise for each $t \in T$ select (nondeterministically) exactly one pair of resources from $next_t(r, s)$ and exactly one pair of resources from $next_t(s, r)$, and create (at most) $2 \cdot |T|$ children of N whose labels are precisely the pairs selected for all $t \in T$.

($*$ We recall that $T \neq \emptyset$; hence if $r \simeq_1 s$, then the set of children of N is nonempty. $*$)

We note some simple facts regarding EXPAND (that are used later, in the proof of Theorem 4.4):

Claim 4.2. (Properties of EXPAND)
Let N be a leaf of CT, labeled with (r, s) where $r \neq s$. Then we have:

1. If $r \simeq s$, then there is at least one application of EXPAND to N such that each arising child of N is labeled with an equivalent pair, i.e. with some (r', s') where $r' \simeq s'$.

2. If $\text{EQLEV}(r, s) = k \in \mathbb{N}_+$ (hence $r \simeq_k s$, $r \not\simeq_{k+1} s$, $k \geq 1$), then each application of EXPAND to N gives rise to at least one child of N that is labeled with some (r', s') where $\text{EQLEV}(r', s') < k$.

Proof:
The claims easily follow from the definitions of relations $\simeq$ and $\simeq_i$.   □

Now we describe the rule REDUCE, and also note its useful properties.

Application of REDUCE to a node N labeled with (r, s) (where $r \neq s$), in a given current tree CT:

If there a node $\text{N}' \neq \text{N}$ on the path from the root to N in CT that is labeled with (r', s') where $(r', s') \leq (r, s)$, then the rule REDUCE is applicable; otherwise it is not applicable.

If the rule is applicable, then $(r', s') \leq (r, s)$ related to one such node N' is selected, and N gets precisely one child, namely $\bar{\text{N}}$ labeled with $(\bar{r}, \bar{s})$ where we have:

- if $|r'| < |s'|$, or $|r'| = |s'|$ and $r' <_{\text{LEX}} s'$, then $(\bar{r}, \bar{s}) = (r, (s - s') + r')$, and
- if $|s'| < |r'|$, or $|r'| = |s'|$ and $s' <_{\text{LEX}} r'$, then $(\bar{r}, \bar{s}) = ((r - r') + s', s)$.

($*$ Since identity-nodes are leaves, we have $r' \neq s'$. But we note that the case $(r', s') = (r, s)$ is not excluded; in this case the child $\bar{\text{N}}$ is an identity-node, labeled with (r, r) or (s, s). $*$)

Claim 4.3. (Properties of REDUCE)
Let us consider a current tree CT and an application of REDUCE to a node N in CT, labeled with (r, s), where the application is based on a node N' labeled with (r', s') (where $(r', s') \leq (r, s)$); let the resulting child $\bar{\text{N}}$ of N be labeled with $(\bar{r}, \bar{s})$. We then have:

1. $|\bar{r} + \bar{s}| < |r + s|$, or $|\bar{r} + \bar{s}| = |r + s|$ and $\bar{r} + \bar{s} <_{\text{LEX}} r + s$;

2. the edges on the path from N' to N could not be created by applications of REDUCE only (i. e., at least one edge has been created by applying EXPAND);

3. if $r' \simeq s'$ and $r \simeq s$, then $\bar{r} \simeq \bar{s}$;

4. if $\text{EQLEV}(r', s') > \text{EQLEV}(r, s)$, then $\text{EQLEV}(\bar{r}, \bar{s}) = \text{EQLEV}(r, s)$.

Proof:
1) If $|r'| < |s'|$, then $|\bar{s}| = |(s - s') + r'| < |s|$ (since $s' \leq s$); hence $|\bar{r} + \bar{s}| = |r + \bar{s}| < |r + s|$.
If $r' <_{\text{LEX}} s'$, then $(s - s') + r' <_{\text{LEX}} s$ (due to the first component i in which r' and s' differ; hence $(r')_i < (s')_i$, and therefore $((s - s') + r')_i < (s)_i$); this entails that $r + ((s - s') + r') <_{\text{LEX}} r + s$. Hence if $|r'| = |s'|$ and $r' <_{\text{LEX}} s'$, then $\bar{r} + \bar{s} <_{\text{LEX}} r + s$ (since $\bar{r} = r$ and $\bar{s} = (s - s') + r'$). The case $|r'| > |s'|$, or $|r'| = |s'|$ and $s' <_{\text{LEX}} r'$ is analogous.

2) Since $(r', s') \leq (r, s)$, we have either $|r' + s'| < |r + s|$, or $(r', s') = (r, s)$. By 1) it is thus obvious that the edges on the path from N' to N could not be all created by applications of REDUCE.

3) Since $\simeq$ is a congruence, $r' \simeq s'$ entails $r' + (s - s') \simeq s' + (s - s')$, hence $r' + (s - s') \simeq s$. Then $r \simeq s$ entails $r \simeq r' + (s - s')$ (by symmetry and transitivity of $\simeq$). The claim is thus clear.

4) We note that $\mathrm{EQLEV}(r', s') \leq \mathrm{EQLEV}(r' + (s - s'), s' + (s - s'))$, since $\simeq_i$ are congruences (by Proposition 3.7(1)). Hence $\mathrm{EQLEV}(r', s') > \mathrm{EQLEV}(r, s)$ entails $\mathrm{EQLEV}(r' + (s - s'), s) > \mathrm{EQLEV}(r, s)$, and thus $\mathrm{EQLEV}(r, r' + (s - s')) = \mathrm{EQLEV}(r, s)$, by Proposition 3.7(3). $\square$

The following theorem asserts the termination and correctness of the algorithm ALG.

Theorem 4.4. (ALG decides resource bisimilarity)
Let $N = (P, T, W, l)$ be a labeled Petri net and $r_0, s_0 \in \mathcal{M}(P)$ be its resources. Then:

1. For the input N, r_0, s_0 there are only finitely many computations of the above nondeterministic algorithm ALG, and each computation finishes by constructing a finite tree $\mathcal{T}$ whose nodes are labeled with pairs of resources; moreover, either one leaf of $\mathcal{T}$ is *unsuccessful*, i.e. labeled with (r, s) where $r \not\simeq_1 s$ (in which case the computation returns NO), or $\mathcal{T}$ is successful, i.e. all leaves of $\mathcal{T}$ are identity-nodes (in which case the computation returns YES).

2. We have $r_0 \simeq s_0$ if, and only if, at least one computation (of ALG on N, r_0, s_0) constructs a successful tree.

Proof:
1) Any constructed tree is finitely branching, since each node has at most $2 \cdot |T|$ children. If there was an infinite computation, it would construct a tree with an infinite branch (by König's lemma); let us fix such a branch B for the sake of contradiction. Each edge on B results by an application of EXPAND or REDUCE. Claim 4.3(1) entails that infinitely many edges in B have arisen by using EXPAND (since we cannot have an infinite row of REDUCE applications). Hence we get an infinite sequence (r_1, s_1), $(r_2, s_2), \ldots$ of labels related to nodes $N_1, N_2, \ldots$ in B where EXPAND was used, and thus REDUCE was not applicable. But Dickson's lemma contradicts this, since there must exist some $i < j$ such that $(r_i, s_i) \leq (r_j, s_j)$ (and thus REDUCE would be applicable to N_j).

2) We analyze the respective two cases:

- Let $r_0 \simeq s_0$. We consider a computation which keeps the property that all nodes are labeled with equivalent pairs (each node is labeled with some (r, s) where $r \simeq s$); there is such a computation by Claim 4.2(1) and Claim 4.3(3). All leaves of the constructed tree are then successful (being identity-nodes).

- Let $r_0 \not\simeq s_0$ (hence $\mathrm{EQLEV}(r_0, s_0) = k \in \mathbb{N}$), and let us consider the tree $\mathcal{T}$ constructed by an arbitrarily chosen computation. By recalling Claim 4.2(2) and Claim 4.3(2,4) we deduce that there must be a branch of $\mathcal{T}$ along which the equivalence level never increases (it drops along each edge related to EXPAND, and remains the same along each edge related to REDUCE). Hence the leaf of this branch must be unsuccessful (labeled with some (r, s) where $r \not\simeq_1 s$). $\square$

5.   Conclusions and additional remarks

In this paper we have investigated the decidability issues for two congruent restrictions of bisimulation equivalence (denoted by $\sim$) in classical labeled Petri nets (P/T-nets).

These congruences are resource similarity (denoted by $\approx$) and resource bisimilarity (denoted by $\simeq$), as defined in [26]; both try to clarify when replacing a resource (submarking) in a Petri net marking with a similar resource does not change the observable system behavior.

Here we have stressed that $\approx$ is the largest congruence included in $\sim$, and that $\simeq$ is the largest congruence included in $\sim$ that is a bisimulation; we thus also have $\simeq \subseteq \approx \subseteq \sim$. Besides a straightforward fact that $\approx$ is a *strict* refinement of $\sim$ in general (i.e., $\approx \subsetneq \sim$ for some labeled Petri nets), we have shown here also that $\simeq$ strictly refines $\approx$ ($\simeq \subsetneq \approx$); the latter fact answered a question that was left open in [26, 30]. We can also notice that deciding the relations $\approx$ and $\simeq$ can be used for deducing bisimilar states, and thus help to increase the efficiency of verification by reducing the state space of the relevant systems. For another application of resource equivalences we can refer to a Petri net reduction, in [36].

Since resource similarity and resource bisimilarity are congruences (w.r.t. addition), they are finitely-based; more specifically, they are generated by their minimal non-identity elements. The existence of such finite bases for $\approx$ and $\simeq$ gives some hope at least for semidecidability of these relations; the decidability proof in [6] for labeled communication-free Petri nets (under the name Basic Parallel Processes), where we have $\simeq = \approx = \sim$, was based on such semidecidability. Nevertheless, the previous research [26, 30] already clarified that resource similarity is undecidable (as well as bisimilarity), while decidability of resource bisimilarity remained open.

In this paper we have shown that resource bisimilarity is decidable for labeled Petri nets. In principle, we could proceed along the lines of two semidecision procedures like [6] but we have chosen to present a tableau-based algorithm deciding the problem. Regarding the computational complexity, we only mention the known PSPACE-completeness of $\sim$ for Basic Parallel Processes [7], where $\sim = \simeq$. (The PSPACE-lower bound was shown in [37].)

An interesting question that we have left open here is if the mentioned finite basis of $\simeq$ is computable (which does not follow from the decidability of $\simeq$). In the case of labeled communication-free Petri nets (i.e., Basic Parallel Processes) the answer is positive: it follows by the fact that [7] shows that a Presburger-arithmetic description of the relation $\sim$ can be computed in this case. We note that for resource similarity we know that the finite basis is not computable, since the relation $\approx$ is undecidable.

References

[1] van Glabbeek R. The Linear Time - Branching Time Spectrum. In: Bergstra J, Ponse A, Smolka S (eds.), Handbook of Process Algebra, pp. 3 – 99. Elsevier Science, Amsterdam. ISBN:978-0-444-82830-9, 2001. doi:10.1016/B978-044482830-9/50019-9.

[2] Park D. Concurrency and automata on infinite sequences. In: Deussen P (ed.), Theoretical Computer Science. Springer Berlin Heidelberg, Berlin, Heidelberg, 1981 pp. 167–183. ISBN:978-3-540-38561-5.

[3] Milner R. Communication and Concurrency. Prentice-Hall, Inc., USA, 1989. ISBN:0131150073.

[4] Jančar P. Undecidability of Bisimilarity for Petri Nets and Some Related Problems. *Theor. Comput. Sci.*, 1995. **148**(2):281–301. doi:10.1016/0304-3975(95)00037-W.

[5] Mayr R. Process Rewrite Systems. *Information and Computation*, 2000. **156**(1):264 – 286. doi:10.1006/inco.1999.2826.

[6] Christensen S, Hirshfeld Y, Moller F. Bisimulation equivalence is decidable for basic parallel processes. In: Best E (ed.), CONCUR'93. Springer Berlin Heidelberg, Berlin, Heidelberg, 1993 pp. 143–157. ISBN:978-3-540-47968-0.

[7] Jančar P. Bisimilarity on Basic Parallel Processes. *Theor. Comput. Sci.*, 2022. **903**:26–38. doi:10.1016/j.tcs.2021.11.027.

[8] Baldan P, Bonchi F, Gadducci F, Monreale GV. Modular encoding of synchronous and asynchronous interactions using open Petri nets. *Science of Computer Programming*, 2015. doi:10.1016/j.scico.2014.11.019.

[9] Heckel R. Open petri nets as semantic model for workflow integration. *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, 2003. doi:10.1007/978-3-540-40022-6_14.

[10] Dong X, Fu Y, Varacca D. Place bisimulation and liveness for open petri nets. In: Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics). ISBN: 9783319476766, 2016 doi:10.1007/978-3-319-47677-3_1.

[11] Lomazova IA, Romanov IV. Analyzing compatibility of services via resource conformance. In: Fundamenta Informaticae. 2013 doi:10.3233/FI-2013-937.

[12] Bashkin VA, Lomazova IA. Decidability of k-soundness for workflow nets with an unbounded resource. In: Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics). ISBN:9783662457290, 2014 doi:10.1007/978-3-662-45730-6_1.

[13] Sidorova N, Stahl C. Soundness for resource-constrained workflow nets is decidable. *IEEE Transactions on Systems, Man, and Cybernetics Part A:Systems and Humans*, 2013. doi:10.1109/TSMCA.2012.2210415.

[14] Girard JY. Linear logic. *Theoretical Computer Science*, 1987. doi:10.1016/0304-3975(87)90045-4.

[15] Farwer B. A Linear Logic view of object Petri Nets. *Fundamenta Informaticae*, 1999. doi:10.3233/fi-1999-37303.

[16] Farwer B, Lomazova I. A systematic approach towards object-based petri net formalisms. In: Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics). ISBN: 354043075X, 2001 doi:10.1007/3-540-45575-2_26.

[17] Olderog ER. Strong bisimilarity on nets: A new concept for comparing net semantics. In: de Bakker JW, de Roever WP, Rozenberg G (eds.), Linear Time, Branching Time and Partial Order in Logics and Models for Concurrency. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989 pp. 549–573. ISBN:978-3-540-46147-0.

[18] Autant C, Belmesk Z, Schnoebelen P. Strong Bisimilarity on Nets Revisited. In: Aarts EHL, van Leeuwen J, Rem M (eds.), Parle '91 Parallel Architectures and Languages Europe. Springer Berlin Heidelberg, Berlin, Heidelberg, 1991 pp. 717–734. ISBN:978-3-662-25209-3.

[19] Autant C, Schnoebelen P. Place bisimulations in Petri nets, pp. 45–61. Springer Berlin Heidelberg, Berlin, Heidelberg. ISBN:978-3-540-47270-4, 1992. doi:10.1007/3-540-55676-1_3.

[20] Autant C, Pfister W, Schnoebelen P. Place bisimulations for the reduction of labeled Petri nets with silent moves. In: Proc. 6th Int. Conf. on Computing and Information, Peterborough, Canada. 1994 .

[21] Quivrin-Pfister W. Des bisimulations de places pour la réduction des résaux de Petri. Phd thesis, I.N.P. de Grenoble, France, 1995.

[22] Schnoebelen P, Sidorova N. Bisimulation and the Reduction of Petri Nets, pp. 409–423. Springer Berlin Heidelberg, Berlin, Heidelberg. ISBN:978-3-540-44988-1, 2000. doi:10.1007/3-540-44988-4_23.

[23] Voorhoeve M. Structural Petri net equivalence. Technical Report 9607, Technische Universiteit Eindhoven, 1996.

[24] Gorrieri R. Team bisimilarity, and its associated modal logic, for BPP nets. *Acta Informatica*, 2020. pp. 1–41.

[25] Gorrieri R. A Study on Team Bisimulations for BPP Nets. In: Janicki R, Sidorova N, Chatain T (eds.), Application and Theory of Petri Nets and Concurrency. Springer International Publishing, Cham, 2020 pp. 153–175. ISBN:978-3-030-51831-8.

[26] Bashkin VA, Lomazova IA. Petri nets and resource bisimulation. *Fundamenta Informaticae*, 2003. **55**(2):101–114.

[27] Corradini F, De Nicola R, Labella A. Models of Nondeterministic Regular Expressions. *Journal of Computer and System Sciences*, 1999. **59**(3):412 – 449. doi:10.1006/jcss.1999.1636.

[28] Bashkin VA, Lomazova IA. Resource Similarities in Petri Net Models of Distributed Systems, pp. 35–48. Springer Berlin Heidelberg, Berlin, Heidelberg. ISBN:978-3-540-45145-7, 2003. doi:10.1007/978-3-540-45145-7_4.

[29] Bashkin VA, Lomazova IA. Similarity of Generalized Resources in Petri Nets, pp. 27–41. Springer Berlin Heidelberg, Berlin, Heidelberg, 2005. ISBN:978-3-540-31826-2. doi:10.1007/11535294_3.

[30] Lomazova IA. Resource Equivalences in Petri Nets. In: van der Aalst W, Best E (eds.), Application and Theory of Petri Nets and Concurrency. Springer International Publishing, Cham, 2017 pp. 19–34. ISBN:978-3-319-57861-3.

[31] Aceto L, Ingolfsdottir A, Srba J. The algorithmics of bisimilarity. In: Advanced Topics in Bisimulation and Coinduction. 2011. doi:10.1017/cbo9780511792588.004.

[32] Hennessy M, Milner R. Algebraic Laws for Nondeterminism and Concurrency. *J. ACM*, 1985. **32**(1):137–161. doi:10.1145/2455.2460.

[33] Christensen S. Decidability and decomposition in process algebras. Ph.D. thesis, University of Edinburgh, UK, 1993. URL http://hdl.handle.net/1842/410.

[34] Rédei L. The theory of finitely generated commutative semigroups. Oxford University Press, New-York, 1965.

[35] Hirshfeld Y. Congruences in commutative semigroups. Technical Report ECS-LFCS-94-291, Department of Computer Science, University of Edinburgh, 1994.

[36] Bashkin VA, Lomazova IA. Reduction of Coloured Petri Nets based on resource bisimulation. *Joint Bulletin of NCC & IIS (Comp. Science)*, 2000. **13**:12–17.

[37] Srba J. Strong bisimilarity of simple process algebras: complexity lower bounds. *Acta Informatica*, 2003. **39**(6-7):469–499. doi:10.1007/s00236-003-0116-9.